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EXISTENCE OF L^2-INTEGRABLE HOLOMORPHIC FORMS AND LOWER ESTIMATES OF T_V^1

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§0. **Introduction.** Let (V,q) be normal isolated singularity of dimension $n \geqslant 2$. It is easy to see that holomorphic functions defined on $V - \{q\}$ can be extended across q. However for holomorphic forms, the situation is completely different. Even if we assume that the holomorphic form ω defined on $V - \{q\}$ is L^2-integrable in a neighborhood of q in the sense of Griffiths ([4], [13]), it is not clear whether ω can be extended across q. In [13], the Griffiths number $g^{(p)}$ was introduced to measure how many L^2-integrable holomorphic forms on $V - \{q\}$ cannot be extended across q. Similarly, let us denote the number of holomorphic p-forms on $V - \{q\}$ which cannot be extended across q by $\delta^{(p)}$. In case of hypersurface singularities, these invariants are computed (cf. Theorem 1.1). One can see among these numbers, $g^{(n)}$, $g^{(n-1)}$, $\delta^{(n)}$ and $\delta^{(n-1)}$ are the most interesting invariants. The following are our main theorems:

THEOREM A. *Let $\pi: M \to V$ be any resolution of the singularity of V. Then*
(a) *For $n = 2$, $g^{(2)} \geqslant 1$ and $\delta^{(2)} \geqslant 1 + \dim H^1(M, \mathcal{O})$*
(b) *For $n \geqslant 3$, $g^{(n)} \geqslant n - 1$ and $\delta^{(n)} \geqslant n - 1 + \dim H^{n-1}(M, \mathcal{O})$ if $\dim H^{n-1}(M, \mathcal{O}) > 0$.*

THEOREM B. *Suppose that (V,q) admits a $\mathbf{C}^*$-action. Then $g^{(n-1)} \geqslant g^{(n)}$ and $\delta^{(n-1)} \geqslant \delta^{(n)}$.*

The invariant $\delta^{(n-1)}$ is of particular interest because in the case of Gorenstein surface singularity, $\delta^{(n-1)}$ is exactly equal to $\dim T_V^1$ where T_V^1 is the set of isomorphism class of first order infinitesimal deformation of V. By Grauert [3], T_V^1 can be thought of as a Zariski tangent space of the moduli space of (V,q). An important application of Theorem A and Theorem B is the following corollary.

COROLLARY C. *Let (V,q) be a Gorenstein surface singularity with C^*-action. Then $\dim T_V^1 \geqslant 1 + \dim H^1(M, \mathcal{O})$.*

We should remark that it is a long standing conjecture that $\dim T_V^1 > 0$ for Gorenstein surface singularities. In case (V,q) admits a C^*-action, our Corollary C is far better than the original conjecture. In fact in [14], we have already proved that $\dim T_V^1 > 0$ for Gorenstein surface singularities with C^*-action. More recently J. Wahl has informed us that he has obtained $\dim T_V^1 > 0$ for two-dimensional normal singularities with C^*-action. The detail of his proof will be available soon.

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§2 is devoted to the questions raised by A. Sommese. He asked whether the canonical bundle K over a strongly pseudoconvex two-dimensional manifold which contains no compact Riemann surface of self-intersection number -1 is generated by its global sections in a neighborhood of the exceptional set. Theorem 2.2 and Example 2.4 give a complete answer to his question. We include this section here because of its later application to §3 and §4.

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§1. **Preliminaries.** Let V be a normal n-dimensional ($n \geq 2$) Stein space with q as its only singularity. Let $\bar{\Omega}_V^p$ be the sheaf of germs of holomorphic p-forms of $V - \{q\}$ which are locally L^2-integrable in the sense of Griffiths [4]. Then actually $\bar{\Omega}_V^p$ is equal to the 0-th direct image sheaf $\pi_* \Omega_M^p$ where M is a strongly pseudoconvex manifold and $\pi : M \to V$ is a resolution of the singularity of V. Let Ω_V^p be the usual sheaf of germs of holomorphic p-forms on V in the sense of Grauert-Grothendieck (i.e., Kähler differentials). Clearly there is a natural map from Ω_V^p to $\bar{\Omega}_V^p$. The coker of this map is a finite dimensional vector space over q. In [13], the p-th Griffiths number $g^{(p)}$ of the singularity q is defined to be the dimension of this finite dimensional vector space. The following theorem which was proved in [13] gives an explicit way to compute $g^{(p)}$ in case q is a hypersurface singularity.

THEOREM 1.1. *Let $f(z_0, z_1, \ldots, z_n)$ be holomorphic in N, a Stein neighborhood of the origin in* $\mathbf{C}^{n+1}$ *with $f(0,0,\ldots,0) = 0$. Let $V = \{(z_0, z_1, \ldots, z_n) \in N : f(z_0, z_1, \ldots, z_n) = 0\}$ have the origin as its only singular point. Let $\pi : M \to V$ be a resolution of the singularity of V with A as the exceptional set in M. Let $\tau =$* $\dim \mathbf{C}\,[[z_0, z_1, \ldots, z_n]]/(f, (\partial f/\partial z_0), \ldots, (\partial f/\partial z_n))$, *$\mu$ be the Milnor number and* $h^{(n-1)} = \dim H^{n-1}(M, \mathcal{O})$ *be the $(n-1)$-th Hironaka number. Then*

(a) $g^{(p)} = 0 \qquad 0 \leq p \leq n-2.$

(b) $g^{(n-1)} = \mu + (-1)^n + (-1)^{n+1}\chi_T(A) + \sum_{p=1}^{n-1}(-1)^{n+p}\chi(\Omega^p) - 2h^{(n-1)}$

(c) $g^{(n)} = \tau - h^{(n-1)}$

where $\chi(\Omega^p) = \sum_{i=1}^{n}(-1)^i \dim H^i(M, \Omega^p)$, $\chi_T(A)$ is the topological Euler characteristic of the exceptional set A.

The theorem above suggests that among all Griffiths numbers, $g^{(n)}$ and $g^{(n-1)}$ are the most interesting numbers. The following lemma is obvious.

LEMMA 1.2. *Let V be a Stein analytic space with q as its only singular point. Let $\pi : M \to V$ be a resolution of the singularity. Then*

$$g^{(p)} = \dim \Gamma(M, \Omega^p)/\pi^*\Gamma(V, \Omega^p).$$

Now we introduce another invariant of the singularity $\delta^{(p)} := \dim_{\mathbf{C}} \Gamma(V - \{q\}, \Omega^p)/\Gamma(V, \Omega^p)$. The fact that $\delta^{(p)}$ is defined depends actually only on the singularity q can be seen as follows. Let $\theta : V - \{q\} \to V$ be the inclusion map. Then the 0-th direct image sheaf $\theta_* \Omega^p_{V-\{q\}} = \bar{\bar{\Omega}}^p_V$ is coherent by Siu's theorem [9]. Clearly we have a natural map $\Omega^p_V \to \bar{\bar{\Omega}}^p_V$. It is easy to see that $\delta^{(p)}$ is the dimension of the coker of this map which is a finite dimension vector space over q. The proof of the following Lemma 1.3(i) can be found for instance in [12] while the statement of Lemma 1.3(ii) is trivial.

LEMMA 1.3.
(i) $\delta^{(p)} = 0$ *for* $1 \leqslant p \leqslant n-1$ *if* q *is a hypersurface singularity.*
(ii) $\delta^{(p)} = g^{(p)} + s^{(p)}$ *where* $s^{(p)} = \dim_{\mathbf{C}}(\bar{\bar{\Omega}}^p_V / \bar{\Omega}^p_V)$.

We have the following conjecture.

CONJECTURE.
(i) $g^{(n-1)} \geqslant n-1$
(ii) $\delta^{(n-1)} \geqslant h^{(n-1)} + n - 1$ *where* $h^{(n-1)} = \dim H^{n-1}(M, \mathcal{O})$ *is the* $(n-1)$*-th Hironaka number as before.*

The above conjecture is proved for singularities which admit a C^*-action.

§2. Canonical bundle on two-dimensional strongly pseudoconvex manifolds. The purpose of this section is to answer a question raised by A. Sommese. If the strongly pseudoconvex two-dimensional manifold M contains no complex curve with self intersection number -1, is the canonical bundle K generated by its global sections in a neighborhood of the exceptional set? We shall give an affirmative answer to the above question in case $H^1(M, \mathcal{O}) = 0$. A counterexample to the above question will be given also in case $H^1(M, \mathcal{O}) \neq 0$.

Let $A = \cup A_i$, $1 \leqslant i \leqslant n$, be the decomposition of the exceptional set A into irreducible components. We shall assume without loss of generality that A is connected. A cycle D on A is an integral combination of the A_i, i.e., $D = \sum d_i A_i$, $1 \leqslant i \leqslant n$, with d_i an integer. There is a natural partial ordering between cycles defined by comparing the coefficients. Let Z be the unique fundamental cycle [1] such that $Z > 0$, $A_i \cdot Z \leqslant 0$ for all A_i, and such that Z is minimal with respect to those two properties. Z may be computed from the intersection matrix as follows [6] via what is called a computation sequence for Z:

$$Z_0 = 0, Z_1 = A_{i_1}, \ldots, Z_j = Z_{j-1} + A_{i_j}, \ldots$$

$$Z_l = Z_{l-1} + A_{i_l} = Z,$$

where A_{i_1} is arbitrary and $A_{i_j} \cdot Z_{j-1} > 0$. $1 < j \leqslant l$ $\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)$ represents the sheaf of germs of sections of a line bundle over A_{i_j} of Chern class $-A_{i_j} \cdot Z_{j-1}$. So $H^0(M, \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)) = 0$ for $j > 1$.

$$0 \to \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j) \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_{j-1}} \to 0$$

is an exact sheaf sequence. From the corresponding long exact cohomology sequence, it follows by induction that

$$H^0(M, \mathcal{O}_{Z_k}) = \mathbf{C} \qquad 1 \leqslant k \leqslant l \tag{2.1}$$

$$\dim H^1(M, \mathcal{O}_{Z_k}) = \sum \dim H^1(M, \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)) \qquad 1 \leqslant j \leqslant k. \tag{2.2}$$

Let K be the canonical divisor on M. Let g_i be the geometric genus of A_i, i.e., the genus of the desingularization of A_i. Then [8, p. 75]

$$A_i \cdot K = -A_i \cdot A_i + 2g_i - 2 + 2\delta_i \tag{2.3}$$

where δ_i is the "number" of nodes and cusps on A_i. Each singular point on A_i other than a node or cusp counts as at least two nodes. The following lemma is a trivial consequence of (2.1), (2.2), and (2.3).

LEMMA 2.1. *With the notations as above, if* $\dim H^1(M, \mathcal{O}) = 0$, *then the exceptional set* A *has normal crossings,* $Z_{j-1}A_{i_j} = 1$ $1 < j \leqslant l$ *and the irreducible components of* A *are nonsingular rational curves.*

THEOREM 2.2. *Let* M *be a two-dimensional strongly pseudoconvex manifold with* $H^1(M, \mathcal{O}) = 0$. *Suppose* M *contains no exceptional curve of the first kind. Then the canonical bundle* K *of* M *is generated by its global sections in a neighborhood of the exceptional set.*

Proof. Let A_1 be an arbitrary irreducible component of the exceptional set. Consider the following exact sheaf sequence

$$0 \to K(-A_1) \to K \to K \otimes \mathcal{O}_{A_1} \to 0.$$

By the adjunction formula, $K \otimes \mathcal{O}_{A_1} = K_{A_1} \otimes \mathcal{O}_{A_1}(-A_1)$ where K_{A_1} is the canonical bundle of A_1. Since M contains no exceptional curve of first kind, the Chern class of the conormal sheaf $\mathcal{O}_{A_1}(-A_1)$ of A_1 is at least 2. It follows that $K \otimes \mathcal{O}_{A_1}$ is generated by its global sections as the Chern class of $K \otimes \mathcal{O}_{A_1}$ is nonnegative. It remains to prove $H^1(M, K(-A_1)) = 0$. Let $0 = Z_0, \ldots, Z_l = Z$ be a computation sequence for the fundamental cycle Z with $A_{i_1} = A_1$. Consider the following sheaf exact sequence

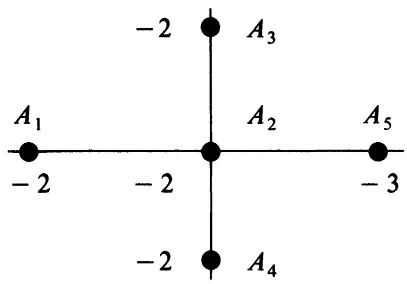
$$\begin{array}{lllll} 0 \to & K(-nZ - Z_1) \to & K(-nZ) \to & K(-nZ) \otimes \mathcal{O}_{A_{i_1}} & \to 0 \\ & \vdots & \vdots & \vdots & \\ 0 \to & K(-nZ - Z_j) \to & K(-nZ - Z_{j-1}) \to & K(-nZ - Z_{j-1}) \otimes \mathcal{O}_{A_{i_j}} & \to 0 \\ & \vdots & \vdots & \vdots & \\ 0 \to & K(-nZ - Z_l) \to & K(-nZ - Z_{l-1}) \to & K(-nZ - Z_{l-1}) \otimes \mathcal{O}_{A_{i_l}} & \to 0. \end{array} \tag{2.4}$$

$K(-nZ - Z_{j-1}) \otimes \mathcal{O}_{A_{i_j}}$ is the sheaf of germs of sections of a line bundle over A_{i_j}

of Chern class $-A_{i_j}(nZ + Z_{j-1}) + c(K \otimes \mathcal{O}_{A_{i_j}})$. Since $A_{i_j}Z_{j-1} = 1$ and $c(K \otimes \mathcal{O}_{A_{i_j}}) \geqslant 0$ for $1 < j \leqslant l$, so $A_{i_j} \cdot (nZ + Z_{j-1}) - c(K \otimes \mathcal{O}_{A_{i_j}}) \leqslant 1$, for all j and all n. Thus $H^1(M, K(-nZ - Z_{j-1}) \otimes \mathcal{O}_{A_{i_j}}) = 0$, and the maps $H^1(M, K(-nZ - Z_j)) \to H^1(M, K(-nZ - Z_{j-1}))$ in (2.4)' are surjective. Composing the maps, we see that $\rho : H^1(M, K(-nZ - Z_j)) \to H^1(M, K(-Z_j))$ is surjective for all $n \geqslant 0$. For sufficiently large n, ρ is 0 map by [2, Section 4, Satz 1, p. 355]. Hence $H^1(M, K(-Z_j)) = 0$. In particular $H^1(M, K(-A_1)) = 0$. Q.E.D.

Remark 2.3. The assumption $\dim H^1(M, \mathcal{O}) = 0$ is important in Theorem 2.2. For $\dim H^1(M, \mathcal{O}) = 1$, the theorem is false even if there is no curve with self intersection number -1 sitting inside M. Let us first recall that the topological nature of the embedding of the exceptional set A in M is described by the weighted dual graph Γ. The vertices of Γ correspond to the A_i. The edges of Γ connecting the vertices corresponding to A_i and A_j, $i \neq j$, correspond to the points of $A_i \cap A_j$. Finally, associated to each A_i is its genus, g_i, as a Riemann surface, and its weighted, A_iA_i, the topological self-intersection number.

Example 2.4. Let $V = \{(x, y, z) \in \mathbb{C}^3 : z^2 = y^3 + x^9\}$ and $\pi : M \to V$ be the minimal resolution of the singularity of V the weighted dual graph of the exceptional set $A = \cup A_i$, $1 \leqslant i \leqslant 5$ can be described as follows:



The genera of A_i, $1 \leqslant i \leqslant 5$ are 0. We claim that the canonical bundle K is not spanned by its global section at a point p in A_5 which is not a point in A_2.

Let $Z_0 = 0$, $Z_1 = A_2$, $Z_2 = A_2 + A_1$, $Z_3 = A_2 + A_1 + A_3$, $Z_4 = A_2 + A_1 + A_3 + A_4$, $Z_5 = 2A_2 + A_1 + A_3 + A_4$, $Z_6 = 2A_2 + A_1 + A_3 + A_4 + A_5 = Z$ be a computation sequence for the fundamental cycle Z. Since $Z_{i-1} \cdot A_{i_j} = 1$, $2 \leqslant j \leqslant 5$, $Z_5A_{i_6} = 2$ and $c(K \otimes A_{i_6}) = 1$ where $A_{i_6} = A_5$, $A_{i_1} = A_2$, it follows from the proof of Theorem 2.2 that

$$H^1(M, K) = 0, \quad H^1(M, K(-Z)) = 0 \tag{2.5}$$

$$H^1(M, K(-A_2)) = 0 \tag{2.6}$$

so the map $H^0(M, K) \to H^0(M, K \otimes \mathcal{O}_{A_2}) = \mathbb{C}$ is surjective. Moreover, the $f \in H^0(M, K)$ which generates the image has non-zero image in $H^0(M, K \otimes \mathcal{O}_{A_5})$ because $A_5 \cdot A_2 = 1$. In fact, the image of f in $H^0(M, K \otimes \mathcal{O}_{A_5})$ has no zeroes near $A_1 \cap A_2$ as a section of $K \otimes \mathcal{O}_{A_5}$. From (2.5), we have the following exact sequence:

$$H^0(M,K) \to H^0(M,K \otimes \mathcal{O}_{A_5}) \to H^1(M,K(-A_5)) \to 0.$$

Since $\dim H^0(M,K \otimes \mathcal{O}_{A_5}) = 2$ to prove our claim, it suffices to prove $\dim H^1(M,K(-A_5) = 1$. To see this, let us look at the following sheaf exact sequences

$$\begin{aligned}
&0 \to K(-Z) \to K(-2A_2 - A_3 - A_4 - A_5)\\
&\quad \to K \otimes \mathcal{O}_{A_1}(-2A_2 - A_3 - A_4 - A_5) \to 0\\
&0 \to K(-2A_2 - A_3 - A_4 - A_5) \to K(-A_2 - A_3 - A_4 - A_5)\\
&\quad \to K \otimes \mathcal{O}_{A_2}(-A_2 - A_3 - A_4 - A_5) \to 0\\
&0 \to K(-A_2 - A_3 - A_4 - A_5) \to K(-A_2 - A_4 - A_5)\\
&\quad \to K \otimes \mathcal{O}_{A_3}(-A_2 - A_4 - A_5) \to 0\\
&0 \to K(-A_2 - A_4 - A_5) \to K(-A_2 - A_5) \to K \otimes \mathcal{O}_{A_4}(-A_2 - A_5) \to 0\\
&0 \to K(-A_2 - A_5) \to K(-A_5) \to K \otimes \mathcal{O}_{A_2}(-A_5) \to 0.
\end{aligned}$$

The corresponding long cohomology exact sequences and (2.5) imply that

$$\begin{aligned}
H^1(M,K(-A_5)) &= H^1(M,K(-A_2 - A_5))\\
&= H^1(M,K(-A_2 - A_4 - A_5))\\
&= H^1(M,K(-A_2 - A_3 - A_4 - A_5))\\
&= H^1(M,K(-2A_2 - A_3 - A_4 - A_5))\\
&= H^1(M,K \otimes \mathcal{O}_{A_1}(-2A_2 - A_3 - A_4 - A_5))\\
&= \mathbf{C}.
\end{aligned}$$

§3. Positivity of Griffiths numbers. In this section we shall prove the positivity of the Griffiths numbers $g^{(n-1)}$ and $g^{(n)}$. For $g^{(n)}$, the answer is more or less complete.

THEOREM 3.1. *If (V,q) is an irreducible isolated singularity of dimension n and $\dim H^{n-1}(M,\mathcal{O}) > 0$, then $g^{(n)} \geqslant n - 1$.*

Proof. By lemma 1.2, it suffices to prove there exist $n - 1$ linearly independent holomorphic n-forms on M which are not obtained by pulling back holomorphic n-forms on V. We can assume without loss of generality that the exceptional set A is a divisor in M with normal crossings. Let $A = \cup A_i$, $1 \leqslant i \leqslant s$ be a decomposition of A into its irreducible components. Recall that $\dim H^{n-1}(M,\mathcal{O}) = \dim H^0(M - A,\Omega^n)/H^0(M,\Omega^n)$. (Cf. Theorem A of [11].) By the assumption there exists a holomorphic n-form ω with pole along at least one irreducible component of A. Let A_1 be the irreducible component of A at which ω has the highest order of pole. Choose a point b in A_1 which is a smooth point of A. Let $(x_1, \ldots, x_n)$ be a coordinate system centered at b such that A_1 is given locally at b by $x_1 = 0$. Embed (V, q) into $(\mathbf{C}^m, 0)$. Let $z_1, \ldots, z_m$ be coordinate functions of $\mathbf{C}^m$. Then the functions $\pi^*(z_i)$ obtained by pulling back z_i to M vanish along A. Take the power series expansion of $\pi^*(z_i)$ around the point b,

$$\pi^*(z_1) = x_1^{r_1}\left(C_0^1 + \sum_{i=1}^{n} C_i^1 x_i + \cdots\right)$$

$$\pi^*(z_2) = x_1^{r_2}\left(C_0^2 + \sum_{i=1}^{n} C_i^2 x_i + \cdots\right)$$

$$\vdots$$

$$\pi^*(z_m) = x_1^{r_m}\left(C_0^m + \sum_{i=1}^{n} C_i^m x_i + \cdots\right),$$

where r_i is the vanishing order of $\pi_*(z_i)$ along A_1, $1 \leqslant i \leqslant m$. Without loss of generality, we assume $r_1 = \min(r_1, \ldots, r_n)$. It is easy to see that the holomorphic n-form

$$d\pi^*(z_{i_1}) \Lambda d\pi^*(z_{i_2}) \Lambda \cdots \Lambda d\pi^*(z_{i_n}) \tag{3.1}$$

has vanishing order at least $nr_1 - 1$ along A_1. Choose a minimal positive integer t such that the vanishing order of $(\pi^*(z_1))^t \cdot \omega$ along A_1 is less than or equal to $r_1 - 1$. So the vanishing orders of the n-forms $\pi^*(z_1)^{t+j} \cdot \omega$, $0 \leqslant j \leqslant n-2$, are at most $(n-1)r_1 - 1$. These n-forms cannot be linear combinations of (3.1). Therefore we have produced at least $n-1$ holomorphic n-forms on M which are not obtained by pulling back of holomorphic n-forms on V. Q.E.D.

THEOREM 3.2. *Let (V, q) be a 2-dimensional irreducible isolated singularity. Then $g^{(2)} \geqslant 1$.*

Proof. This is an easy consequence of Theorem 2.2 and Theorem 3.1.

Let V be an n-dimensional Stein normal analytic space with $q \in V$ as an isolated singularity. We say that $q \in V$ admits a $\mathbf{C}^*$-action if there exists an embedding $j: (V, q) \to (\mathbf{C}^m, 0)$ for some n such that $j(V)$ is closed in $\mathbf{C}^m$ and is invariant under the $\mathbf{C}^*$-action $\tilde{\sigma}$ where $\tilde{\sigma}: \mathbf{C}^* \times \mathbf{C} \to \mathbf{C}^m$ is defined by

$$\tilde{\sigma}(t, (z_1, \ldots, z_m)) = (t^{q_1} z_1, \ldots, t^{q_m} z_m) \quad q_i \text{ integer}.$$

THEOREM 3.3. *If (V, q) is an n-dimensional irreducible isolated singularity with $\mathbf{C}^*$-action and $\dim H^{n-1}(M, \mathcal{O}) \geqslant 1$, then $g^{(n-1)} \geqslant g^{(n)} \geqslant n-1$.*

Proof. Let $\overline{\Omega}_V^p$ be the sheaf of germs of holomorphic p-forms on $V - \{q\}$ which are locally L^2-integrable in the sense of Griffiths [4]. Then actually $\overline{\Omega}_V^p$ is equal to the 0-th direct image sheaf $\pi_* \Omega^p_M$ where $\pi: M \to V$ is a resolution of the singularity of V. There is a natural map $\phi_p: \Omega^p_V \to \bar{\Omega}^p_V$. In [13], we define p-th Griffiths number $g^{(p)}$ to be the dimension of $H_V^{(p)} = \operatorname{coker} \phi_p$ which is a coherent sheaf supported on the singular point q. Let ξ denote the generating vector field of the $\mathbf{C}^*$-action, i_ξ the inner multiplication. Clearly the map $i_\xi: \bar{\Omega}^n_{V,q} \to \bar{\Omega}^{n-1}_{V,q}$ is injective and we have the following commutative diagram:

$$\begin{array}{ccccccc}
\Omega^n_{V,q} & \xrightarrow{\phi_n} & \bar{\Omega}^n_{V,q} & \longrightarrow & H^n_{V,q} & \longrightarrow & 0 \\
\downarrow i_\xi & & \downarrow i_\xi & & \downarrow & & \\
\Omega^{n-1}_{V,q} & \xrightarrow{\phi_{n-1}} & \bar{\Omega}^{n-1}_{V,q} & \longrightarrow & H^{n-1}_{V,q} & \longrightarrow & 0 \\
\downarrow i_\xi & & \downarrow i_\xi & & & & \\
\Omega^{n-2}_{V,q} & \xrightarrow{\phi_{n-2}} & \bar{\Omega}^{n-2}_{V,q}. & & & &
\end{array} \tag{3.2}$$

There is therefore induced a quotient map $H^n_{V,q} \to H^{n-1}_{V,q}$ which we denote by i_ξ again. We claim that this map is injective. Let $\beta \in \bar{\Omega}^n_{V,q}$ such that $i_\xi(\beta) = \phi_{n-1}(\alpha)$ for some α in $\Omega^n_{V,q}$. Write α as a sum $\sum \alpha^j$ of quasi-homogeneous elements where α^j is a quasi-homogeneous element of (quasi-homogeneous) degree $l_j > 0$. Since

$$\phi_{n-2}(i_\xi(\alpha)) = i_\xi(\phi_{n-1}(\alpha)) = i_\xi(i_\xi(\beta)) = 0, \tag{3.3}$$

therefore $i_\xi(\alpha) = \sum i_\xi(\alpha^j)$ is a torsion element of $\Omega^{n-2}_{V,q}$. This implies that $i_\xi(\alpha^j)$ are torsion element of $\Omega^{n-2}_{V,q}$ for all j. Hence we have

$$\phi_{n-2}(i_\xi(\alpha^j)) = 0 \qquad \text{for all} \quad j. \tag{3.4}$$

Let $L_\xi = i_\xi d + d i_\xi$ be the Lie derivation. Then

$$\begin{aligned} l_j \alpha^j = L_\xi \alpha^j &= d i_\xi(\alpha^j) + i_\xi d(\alpha^j) \\ &\Rightarrow \alpha_j = 1/l_j d i_\xi(\alpha^j) + 1/l_j i_\xi d(\alpha^j). \end{aligned} \tag{3.5}$$

Set $\gamma = \sum 1/l_j d(\alpha^j) \in \Omega^n_{V,q}$. Then from (3.5), we have

$$\begin{aligned} \sum \phi_{n-1}(\alpha^j) &= \sum 1/l_j \phi_{n-1} d i_\xi(\alpha^j) + \phi_{n-1} i_\xi(\gamma) \\ &= \sum 1/l_j d \phi_{n-2} i_\xi(\alpha^j) + \phi_{n-1} i_\xi(\gamma). \end{aligned} \tag{3.6}$$

Since $\sum \phi_{n-1}(\alpha^j) = i_\xi(\beta)$, (3.4) and (3.6) together imply

$$\begin{aligned} i_\xi(\beta) &= \phi_{n-1} i_\xi(\gamma) \\ &= i_\xi \phi_n(\gamma). \end{aligned}$$

Therefore $\beta = \phi_n(\gamma)$. This proves our claim. It follows that $g^{(n-1)} \geqslant g^{(n)} \geqslant n-1$ by Theorem 3.1. Q.E.D.

THEOREM 3.4. *If* (V,q) *is a* 2*-dimensional irreducible isolated singularity with* C^**-action, then* $g^{(1)} \geqslant g^{(2)} \geqslant 1$.

Proof. The proof is exactly the same as the proof of Theorem 3.3, but we use Theorem 3.2 instead of Theorem 3.1. Q.E.D.

§4. **Lower estimates for** $\delta^{(n-1)}$ **and** $\dim T_V^1$. We shall first get a lower estimate for $\delta^{(n)}$ and $\delta^{(n-1)}$ in terms of $\dim H^{n-1}(M,\mathcal{O})$ where $\pi: M \to V$ is a resolution of the singularity of V. Our estimate for the $\delta^{(n-1)}$ given here depends on the fact that (V,q) admits a C^*-action. The same lower estimate may be true even for singularities without C^*-action.

Let $\theta: V - \{q\} \to V$ be the inclusion map. Then the 0-th direct image sheaf $\theta_*\Omega^p_{V-\{q\}} = \bar{\bar{\Omega}}^p_V$ is coherent by Siu's Theorem [9]. Clearly we have an inclusion $\bar{\Omega}^p_V \hookrightarrow \bar{\bar{\Omega}}^p_V$. Then p-th Siu number $s^{(p)}$ is equal to $\dim(\bar{\bar{\Omega}}^p_{V,q}/\bar{\Omega}^p_{V,q})$ (cf. [13], [14]).

LEMMA 4.1. *Let* (V,q) *be an* n*-dimensional irreducible isolated singularity. If* $\dim H^{n-1}(M,\mathcal{O}) \neq 0$, *then* $\delta^{(n)} \geqslant n-1+\dim H^{n-1}(M,\mathcal{O})$.

Proof. By Lemma 1.3, $\delta^{(n)} = g^{(n)} + s^{(n)}$. In [11] we have proved $s^{(n)} = \dim H^{n-1}(M,\mathcal{O})$. So Lemma 4.1 follows from Theorem 3.1

LEMMA 4.2. *Let* (V,q) *be a* 2*-dimensional irreducible isolated singularity. Then* $\delta^{(2)} \geqslant 1 + \dim H^1(M,\mathcal{O})$.

Proof. The proof is exactly the same as those given in Lemma 4.1. Here we apply Theorem 3.2 instead of Theorem 3.1.

THEOREM 4.3. *Let* (V,q) *be an* n*-dimensional irreducible isolated singularity with* C^**-action and* $\dim H^{n-1}(M,\mathcal{O}) \geqslant 1$. *Then* $\delta^{(n-1)} \geqslant \delta^{(n)} \geqslant n-1+\dim H^{n-1}(M,\mathcal{O})$.

Remark 4.4. It is not true that $s^{(n-1)} \geqslant s^{(n)}$. For example, simple elliptic singularities have $s^{(1)} = 0$ and $s^{(2)} = 1$ (cf. [13]).

Proof of Theorem 4.3. For each p, there is a natural map $\psi_p: \Omega^p_V \to \bar{\bar{\Omega}}^p_V$. The invariant $\delta^{(p)}$ is exactly the dimension of $I^p_V = \operatorname{coker}\psi_p$ which is a coherent sheaf supported on the singular point q. Let ξ denote the generating vector field of the C^*-action, i_ξ the inner multiplication. As in the proof of Theorem 3.3, there is a natural map $i_\xi: I^n_{V,q} \to I^{n-1}_{V,q}$. This map is actually injective. The argument here is exactly the same as those given in the proof of Theorem 3.3. Our theorem follows from Lemma 4.1. Q.E.D.

THEOREM 4.5. *Let* (V,q) *be a* 2*-dimensional irreducible isolated singularity with* C^**-action. Then* $\delta^{(1)} \geqslant \delta^{(2)} \geqslant 1 + \dim H^1(M,\mathcal{O})$.

Proof. The proof is the same as those given in the proof of Theorem 4.3. Here we apply Lemma 4.2 instead of Lemma 4.1. Q.E.D.

Following Schlessinger, let T_V^1 be the set of isomorphism classes of first order infinitesimal deformation of V. It is well-known that the T_V^1 may be replaced by $\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V^1, \mathcal{O}_V)$ when V has positive depth along singular locus, e.g., when V is reduced of positive dimension. In [3] Grauert constructs a versal deformation $X \to S$ of V from which every other deformation $W \to T$ may be induced, up to isomorphism, by a map $\phi: T \to S$, with $\phi^*(X) \cong W$. Morever, the map $t_\phi: t_T \to t_s$ between Zariski tangent spaces is uniquely determined by the isomorphism class of W. As Grauert shows, the Zariski tangent space of S is isomorphic to T_V^1. However in general T_V^1 is extremely hard to compute. The following theorem gives a lower estimate of $\dim T_V^1$ in case of Gorenstein surface singularities with C^*-action.

THEOREM 4.6. *Let* (V, q) *be a Gorenstein surface singularity with* $\mathbf{C}^*$*-action. Then* $\dim T_V^1 \geqslant 1 + \dim H^1(M, \mathcal{O})$.

Proof.

$$
\begin{aligned}
\dim T_V^1 &= \dim \mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V^1, \mathcal{O}_V) \\
&= \dim H^1_{\{q\}}(V, \Omega_V^1) \qquad \text{(by local duality)} \\
&= \dim \Gamma(V - \{q\}, \Omega_V^1)/\Gamma(V, \Omega_V^1) \\
&= \delta^{(1)} \\
&\geqslant 1 + \dim H^1(M, \mathcal{O}) \qquad \text{(by Theorem 4.5).}
\end{aligned}
$$

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