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Mathematics

Criterion for biholomorphic equivalence of isolated hypersurface singularities

(moduli algebra/finite determinancy)

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ABSTRACT Two germs of complex analytic hypersurfaces with isolated singularities are biholomorphically equivalent if and only if they have the same dimension and their moduli algebras are isomorphic.

In this note, we announce a criterion for two germs of complex analytic hypersurfaces with isolated singularities to be biholomorphically equivalent. Our criterion was conjectured by H. Hironaka (personal communication).

Let $\mathcal{O}_{n+1}$ denote the ring of germs at the origin of holomorphic functions $(\mathbf{C}^{n+1}, 0) \to \mathbf{C}$. If $(V,0)$ is a germ at the origin of a hypersurface in $\mathbf{C}^{n+1}$, the ideal $I(V)$ of functions in $\mathcal{O}_{n+1}$ vanishing on V is principal. Let f be a generator of it. It is well known that $V - 0$ is nonsingular if and only if the $\mathbf{C}$-vector space

$$A(V) = \mathcal{O}_{n+1}\Big/\left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right)\mathcal{O}_{n+1}$$

is finite dimensional. The germ at 0 of $A(V)$ is the base space for the miniversal deformation of $(V,0)$, so it seems natural to call $A(V)$, provided with the obvious $\mathbf{C}$-algebra structure, the moduli algebra of V. Our theorem will state that V is determined by $A(V)$ and also by the following $\mathbf{C}$-algebra:

$$B(V) = \mathcal{O}_{n+1}\Big/\left(f, z_i\frac{\partial f}{\partial z_j}\right)\mathcal{O}_{n+1}, \quad 0 \le i, j \le n.$$

THEOREM. *Suppose* $(V,0)$ *and* $(W,0)$ *are germs of hypersurfaces in* $\mathbf{C}^{n+1}$, *and* $V - 0$ *is nonsingular. Then the following conditions are equivalent:*

(i) $(V,0)$ *is biholomorphically equivalent to* $(W,0)$.
(ii) $A(V)$ *is isomorphic to* $A(W)$ *as a* $\mathbf{C}$-*algebra.*
(iii) $B(V)$ *is isomorphic to* $B(W)$ *as a* $\mathbf{C}$-*algebra.*

Remark: When V is a homogeneous hypersurface, the above theorem was proved previously by Max Benson (personal communication) with a different method.

Outline of the proof

(*i*) → (*ii*) and (*i*) → (*iii*) are easily checked.

Section 2 in ref. 1 defines a group $\mathcal{K}$ of germs of C^∞ mappings associated to a pair (n,p) of positive integers. In this note, we let $\mathcal{K}$ denote the analogous group of holomorphic mappings associated to $(n+1,1)$. Then $\mathcal{K}$ acts on $\mathcal{O}_{n+1}$ and $f,g \in \mathcal{O}_{n+1}$ define biholomorphically equivalent germs of hypersurfaces if and only if they lie in the same $\mathcal{K}$-orbit.

Let J^k denote the $\mathbf{C}$-vector space of k-jets of elements of $\mathcal{O}_{n+1}$. Let $\mathcal{K}^k$ denote the Lie group of k-jets of members of $\mathcal{K}$. Then $\mathcal{K}^k$ acts on J^k. Let f and g generate $I(V)$ and $I(W)$, respectively. The hypothesis that $V - 0$ is nonsingular implies that f is finitely determined with respect to the group $\mathcal{K}$. This means that if k is a sufficiently large positive integer, then $g^{(k)} \in \mathcal{K}^k f^{(k)}$ implies $g \in \mathcal{K}f$, where $f^{(k)}$ denotes the k-jet of f at the origin. The finite determinancy of f follows from the Nullstellensatz and the methods of ref. 1.

To prove (*iii*) → (*i*), it suffices to prove (*i*) under the stronger hypothesis

$$\left(f, z_i\frac{\partial f}{\partial z_j}\right)\mathcal{O}_{n+l} = \left(g, z_i\frac{\partial g}{\partial z_j}\right)\mathcal{O}_{n+1}. \qquad [1]$$

For, we may reduce the general assertion to this case by a coordinate change that maps the ideal $\left(g, z_i\frac{\partial g}{\partial z_j}\right)\mathcal{O}_{n+1}$ onto $\left(f, z_i\frac{\partial f}{\partial z_j}\right)\mathcal{O}_{n+1}$.

Because f is finitely determined, it is enough to prove that $g^{(k)} \in \mathcal{K}^k f^{(k)}$ for every positive integer k. The assertion that $g^{(k)} \in \mathcal{K}^k f^{(k)}$ follows from lemma 3.1 of ref. 2, which we apply by taking its manifold V to be the complex line in J^k containing $f^{(k)}$ and $g^{(k)}$, minus at most finitely many points where lemma 3.1b of ref. 2 fails to hold. The fact that the hypotheses of lemma 3.1 of ref. 2 hold at all but at most finitely many points of the complex line joining $f^{(k)}$ and $g^{(k)}$ follows easily from Eq. 1 and the complex analogue of proposition 7.4a in ref. 1.

Thus, (*iii*) → (*i*). Finally, we show (*ii*) → (*iii*). Again, by coordinate change, we may reduce to the case when

$$\left(f, \frac{\partial f}{\partial z_i}\right)\mathcal{O}_{n+1} = \left(g, \frac{\partial g}{\partial z_i}\right)\mathcal{O}_{n+1}. \qquad [2]$$

Then $f = ag + \sum_{i=0}^{n} a_i \frac{\partial g}{\partial z_i}$, where $a, a_0, \ldots, a_n \in \mathcal{O}_{n+1}$. We may also assume the multiplicity of $g \ge 2$, for otherwise (*ii*) → (*iii*) is obvious. We will show that $a_i(0) = 0$ for $i = 0, \ldots, n$. If not, a further change of coordinates permits us to suppose $a_0(0) = 1$, $a_i(0) = 0$, for $i \ge 1$. Let $\tau: (\mathbf{C},0) \to (\mathbf{C}^{n+1},0)$ be a nonvanishing germ of a mapping such that $\frac{\partial g}{\partial z_i} \cdot \tau = 0$ for $i \ge 1$. Then $\frac{d(g \cdot \tau)}{dt} = \left(\frac{\partial g}{\partial z_0} \cdot \tau\right)\frac{d\tau_0}{dt}$. Consequently, multiplicity of $\frac{\partial g}{\partial z_0} \cdot \tau <$ multiplicity of $g \cdot \tau$, and we obtain multiplicity of $f \cdot \tau =$ multiplicity of $\frac{\partial g}{\partial z_0} \cdot \tau$. Because $\frac{d(f \cdot \tau)}{dt} = \sum_{i=0}^{n}\left(\frac{\partial f}{\partial z_i} \cdot \tau\right)\frac{d\tau_i}{dt}$, we obtain multiplicity of $\frac{\partial f}{\partial z_i} \cdot \tau <$ multiplicity of $f \cdot \tau$ for at least one i. Hence the multiplicity of the ideal $\left(f \cdot \tau, \frac{\partial f}{\partial z_i} \cdot \tau\right)\mathcal{O}_1$ is $<$ the multiplicity of the ideal $\left(g \cdot \tau, \frac{\partial g}{\partial z_i} \cdot \tau\right)\mathcal{O}_1 =$ multiplic-

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ity of $\frac{\partial g}{\partial z_0} \cdot \tau$. But this contradicts Eq. 2.

Hence $a_i(0) = 0$ for $i = 0, \ldots, n$, and $f \in \left(g, z_i \frac{\partial g}{\partial z_j}\right)$. A similar argument shows that $g \in \left(f, z_i \frac{\partial f}{\partial z_j}\right)$. From these facts and Eq. 2, we get Eq. 1. Hence *(ii)* $\rightarrow$ *(iii)*.

Our theorem is not true over the real numbers and over fields of characteristic $p > 0$, as the following examples show.

Example 1: Let $V = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 0\}$, $W = \{(x,y) \in \mathbf{R}^2 : xy = 0\}$. Then $A(V) = A(W)$, $B(V) = B(W)$, but $(V,0)$ is not real analytically equivalent to $(W,0)$.

The most natural way to formulate a result in characteristic $p > 0$ analogous to our theorem seems to be in terms of the local rings of formal power series associated to the varieties. Thus, over any field k, given a "hypersurface" $V = \{f=0\}$, where $f \in k[[x_0, \ldots, x_n]]$, we define $\mathcal{F}(V) = k[[x_0, \ldots, x_n]]/(f)k[[x_0, \ldots, x_n]]$ and $A(V)$, $B(V)$, as before, using the ring of formal power series in place of the ring of convergent power series. If k is an algebraically closed field of characteristic 0, we have $\mathcal{F}(V) \approx \mathcal{F}(W) \leftrightarrow A(V) \approx A(W) \leftrightarrow B(V) \approx B(W)$, where $\approx$ denotes isomorphism of k-algebras. Example 1 shows that this result is not true when $k = \mathbf{R}$. The following example, due to Max Benson and H. Hironaka (personal communication), shows that it is not true when k is a field of characteristic $p > 0$.

Example 2: Let k be a field of characteristic $p > 0$. Let $V = \{f=0\}$, $W = \{g=0\}$, where $f = f(x,y) = x^{p+1} + y^{p+1}$, $g = g(x,y) = x^{p+1} + x^p y + y^{p+1}$. Then $A(V) \approx A(W)$ and $B(V) \approx B(W)$, but $\mathcal{F}(V) \neq \mathcal{F}(W)$.

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2. Mather, J. N. (1970) *Inst. Hautes Études Sci. Publ. Math.* **37**, 223–248.