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Mathematics

Continuous family of finite-dimensional representations of a solvable Lie algebra arising from singularities

(moduli algebra/derivation algebra)

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ABSTRACT **A natural mapping from the set of complex analytic isolated hypersurface singularities to the set of finite dimensional Lie algebras is first defined. It is proven that the image under this natural mapping is contained in the set of solvable Lie algebras. This approach gives rise to a continuous inequivalent family of finite dimensional representations of a solvable Lie algebra.**

Let G be a semi-simple Lie group acting on its Lie algebra $\mathfrak{G}$ by the adjoint action and let $\mathfrak{G}/G$ be the variety corresponding to the G-invariant polynomials on $\mathfrak{G}$. The quotient morphism $\chi:\mathfrak{G} \to \mathfrak{G}/G$ was intensively studied by Kostant (1, 2).

Let $\mathfrak{H} \subset \mathfrak{G}$ be a Cartan subalgebra of $\mathfrak{G}$ and W be the corresponding Weyl group.

(*i*) The space $\mathfrak{G}/G$ may be identified with the set of semi-simple G classes in $\mathfrak{G}$ such that χ maps an element $x \in \mathfrak{G}$ to the class of its semi-simple part x_s.

(*ii*) By a theorem of Chevalley, the space $\mathfrak{G}/G$ is isomorphic to $\mathfrak{H}/W$, an affine space of dimension r = rank $\mathfrak{G}$, the isomorphism being given by the map of a semi-simple class to its intersection with $\mathfrak{H}$ (a W orbit).

The following beautiful theorem of Brieskorn (3) conjectured by Grothendieck (4) establishes connections between the simple singularities and the simple Lie algebras.

THEOREM. *Let $\mathfrak{G}$ be a simple Lie algebra over* **C** *of type* A_r, D_r, E_r. *Then*

(i) *The intersection of the variety* $N(\mathfrak{G})$ *of the nilpotent elements of $\mathfrak{G}$ with a transverse slice S to the subregular orbit, which has codimension 2 in* $N(\mathfrak{G})$, *is a surface* $S \cap N(\mathfrak{G})$ *with an isolated rational double point of the type corresponding to the algebra* $\mathfrak{G}$.

(ii) *The restriction of the quotient* $\chi:\mathfrak{G} \to \mathfrak{H}/W$ *to the slice S is a realization of a semi-universal deformation of the singularity in* $S \cap N(\mathfrak{G})$.

The complete detail of this theory can be found in Slodowy's papers (5, 6).

Among many other things, Brieskorn's theory gives construction of rational double points from simple Lie algebras. A natural question to ask is whether one can construct Lie algebras from isolated hypersurface singularities. The purpose of this note is to announce a way to establish connections between the set of isolated singularities and the set of finite dimensional Lie algebras. In fact, a natural mapping L from the set of isolated hypersurface singularities to the set of finite dimensional Lie algebras can be defined. The mapping L is defined as follows. Let $(V,0)$ be an isolated hypersurface singularity in $(\mathbf{C}^{n+1},0)$ defined by the zero set of a holomorphic function f. Recall that the moduli algebra $A(V)$ of V is $\mathbf{C}\{z_0,z_1,\ldots,z_n\}\Big/\left(f,\frac{\partial f}{\partial z_0},\ldots,\frac{\partial f}{\partial z_n}\right)$. $L(V)$ is defined to be the algebra of derivations of $A(V)$, since $A(V)$ is finite dimensional as **C**-vector space and $L(V)$ is contained in the endomorphism algebra of $A(V)$; consequently, $L(V)$ is a finite dimensional Lie algebra. Let $\mathcal{O}_{n+1}$ denote the ring of germs of the origin of holomorphic functions $(C^{n+1},0) \to \mathbf{C}$. Let $\mathcal{O}_{V,0} = \mathcal{O}_{n+1}/(f)\mathcal{O}_{n+1}$ be the local ring of V at 0. It is clear that we have the following lemma.

LEMMA 1. *A derivation of* $\mathcal{O}_{V,0}$ *induces a derivation of* A(V). *Hence, there is a natural map from the algebra of derivations of* $\mathcal{O}_{V,0}$ *to* L(V).

Remark 2: The above natural map is highly nonsurjective. This can be seen as follows:

Let us assume for a moment that f is a weighted homogenous function—i.e., there exist $q_0,\ldots,q_n$, $d \in \mathbf{N}$ (the set of positive integers) such that

$$f(t^{q_0}z_0,\ldots,t^{q_n}z_n) = t^d f(z_0,\ldots,z_n) \qquad [1]$$

for all $(z_0,\ldots,z_n) \in \mathbf{C}^{n+1}$ and $t \in \mathbf{C}^* = C - \{0\}$. Then $\Sigma_{i=0}^n q_i z_i \frac{\partial}{\partial z_i}$ is a derivation of the local ring $\mathcal{O}_{V,0}$. This distinguished derivation is called the Euler derivation. The following proposition is well known.

PROPOSITION 3. *Let* (V,0) *be an isolated singularity with* $\mathbf{C}^*$-*action—i.e.,* $\mathcal{O}_{V,0} = \mathcal{O}_{n+1}/(f)\mathcal{O}_{n+1}$, *where* f *is a weighted homogenous holomorphic function. Then the algebra of derivations of* $\mathcal{O}_{V,0}$ *is generated as an* $\mathcal{O}_{V,0}$ *module by the Euler derivation* E *and the following derivations*

$$\frac{\partial f}{\partial z_j}\frac{\partial}{\partial z_i} - \frac{\partial f}{\partial z_i}\frac{\partial}{\partial z_j}.$$

Example 1: Let $V = \{(x,y,z) \in \mathbf{C}^3 : x^3 + y^3 + z^3 = 0\}$. Then the moduli algebra

$$A(V) := \mathbf{C}(x,y,z)/(x^2,y^2,z^2)$$
$$\cong \mathbf{C}\text{-vector space spanned by } \langle 1,x,y,z,xy,yz,zx,xyz\rangle$$
$$\text{with multiplication rules } x^2 = 0,\ y^2 = 0,\ z^2 = 0.$$

The Lie algebra associated to the singularity is

$$L(V) \cong \mathbf{C}\text{-vector space spanned by } \left\langle x\frac{\partial}{\partial x}, xy\frac{\partial}{\partial x}, zx\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, xy\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, xyz\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, yz\frac{\partial}{\partial z}, zx\frac{\partial}{\partial z}, xyz\frac{\partial}{\partial z}\right\rangle.$$

Clearly, the natural mapping from the algebra of derivations of $\mathcal{O}_{V,0}$ to $L(V)$ is highly nonsurjective in view of *Proposition* 3.

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The following proposition says that the Lie algebra obtained in this way is a nontrivial invariant.

PROPOSITION 4. *Let* $f(z_0,\ldots,z_n)$ *be a weighted homogenous function. Suppose* $V = \{(z_0,\ldots,z_n) \in \mathbf{C}^{n+1}: f(z_0,\ldots,z_n) = 0\}$ *has an isolated singularity of the origin. Then the Lie algebra* L(V) *associated to the singularity* (V,0) *is abelian if and only if* (V,0) *is either* A_1 *or* A_2.

Sketch of the Proof: "if" part is easy.

"only if." The idea is to construct a monomial derivation which does not commute with the Euler derivation in case (V,0) is neither A_1 nor A_2.

In ref. 7 it was proved that the natural mapping

$$\left\{\begin{matrix}\text{isolated hypersurface singu-}\\ \text{larities of dimension } n\end{matrix}\right\} \to \left\{\begin{matrix}\text{commutative local}\\ \text{Artinian algebras}\end{matrix}\right\}$$

$$\begin{matrix} \Cup & & \Cup \\ (V,0) & \to & A(V) = \text{moduli algebra of } V\end{matrix}$$

is one-to-one. The natural question to ask is the recognition problem—i.e., when a commutative local Artinian algebra is a moduli algebra of an isolated hypersurface singularity or, in other words, characterizes the image of the above map. The following theorem gives a necessary condition for an Artinian commutative algebra to be a moduli algebra.

THEOREM 5. *Let* (V,0) *be an isolated hypersurface singularity. Then* L(V) *is a finite dimensional solvable Lie algebra.*

The main idea in the proof is to observe that the derivation algebra acts on $\bigoplus_{k=1}^{\infty} m^k/m^{k+1}$ and leaves the initial ideal of the moduli ideal $\left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right)$ invariant. In the course of the proof, the length filtration on space of homogenous polynomials of fixed degree with respect to first-order derivations has to be introduced. The proof is quite technical and is proved by induction.

The following example shows that even for singularity without $\mathbf{C}^*$-action, not all elements in $L(V)$ have coefficients in m^2.

Example 2: Let $V = \{(x,y,z) \in \mathbf{C}^3: z^2 = y^3 + x^2y^2 + x^6y + x^8\}$. The resolution graph is

$-3 \qquad -2$

The irreducible components of the exceptional set are rational curves. The moduli algebra

$$A(V) = \mathbf{C}\{x,y\}/(y^3 + x^2y^2 + x^6y + x^8,\ 2xy^2 + 6x^5y + 8x^7,\ 3y^2 + 2x^2y + x^6)$$
$$\cong \langle 1, x, y, x^2, xy, x^3, x^2y, x^4, x^5, x^6, x^7\rangle$$

with multiplication rules

$$y^2 = -{}^2\!/_3x^2y - {}^1\!/_3x^6,\ y^3 = 0,\ xy^2 = -4x^7,$$
$$x^3y = {}^{11}\!/_2x^7,\ x^2y^2 = 0,\ x^4y = 0,\ x^8 = 0.$$

$$L(V) \cong \left\langle y\frac{\partial}{\partial x} + 8x^5\frac{\partial}{\partial y},\ xy\frac{\partial}{\partial x},\ x^2\frac{\partial}{\partial x} - 14x^5\frac{\partial}{\partial y},\ x^2y\frac{\partial}{\partial y},\ x^2y\frac{\partial}{\partial x},\ x^3\frac{\partial}{\partial x},\ x^4\frac{\partial}{\partial x},\ x^5\frac{\partial}{\partial x},\ x^6\frac{\partial}{\partial x},\ x^7\frac{\partial}{\partial x},\ x^6\frac{\partial}{\partial y},\ x^7\frac{\partial}{\partial y},\ (xy + {}^{13}\!/_2x^5)\frac{\partial}{\partial y}\right\rangle.$$

Let us consider a family of nonsingular elliptic curves in $\mathbf{CP}^2$ defined by

$$x^3 + y^3 + z^3 + \nu xyz = 0, \qquad [2]$$

where $\nu^3 + 27 \neq 0$. The complex structure of the elliptic curve depends on ν. In fact, $j = -\frac{1}{27.4}\frac{\nu^6}{\nu^3 + 27}$. If we view Eq. 2 as an equation in affine 3-space, we have a family of so-called simple elliptic singularities V_ν. For each fixed ν with $\nu^3 + 27 \neq 0$, the moduli algebra

$$A(V_\nu) \cong \langle 1, x, y, z, xy, yz, xyz\rangle$$

with multiplication rules

$$x^2 = -\frac{\nu}{3}yz,\ y^2 = -\frac{\nu}{3}xz,\ z^2 = -\frac{\nu}{3}xy$$
$$x^2y = xy^2 = y^2z = yz^2 = x^2z = xz^2 = 0$$
$$x^3 = y^3 = z^3 = -\frac{\nu}{3}xyz$$
$$x^iy^jz^k = 0 \quad \text{for} \quad i + j + k \geq 4.$$

We shall assume $\nu \neq 0$ and $\frac{\nu^6}{27} - 7\nu^3 - 216 \neq 0$. Under these assumptions

$$L(V_\nu) = \left\langle xy\frac{\partial}{\partial x} - \frac{\nu}{6}zx\frac{\partial}{\partial y},\ zx\frac{\partial}{\partial x} - \frac{\nu}{6}xy\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial x},\ -\frac{\nu}{6}yz\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\ yz\frac{\partial}{\partial y} - \frac{\nu}{6}xy\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial y},\ -\frac{\nu}{6}zx\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z},\ -\frac{\nu}{6}yz\frac{\partial}{\partial x} + zx\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial z},\ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right\rangle.$$

Remark 6: $\dim L(V_\nu) = 10$ for $\nu \neq 0$ and $\frac{\nu^6}{27} - 7\nu^3 - 216 \neq 0$. In *Example* 1, I have computed $\dim L(V_0) = 12$. We see that $\dim L(V)$ is already an interesting analytic invariant of the singularity (V,0).

PROPOSITION 7. *For* $\nu \neq 0$ *and* $\frac{\nu^6}{27} - 7\nu^3 - 216 \neq 0$, $L(V_\nu)$ *are isomorphic as Lie algebra.*

Propositions 10 and 11 below say that the above solvable Lie algebra has a continuous family of inequivalent finite dimensional representation.

Definition 8: The Lie algebra $L(V)$ acts naturally on the moduli algebra. This representation is called the natural representation.

Definition 9: The solvable Lie algebra in *Proposition* 6 is called $L(\tilde{E}_6)$.

PROPOSITION 10. $L(\tilde{E}_6)$ *is isomorphic to the Lie algebra generated by the following matrices*

where ν *is fixed with* $\nu \neq 0$, *and* $\frac{\nu^6}{27} - 7\nu^3 - 216 \neq 0$.

PROPOSITION 11. *The matrix representations of* $L(\tilde{E}_6)$ *given by* Proposition *10 above are all inequivalent.*

Remark 12: *Propositions* 10 and 11 give a one-parameter family of inequivalent finite dimensional representation of the Lie algebra $L(\tilde{E}_6)$. I suspect that this family is not obtainable by letting the automorphism group of the Lie algebra act on a representation.

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$$e_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\nu}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\nu}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad e_4 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\nu}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e_5 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\nu}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad e_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e_7 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\nu}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad e_8 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\nu}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad e_{10} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

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