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HIGHER ORDER BERGMAN FUNCTIONS AND EXPLICIT CONSTRUCTION OF MODULI SPACE FOR COMPLETE REINHARDT DOMAINS

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Abstract

In this article we introduce higher order Bergman functions for bounded complete Reinhardt domains in a variety with possibly isolated singularities. These Bergman functions are invariant under biholomorhic maps. We use Bergman functions to determine all the biholomorhic maps between two such domains. As a result, we can construct an infinite family of numerical invariants from the Bergman functions for such domains in A_n variety $\{(x,y,z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. These infinite family of numerical invariants are actually a complete set of invariants for either the set of all bounded strictly pseudoconvex complete Reinhardt domain in A_n variety or the set of all bounded pseudoconvex complete Reinhardt domains with real analytic boundaries in A_n variety. In particular the moduli space of these domains in A_n variety is constructed explicitly as the image of this complete family of numerical invariants. It is well known that A_n variety is the quotient of cyclic group of order $n+1$ on $\mathbb{C}^2$. We prove that the moduli space of bounded complete Reinhardt domains in A_n variety coincides with the moduli space of the corresponding bounded complete Reinhardt domains in $\mathbb{C}^2$. Since our complete family of numerical invariants are computable, we have solved the biholomorphically equivalent problem for large family of domains in $\mathbb{C}^2$.

1. Introduction

Let D_1 and D_2 be two domains in $\mathbb{C}^n$. One of the most fundamental problems in complex geometry is to determine conditions which will imply that D_1 and D_2 are biholomorphically equivalent. For $n = 1$, the celebrated Riemann mapping theorem states that any simply connected domains in $\mathbb{C}$ are biholomorphically equivalent. For $n \geqslant 2$, it is well known that there are lots of domains which are topologically equivalent to the ball but not necessarily biholomorphically equivalent

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to the ball. This problem was first studied by Poincaré in 1907 [Po]. He worked on the perturbations of the unit ball in $\mathbb{C}^2$ of a particular kind, and found necessary and sufficient conditions on a first order perturbation that the perturbed domain be biholomorphically equivalent to the ball. More generally, Poincaré studied the invariance properties of a CR manifold X of real dimension $2n-1$ which is a real hypersurface in $\mathbb{C}^n$ with respect to the infinite pseudo-group of biholomorphic transformations. The systematic study of such properties for hypersurfaces with nondegenerate Levi form was made by Cartan [Ca] in 1932, and later by Chern and Moser [Ch-Mo]. A main result of the theory is the existence of a complete system of local differential invariants for CR-structures on real hypersurfaces. In 1974, Fefferman [Fe1] proved that a biholomorphic mapping between two strictly pseudoconvex domains is smooth up to the boundaries and the induced boundary mapping is a CR-equivalence on the boundary. Thus, by the fundamental theorem of Fefferman, the biholomorphically equivalent problem of bounded strictly pseudoconvex domains in $\mathbb{C}^n$ is the same as the CR equivalent problem of strictly pseudoconvex compact CR manifolds of real dimension $2n-1$ in $\mathbb{C}^n$.

In 1978, Burns, Shnider and Wells [B-S-W] used the Chern-Moser theory to distinguish generic perturbations of a given strictly pseudoconvex domain. They were able to construct perturbations with arbitrarily large parameters in such a way that the domains are biholomorphically inequivalent if the parameter values of these domains are different. As a result the " number of moduli " of a " moduli space " of a bounded strictly pseudoconvex domain has to be infinite. On the other hand, Webster [We] gave a complete characterization when two ellipsoids in $\mathbb{C}^n$ are CR equivalent by Chern-Moser's theory and Cartan method of equivalence. In 1988, Lempert [Le] made a significant progress in the subject. He considered smoothly bounded strictly convex domains containing the origin of $\mathbb{C}^n$. He called two such domains equivalent if there is an origin preserving biholomorphic mapping between them whose differential at the origin is the identity. With each smoothly bounded strictly convex domain D, Lempert associated a triple (I, H, Q) of invariants where I is the Kobayashi indicatrix of D at the origin, and the smooth hermitian form H and the smooth quadratic form Q are defined on the rank $n-1$ vector bundle of $(1,0)$ vectors tangent to the boundary of the Kobayashi indicatrix I. He showed that if two marked convex domains share the same invariants, then they are equivalent. In dimension 2, the construction may be simplified and it is possible to reduce the number of invariants. This allows the explicit description of the moduli space of marked strictly convex domains in $\mathbb{C}^2$ as a subdomain of a suitable Fréchet bundle. Although the theory established by Lempert is extremely beautiful, the computation of his invariants is quite a hard problem.

Despite the success of the Chern-Moser theory and the Lempert theory, the fundamental question of distinguishing two strictly pseudoconvex CR manifolds remains unsolved. Let X be a compact connected strictly pseudoconvex CR manifold of real dimension $2n-1$. In 1974, Boutel de Monvel [Bo] (see also Kohn [Ko]) proved that X is CR-embeddable in some $\mathbb{C}^N$ if $\dim X \geqslant 5$. Throughout this paper, we shall only consider CR embeddable strictly pseudoconvex CR manifolds. In view of a beautiful theorem of Harvey-Lawson [Ha-La], there exists a complex variety V in $\mathbb{C}^N$ for N sufficiently large such that $\partial V = X$ and V has only normal isolated singularities. It is well known that one can use the structures of the singularities of V to distinguish the CR structure of X (see for example Theorem 3.1 of [Ya]). Thus if two strictly pseudoconvex CR manifolds bound non-isomorphic singularities, then their CR structures are different. The difficult unsolved CR equivalence problem is: how can one distinguish between strictly pseudoconvex CR manifolds X_1 and X_2 when they are lying in the same variety V. If V is $\mathbb{C}^N$, this difficult problem is just the classical problem discussed above and has been considered by many leading mathematicians Chern-Moser [Ch-Mo], Fefferman [Fe1], Webster [We], Burns-Shnider-Wells [B-S-W], etc. Even in this case, it seems that the biholomorphically equivalence problem or moduli problem for complete Reinhardt domains remains open, although there is a beautiful theorem of Sunada [Su] which relates two such domains by a special linear map. For example, consider the following natural family of complete Reinhardt domains

$$D(a_1, \cdots, a_k; b_1, \cdots, b_k; c_1, \cdots, c_k) =$$

$$\Big\{(z_1, z_2) \in \mathbb{C}^2 : \sum_{i=1}^{k} [a_i|z_1|^{4i} + b_i|z_2|^{4i} + c_i|z_1 z_2|^{2i}] < d, \text{ where}$$

$$a_1, \cdots, a_k; b_1, \cdots, b_k; c_1, \cdots, c_k \text{ and } d \text{ are positive real numbers}\Big\}.$$

It is not known how to solve the problem of biholomorphic equivalence of this family of complete Reinhardt domains by Suanda's theorem. On the other hand, when V is a singular variety, the CR equivalence problems is wide open. In [Ya], the second author discovered a novel technique to attack CR equivalence problem. He constructed a new biholomorphically invariant called Bergman function. The Bergman functions put a lot of restriction on biholomorphic maps between bounded complete Reinhardt domains, from which new holomorphic invariants can be constructed and the automorphism groups of the bounded complete Reinhardt domains can be determined. He illustrated how his new technique works in a concrete example of A_1-variety $V = \big\{(x, y, z) \in$

$\mathbb{C}^3 : xy = z^2\big\}$. He constructed a fundamental holomorphic numerical invariant ν_D (cf. Theorem 3.6 of [Ya]), for bounded complete Reinhardt domains D in V. He also computed the automorphism group of D. For a one parameter family of complete Reinhardt domains $D_a = \big\{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a|x|^2 + |y|^2 + |z|^2 < \varepsilon_0$, where $a > 0$ and ε_0 is a fixed positive constant$\big\}$, he showed that the holomorphic numerical invariant ν_{D_a} is a complete invariant in the sense that D_{a_1} is biholomorphically equivalent to D_{a_2} if and only if $\nu_{D_{a_1}} = \nu_{D_{a_2}}$.

In this paper, we introduce a higher order Bergman functions on domains in varieties which are global invariants. These Bergman functions are used to prove that biholomorphisms between two bounded complete Reinhardt domains are necessarily special linear maps. We construct an infinite family of numerical invariants for complete Reinhardt domains in A_n-variety. Our numerical invariants are able to distinguish any two bounded complete Reinhardt pseudoconvex domains with real analytic boundaries or any two bounded complete Reinhardt strictly pseudoconvex domains in A_n-variety. Thus the moduli spaces of bounded complete Reinhardt pseudoconvex domains with real analytic boundaries or any two bounded complete Reinhardt strictly pseudoconvex domains in A_n-variety are constructed explicitly as the image of this complete family of numerical invariants. Because each bounded complete Reinhardt domain in A_n-variety corresponds to a unique bounded complete Reinhardt domain in $\mathbb{C}^2$, we have also constructed the moduli space of a large class of bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^2$.

Before stating our result, let us recall some notations.

An open subset $D \subseteq \mathbb{C}^n$ is a complete Reinhardt domain if, whenever $(z_1, \cdots, z_n) \in D$ then $(\xi_1 z_1, \cdots, \xi_n z_n) \in D$ for all complex numbers ξ_j with $|\xi_j| \leqslant 1$.

Let $\widetilde{V_n} = \big\{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\big\}$. It is well known that $\widetilde{V_n}$ is the quotient of $\mathbb{C}^2$ by a cyclic group of order $n+1$, i.e. $\delta.(z_1, z_2) = (\delta z_1, \delta^n z_2)$, where δ is a primitive $(n+1)$-th root of unit. The quotient map $\pi : \mathbb{C}^2 \to \widetilde{V}$ is given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$.

Definition 1.1. An open set V in the A_n-variety $\widetilde{V_n} = \big\{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\big\}$ is called a complete Reinhardt domain if $\pi^{-1}(V)$ is a complete Reinhardt domain in $\mathbb{C}^2$.

Recall that the minimal resolution $\widetilde{M_n}$ of $\widetilde{V_n}$ consists of $n+1$ coordinate charts $\widetilde{W_k} = \mathbb{C}^2 = \{(u_k, v_k)\}, k = 0, 1, \cdots, n$. The space of holomorphic two forms on $\widetilde{M_n}$ has a basis

$$\left\{\phi_{\alpha\beta} = u_0^\alpha v_0^\beta \, du_0 \wedge dv_0 : \alpha \geqslant \frac{n}{n+1}\beta\right\}.$$

Let M $(\subseteq \widetilde{M_n})$ be the resolution of complete Reinhardt domain V in $\widetilde{V_n}$. In what follows, we shall use notation $\|\phi_{\alpha\beta}\|_M^2$ for $\int_M \phi_{\alpha\beta}\wedge\overline{\phi_{\alpha\beta}}$.

Let

$$g^{(\alpha,\,\beta)} = \frac{\|\phi_{10}\|^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|\,\|\phi_{00}\|^{\alpha-\frac{n-1}{n+1}\beta-1}}.$$

Theorem A. *Let V_i, $i=1,2$, be two bounded complete Reinhardt domains in A_n-variety $\widetilde{V_n}=\{(x,y,z)\in\mathbb{C}^3 : xy=z^{n+1}\}$. If V_1 is biholomorphic to V_2, then*

$$\xi^{(\alpha,\,\beta)} := g^{(\alpha,\,\beta)}\cdot g^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}\,,$$
$$\zeta^{(\alpha,\,\beta)} := g^{(\alpha,\,\beta)} + g^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}\,,$$

$$\eta^{(\alpha,\,p,\,q)} := \left(g^{(\alpha,\,p)} - g^{(n\alpha-(n-1)p,(n+1)\alpha-np)}\right)\cdot \left(g^{(\alpha,\,q)} - g^{(n\alpha-(n-1)q,(n+1)\alpha-nq)}\right)$$

and

$$\omega^{(\alpha_1,\,\alpha_2,\,p_1,\,p_2)} := \left(g^{(\alpha_1,\,p_1)} - g^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_1-np_1)}\right)\cdot \left(g^{(\alpha_2,\,p_2)} - g^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)}\right),$$

where

$$\alpha\geqslant 1, \alpha\geqslant \frac{n}{n+1}\beta, 0\leqslant p,\, q\leqslant \left[\frac{n+1}{n}\alpha\right], p\neq q,$$
$$0\leqslant p_i\leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i\geqslant 1, \alpha_1\neq\alpha_2, i=1,2,$$

are all invariants, i.e.

$$\xi_{V_1}^{(\alpha,\beta)}=\xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)}=\zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)}=\eta_{V_2}^{(\alpha,p,q)},$$
$$\omega_{V_1}^{(\alpha_1,\,\alpha_2,\,p_1,\,p_2)}=\omega_{V_2}^{(\alpha_1,\,\alpha_2,\,p_1,\,p_2)},$$

where

$$\alpha\geqslant 1, \alpha\geqslant \frac{n}{n+1}\beta, 0\leqslant p,\, q\leqslant \left[\frac{n+1}{n}\alpha\right], p\neq q,$$
$$0\leqslant p_i\leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i\geqslant 1, \alpha_1\neq\alpha_2, i=1,2.$$

The invariants in Theorem A determine completely the Bergman function up to automorphisms of A_n-variety.

Theorem B. *Let V_i, $i=1,2$, be two bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domains in $\widetilde{V_n}=\{(x,y,z)\in\mathbb{C}^3\colon xy=z^{n+1}\}$. If*

$$\xi_{V_1}^{(\alpha,\beta)}=\xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)}=\zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)}=\eta_{V_2}^{(\alpha,p,q)},$$

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)} = \omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)},$$

where

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

then there exists an automorphism $\Psi = (\psi_1, \psi_2, \psi_3)$ *of* A_n*-variety* $\widetilde{V_n} = \{(x,y,z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ *given by either*

$$(\psi_1,\psi_2,\psi_3) = \left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y, \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right),$$

or

$$(\psi_1,\psi_2,\psi_3) = \left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x, \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right),$$

such that Ψ *sends* V_1 *to* V_2.

As an immediate corollary of Theorem B above, we have the following theorem.

Theorem C. *The moduli space of bounded complete Reinhardt strictly pseudoconvex (respectively* C^ω*-smooth pseudoconvex) domains in* A_n*-variety* $\widetilde{V_n} = \{(x,y,z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ *is given by the image of the map* $\Phi : \{V : V$ *a bounded complete Reinhardt strictly pseudoconvex (respectively* C^ω*-smooth pseudoconvex) domain in* $\widetilde{V_n}\} \to \mathbb{R}^\infty$*, where the component function of* Φ *are the invariant functions*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}\omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2.$$

defined in Theorem A.

We are now ready to study a large class of complete Reinhardt domains in $\mathbb{C}^2$. The following theorem says that the biholomorphic equivalence problem for bounded complete Reinhardt domains in A_n-variety $\widetilde{V_n}$ is the same as the biholomorphic equivalence problem for the corresponding bounded complete Reinhardt domains in $\mathbb{C}^2$.

Theorem D. *Let $\pi : \mathbb{C}^2 \to \widetilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ be the quotient map given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$. Let V_i, $i = 1, 2$, be bounded complete Reinhardt domains in $\widetilde{V_n}$ such that $W_i := \pi^{-1}(V_i), i = 1, 2$, are bounded complete Reinhardt domain in $\mathbb{C}^2$. Then V_1 is biholomorphic to V_2 if and only if W_1 is biholomorphic to W_2. In particular, V_1 is biholomorphic to V_2 if and only if there exists a biholomorphism $\Phi : V_1 \to V_2$ given by $\Phi(x, y, z) = (a^{n+1}x, b^{n+1}y, abz)$ or $\Phi(x, y, z) = (a^{n+1}y, b^{n+1}x, abz)$ where $a, b > 0$.*

As a corollary of Theorem C and Theorem D, we have the following theorem.

Theorem E. *(1) Let $\mathcal{W} = \{W : W = \pi^{-1}(V)$ where V is a bounded complete Reinhardt domain in A_n-variety$\}$ be the space of bounded complete Reinhardt domains in $\mathbb{C}^2$ which are invariant under the action of the cyclic group of order $n + 1$ on $\mathbb{C}^2$. Then*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\, q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

defined in Theorem A are invariants of $\mathcal{W}$.

(2) Let $\mathcal{W}_P = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt pseudoconvex C^ω-smooth domain in A_n-variety$\}$ and $\mathcal{W}_{SP} = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt strictly pseudoconvex domain in A_n-variety$\}$. Then the moduli space of $\mathcal{W}_{\mathcal{P}}$ (respectively $\mathcal{W}_{SP}$) is given by the image of the map $\widetilde{\Phi_P} : \mathcal{W}_P \to \mathbb{R}^\infty$ (respectively $\widetilde{\Phi_{SP}} : \mathcal{W}_{SP} \to \mathbb{R}^\infty$), where the component functions of $\widetilde{\Phi_P}$ (respectively $\widetilde{\Phi_{SP}}$) are the invariant functions

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\, q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

defined in Theorem A. In particular, the moduli space of $\mathcal{W}_P$ (respectively $\mathcal{W}_{SP}$) is the same as the moduli space of bounded complete Reinhardt pseudoconvex C^ω-smooth domains (respectively bounded complete Reinhardt strictly pseudoconvex domains) in A_n-variety $\widetilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$.

It is an interesting question to study the geometry of the moduli space of bounded complete Reinhardt domains in A_n-variety. Here we study two families of domains in A_n-variety and solve the biholomorphic equivalent problem. In fact we are able to construct the moduli space of these families explicitly. More specifically, consider

$$V^{(d)}_{(a,b,c)} = \left\{(x, y, z) \colon xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\right\}.$$

Here we assume that a, b, c are strictly greater than zero, and d is a fixed integer greater than or equal to 1. This is a 3 parameters family of pseudoconvex domains in A_1-variety $\widetilde{V_1} = \left\{(x, y, z) \in \mathbb{C}^3 : xy = z^2\right\}$. Using our Bergman function theory, we can write down the explicit moduli space of this family as shown in the following theorem by means of the invariant (cf. Corollary 5.6)

$$\nu^{(\alpha,\beta)} = \frac{\sqrt{\xi^{(\alpha,\beta)} \cdot \xi^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}}}{\sqrt{\xi^{(\alpha,\alpha)}}}, \qquad for \qquad n = 1.$$

Theorem F. *Let*

$$V^{(d)}_{(a,b,c)} = \left\{(x, y, z) \in \mathbb{C}^3 \colon xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\right\}.$$

Let $\sim$ *denote the biholomorphic equivalence. Then the map*

$$\varphi\colon \left\{V^{(d)}_{(a,b,c)}\right\} \to \mathbb{R}_+,\ V^{(d)}_{(a,b,c)} \mapsto \nu^{(2d-1,d-1)}$$

is injective up to a biholomorphism $\sim$. *More precisely the induced map* $\widetilde{\varphi}\colon \left\{V^{(d)}_{(a,b,c)}\right\}/\!\sim\, \to \mathbb{R}_+$ *is one-to-one map from* $\left\{V^{(d)}_{(a,b,c)}\right\}/\!\sim$ *onto* $\left(0, \frac{2}{\pi}\right)$. *So the moduli space of* $\left\{V^{(d)}_{(a,b,c)}\right\}$ *is an open interval* $\left(0, \frac{2}{\pi}\right)$.

The biholomorphically equivalent problem of domains in A_1-variety is not only interesting in its own right, but also has application to the classical biholomorphically equivalent problem of domains in $\mathbb{C}^2$. In fact, let

$$W^{(d)}_{(a,b,c)} = \left\{(x, y) \colon a|x|^{2d} + b|y|^{2d} + c|xy|^{d} < \varepsilon_0\right\}.$$

Corollary G. *The moduli space of* $W^{(d)}_{(a,b,c)}$ *is the same as the moduli space of* $V^{(d)}_{(a,b,c)}$, *which is* $\left(0, \frac{2}{\pi}\right)$.

As an application to our theory, we compute explicitly the invariant $\nu^{(3,1)}$ for two domains $V^{(1)}_{(1,1,1)}$ and $V^{(2)}_{(1,1,1)}$ in A_1-variety. As a consequence, we see that $V^{(1)}_{(1,1,1)}$ is not biholomorphic to $V^{(2)}_{(1,1,1)}$ and the domain $W^{(1)}_{(1,1,1)}$ in $\mathbb{C}^2$ is not biholomorphic to the domain $W^{(2)}_{(1,1,1)}$ in $\mathbb{C}^2$.

Our paper is organized as follow. In Section 2, following the idea of [Ya], we introduce the higher order Bergman functions which are biholomorphic invariants. In Section 3, we show how to write down the k-th order Bergman functions for domains on A_n-variety and recall the fundamental CR-invariant $\nu_V^{(1,0)}$ which we need to use later. In Section 4, we determines all possible biholomorphisms between two domains in A_n-variety. In Section 5, we use higher order Bergman functions to construct numerical invariants in Theorem A. Moreover, in Theorem B, we prove that the invariants in the Theorem A determine the Bergman function up to automorphism of A_n-variety. Theorem C, Theorem D and Theorem E are proved also in this section.

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This paper is dedicated to Professor Joseph Kohn on the occasion of his 75-th Birthday.

2. Preliminaries

In this section, we shall recall some basic definitions and results in our previous paper [Ya] which will facilitate our subsequent discussion. We also take this opportunity to correct some small mistakes in [Ya].

Recall that a complex manifold M is called strictly pseudoconvex if there is a compact set B in M, and a continuous real valued function ϕ on M, which is strictly plurisubharmonic outside B and such that for each $c \in \mathbb{R}$, the set $M_c = \{x \in M : \phi(x) < c\}$ is relatively compact in M. Note that a strictly pseudoconvex complex manifold is a modification of a Stein space at a finite many points.

Let V be a Stein variety of dimension $n \geqslant 2$ in $\mathbb{C}^N$ with only irreducible isolated singularities. We assume that ∂V is a smooth CR manifold. Let $\pi\colon M \to V$ be a resolution of singularity with E as an exceptional set. We shall define the k-th order Bergman function $B^{(k)}_M(z)$ on M which is a biholomorphic invariant of M. The 1$^{\text{st}}$ order Bergman function $B^{(1)}_M(z)$ is the Bergman function $B_M(z)$ introduced in our previous paper [Ya].

Definition 2.1. Let F (respectively, F_k) be the set of all L^2 integrable holomorphic n-forms Ψ on M (respectively, vanishing at least the k-th order on the exception set E of M). Let $\{w_j\}$ (respectively, $\{w^{(k)}_j\}$) be a complete orthonormal basis of F (respectively, F_k). The Bergman kernel (respectively Bergman kernel vanishing on the exceptional set of k-th order) is defined to be $K(z) = \sum w_j(z) \wedge \overline{w_j(z)}$ (respectively, $K^{(k)}(z) = \sum w_j^{(k)}(z) \wedge \overline{w_j^{(k)}(z)}$).

The proofs of the following two Lemmas are exactly the same as those in [Ya].

Lemma 2.2. *F/F_k is a finite dimensional vector space.*

Lemma 2.3. *Bergman kernel vanishing on the exceptional set of k-th order $K^{(k)}(z)$ is independent of the choice of the complete orthonormal basis of F_k and $K^{(k)}(z)$ is invariant under biholomorphic maps.*

Definition 2.4. Let M be a resolution of a Stein variety V of dimension $n \geqslant 2$ in $\mathbb{C}^N$ with only irreducible isolated singularity at the origin. The k-th order Bergman function $B_M^{(k)}$ on M is defined to be $K_M^{(k)}/K_M$.

The proof of the following Theorem 2.5 is the same as the proof in Theorem 2.5 of [Ya]. However, the last statement of Theorem 2.5 of [Ya] is not true in general. It is true when the canonical bundle is generated by its global sections in a neighborhood of the exceptional set, which is automorphically satisfied if V has only rational surface singularities.

Theorem 2.5. *$B_M^{(k)}$ is a global function defined on M which is invariant under biholomorphic maps. Moreover, $B_M^{(k)}$ is nowhere vanishing outside the exceptional set of M. If the canonical bundle is generated by its global sections in a neighborhood of the exceptional set, then the zero set of the k-th order Bergman function $B_M^{(k)}$ is precisely the exceptional set of M.*

The same argument of the proof of Theorem 1 in [L-Y-Y] will prove the following theorem.

Theorem 2.6. *Let M be a strictly pseudoconvex complex manifold of dimension $n \geqslant 2$ with exceptional set E. Let A be a compact submanifold contained in E. Let $\pi\colon M_1 \to M$ be the blow up of M along A. Then we have $K_{M_1}^{(k)}(z) = \pi^* K_M^{(k)}(z)$ and $K_{M_1}(z) = \pi^* K_M(z)$. Consequently $B_{M_1}^{(k)}(z) = \pi^* B_M^{(k)}(z)$.*

Let $\pi_i\colon M_i \to V$, $i = 1, 2$, be two resolutions of singularities of V. By Hironaka's theorem [Hi], there exists a resolution $\widetilde{\pi}\colon \widetilde{M} \to V$ of singularities of V such that $\widetilde{M}$ can be obtained from M_i, $i = 1, 2$, by successive blowing up along submanifolds in exceptional set. In view of Theorem 2.5 and Theorem 2.6, the following definition is well defined if the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Moreover we can get Theorem 2.8 easily.

Definition 2.7. Let V be a Stein variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Let $\pi\colon M \to V$ be a resolution of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Define the k-th order Bergman function $B_V^{(k)}$ on V to be the push forward of the k-th order Bergman function $B_M^{(k)}$ by the map π.

Theorem 2.8. *Let V be a Stein variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Assume that there exists a resolution M of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Then the k-th order Bergman function $B_V^{(k)}$ on V is invariant under biholomorphic maps and $B_V^{(k)}$ vanishes precisely on the singular set of V.*

For the convenience of the readers, we recall the following two important theorems.

Theorem 2.9. *([Fe1]) A biholomorphic mapping between two strictly pseudoconvex domains is smooth up to boundary and the induced boundary mapping gives a CR-equivalence between the boundaries.*

Theorem 2.10. *([Su]) Two n-dimensional bounded Reinhardt domains D_1 and D_2 are mutually equivalent if and only if there exists a transformation $\phi : \mathbb{C}^n \to \mathbb{C}^n$ given by $z_i \mapsto r_i z_{\sigma(i)} (r_i > 0, i = 1, \cdots, n$ and σ being a permutation of the indices $i)$ such that $\phi(D_1) = D_2$.*

3. Continuous invariant k^{th} order Bergman function

Let X be a strictly pseudoconvex CR manifold of real dimension $2n-1$. It is well known [Bo] that X can be CR embedded into $\mathbb{C}^N$ if $n \geqslant 3$. For any embeddable strictly pseudoconvex CR manifold of real dimension at least 3, the famous theorem of Harvey and Lawson [Ha-La] implies that X is a boundary of a variety V in $\mathbb{C}^N$ for some N such that V has only isolated normal singularities.

Proposition 3.1. *([Ya]) Let X_1, X_2 be two strictly pseudoconvex CR manifolds of dimension $2n-1$ which bound varieties V_1, V_2 respectively in $\mathbb{C}^N$ with only isolated normal singularities. If $\Phi\colon X_1 \to X_2$ is a CR-isomorphism, then Φ can be extended to a biholomorphic map from V_1 to V_2.*

In view of the above Proposition 3.1, if X_1 and X_2 are two strictly pseudoconvex CR manifolds which bound varieties V_1 and V_2 respectively with non-isomorphic singularities, then X_1 and X_2 are not CR equivalent. Therefore to study the CR equivalence of two strictly pseudoconvex CR manifolds X_1 and X_2, it remains to consider the case when X_1 and X_2 are lying on the same variety V. The purpose of this section is to show that our global invariant Bergman function of k-th order defined in Section 2 can be used to study the CR equivalence problem of smooth CR manifolds lying on the same variety. As an example, we shall show explicitly how CR manifolds varies in the A_n-variety $\widetilde{V_n} = \{(x,y,z) \in \mathbb{C}^3 \colon f(x,y,z) = xy - z^{n+1} = 0\}$. An explicit resolution $\widetilde{\pi}\colon \widetilde{M_n} \to \widetilde{V_n}$ can be given in terms of coordinate charts and transition functions as follows:

$$\text{Coordinate charts:} \quad \widetilde{W_k} = \mathbb{C}^2 = \{(u_k, v_k)\},\ k = 0, 1, \cdots, n.$$

$$\text{Transition functions:} \quad \begin{cases} u_{k+1} = \frac{1}{v_k} \\ v_{k+1} = u_k {v_k}^2 \end{cases} \qquad \text{or} \qquad \begin{cases} u_k = {u_{k+1}}^2 v_{k+1} \\ v_k = \frac{1}{u_{k+1}} \end{cases}$$

$$\text{Resolution map:} \quad \widetilde{\pi}(u_k, v_k) = \left(u_k^{k+1} v_k^k, u_k^{n-k} v_k^{n+1-k}, u_k v_k\right) \quad \text{or}$$
$$(x,y,z) = (u_0, u_0^n v_0^{n+1}, u_0 v_0) = \cdots = ({u_n}^{n+1} v_n^n, v_n, u_n v_n).$$

$$\text{Exceptional set:} \quad E = \widetilde{\pi}^{-1}(0) = C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\},$$
$$k = 1, \cdots, n.$$

From now on, we suppose V to be a bounded complete Reinhardt domain in $\widetilde{V_n}$. Then let $M = \widetilde{\pi}^{-1}(V) = \cup_{k=0}^n W_k$, where $W_k = \widetilde{\pi}^{-1}(V) \cap \widetilde{W_k}$, $k = 0, 1, \cdots, n$. Observe that under $\pi := \widetilde{\pi}|_M \colon M \to V$, $W_0 \backslash C_1$ is mapped biholomorphically onto $V \backslash y$-axis. In particular $M \backslash W_0$ is of measure zero in the obvious sense. Hence, we may compute integrals on M using the (u_0, v_0) coordinate on the chart W_0 alone.

The following proposition is a general consequence of the proof of Proposition 3.2 of [Ya] (also cf. Proposition 8 in [L-Y-Y]).

Proposition 3.2. *In the above notations, let* $\phi_{\alpha\beta} = u_0^\alpha v_0^\beta\, du_0 \wedge dv_0$, $\alpha, \beta = 0, 1, 2, \ldots$. *Then* $\left\{\frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} \colon \alpha \geqslant \frac{n}{n+1}\beta\right\}$ *is a complete orthonormal base of* F *and* $\left\{\frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} \colon \alpha \geqslant \frac{n}{n+1}\beta \text{ and } \alpha \geqslant k\right\}$ *is a complete orthonormal base of* F_k. *Therefore the Bergman kernel vanishing on the exceptional set of k-th order* $K_M^{(k)}$ *and the Bergman kernel* K_M *are given respectively by:*

$$K_M^{(k)}(u_0, v_0) = \Theta_M^{(k)}\, du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0}$$

where

$$\Theta_M^{(k)} = \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant k}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2},$$

and

$$K_M(u_0, v_0) = \tag{3.1}$$

$$\left(\frac{1}{\|\phi_{00}\|_M^2} + \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ 1 \leqslant \alpha \leqslant k-1}} \frac{|u_0|^{2\alpha}|v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2} + \Theta_M^{(k)}\right) du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0}.$$

The following results generalize Theorem 3.3 in [Ya].

Theorem 3.3. *In the above notations, the k-th order Bergman function for the strongly pseudoconvex complex manifold M is given by*

$$B_M^{(k)}(u_0, v_0) = \frac{\Theta_M^{(k)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant 1}} \frac{|u_0|^{2\alpha}|v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2}\right)}. \tag{3.2}$$

The k-th order Bergman function for the variety is given by

$$B_V^{(k)}(x, y) = \frac{\Theta_V^{(k)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \Theta_V^{(1)}\right)}, \tag{3.3}$$

where

$$\Theta_V^{(k)} = \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant k}} \frac{|x|^{2\alpha - \frac{2n\beta}{n+1}}|y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_M^2}. \tag{3.4}$$

Proof.

$$B_M^{(k)}(u_0, v_0) = \frac{K_M^{(k)}}{K_M} = \frac{\|\phi_{00}\|_M^2 \Theta_M^{(k)}}{1 + \|\phi_{00}\|_M^2 \Theta_M^{(1)}},$$

so (3.2) follows immediately. Recall that the resolution map is given by $(x, y, z) = (u_0, u_0^n v_0^{n+1}, u_0 v_0)$. Then (3.3) and (3.4) follow from (3.2).

q.e.d.

Lemma 3.4. *Let V be a complete Reinhardt domain in the A_n-variety $\widetilde{V_n}$. Any biholomorphism $\Psi = (\psi_1, \psi_2, \psi_3)\colon V \to V$ has the following representation*

$$\begin{pmatrix} \psi_1(x,y,z) \\ \psi_2(x,y,z) \\ \psi_3(x,y,z) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$+ \text{ higher order terms in } x, y \text{ and } z.$$

If $n = 1$, the constants satisfy the following equations

$$a_{11}a_{21} - {a_{31}}^2 = 0 \tag{3.5}$$

$$a_{12}a_{22} - {a_{32}}^2 = 0 \tag{3.6}$$

$$a_{13}a_{23} - {a_{33}}^2 + a_{11}a_{22} + a_{12}a_{21} - 2a_{31}a_{32} = 0 \tag{3.7}$$

$$a_{11}a_{23} + a_{13}a_{21} - 2a_{31}a_{33} = 0 \tag{3.8}$$

$$a_{12}a_{23} + a_{13}a_{22} - 2a_{32}a_{33} = 0 \tag{3.9}$$

$$det(a_{ij}) \neq 0. \tag{3.10}$$

If $n > 1$, the constants satisfy the following equations

$$a_{11}a_{21} = 0 \tag{3.11}$$

$$a_{12}a_{22} = 0 \tag{3.12}$$

$$a_{13}a_{23} = 0 \tag{3.13}$$

$$a_{11}a_{23} + a_{13}a_{21} = 0 \tag{3.14}$$

$$a_{12}a_{23} + a_{13}a_{22} = 0 \tag{3.15}$$

$$a_{11}a_{22} + a_{12}a_{21} - a_{33}^{n+1} = 0 \tag{3.16}$$

$$det(a_{ij}) \neq 0. \tag{3.17}$$

Proof. The case of $n = 1$ has been proved in [Ya]. For $n > 1$, since $\Psi\colon V \to V$, we have $\psi_1(x,y,z)\psi_2(x,y,z) - \psi_3^{n+1}(x,y,z) = 0$. By looking at the quadratic part of this equation and the fact that $xy = z^{n+1}$, we obtain $(a_{11}x + a_{12}y + a_{13}z)(a_{21}x + a_{22}y + a_{23}z) - a_{33}^{n+1}xy = 0$. Then the lemma follows easily. q.e.d.

Proposition 3.5. *Let V_i, $i = 1, 2$, be two complete Reinhardt domains in $\widetilde{V_n} = \{(x,y,z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. Let $M_i = \widetilde{\pi}^{-1}(V_i)$, $i = 1, 2$. Suppose that $\Psi\colon V_1 \to V_2$ is a biholomorphic map given by $\Psi(x,y,z) = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z) +$ higher order term. Then*

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}|a_{11}|^2 + \frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}|a_{31}|^2 + \frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{n,n+1}\|^2_{M_2}}|a_{21}|^2 = \frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{10}\|^2_{M_1}} \tag{3.18}$$

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}|a_{12}|^2 + \frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}|a_{32}|^2 + \frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{n,n+1}\|^2_{M_2}}|a_{22}|^2 = \frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{n,n+1}\|^2_{M_1}} \tag{3.19}$$

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}|a_{13}|^2 + \frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}|a_{33}|^2 + \frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{n,n+1}\|^2_{M_2}}|a_{23}|^2 = \frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{11}\|^2_{M_1}} \tag{3.20}$$

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}a_{11}\overline{a_{12}} + \frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}a_{31}\overline{a_{32}} + \frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{n,n+1}\|^2_{M_2}}a_{21}\overline{a_{22}} = 0 \tag{3.21}$$

(3.22)
$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}a_{11}\overline{a_{13}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}a_{31}\overline{a_{33}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{n,n+1}\|^2_{M_2}}a_{21}\overline{a_{23}}=0$$
(3.23)
$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}a_{12}\overline{a_{13}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}a_{32}\overline{a_{33}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{n,n+1}\|^2_{M_2}}a_{22}\overline{a_{23}}=0.$$

Proof. Since $B_{V_1}(x,y,z)=B_{V_2}(\Psi(x,y,z))$, we have
$$\begin{aligned}&\frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{10}\|^2_{M_1}}|x|^2+\frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{n,n+1}\|^2_{M_1}}|y|^2+\frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{11}\|^2_{M_1}}|z|^2\\ =&\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}|a_{11}x+a_{12}y+a_{13}z|^2+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{n,n+1}\|^2_{M_2}}|a_{21}x+a_{22}y+a_{23}z|^2\\ &+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}|a_{31}x+a_{32}y+a_{33}z|^2.\end{aligned}$$
By comparing the coefficients of $|x|^2$, $|y|^2$, $|z|^2$, $x\overline{y}$, $x\overline{z}$, $y\overline{z}$, $\overline{x}y$, $\overline{x}z$ and $\overline{y}z$, we can get the identities immediately. q.e.d.

Next we recall some results in [Ya].

Theorem 3.6. *([Ya]) Let V be a bounded complete Reinhardt domain in $\widetilde{V_1}=\left\{(x,y,z)\in\mathbb{C}^3 : xy=z^2\right\}$ such that ∂V is a smooth CR manifold. Let $\widetilde{\pi}:\widetilde{M}\to\widetilde{V_1}$ be a resolution of $\widetilde{V_1}$ and $M=\widetilde{\pi}^{-1}(V)$. With the notation in Proposition* 3.2, *$\nu_V^{(1,0)}:=\frac{\|\phi_{11}\|^2_M}{\|\phi_{10}\|_M\|\phi_{12}\|_M}$ is a holomorphic invariant of V in $\widetilde{V_1}$, i.e., if V_1 and V_2 are two such bounded complete Reinhardt domains in $\widetilde{V_1}$ which are biholomorphically equivalent, then $\frac{\|\phi_{11}\|^2_{M_1}}{\|\phi_{10}\|_{M_1}\|\phi_{12}\|_{M_1}}=\frac{\|\phi_{11}\|^2_{M_2}}{\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}$, where $M_i=\widetilde{\pi}^{-1}(V_i)$, $i=1,2$.*

Corollary 3.7. *([Ya]) Let $V_i, i=1,2$, be two bounded complete Reinhardt domains in $\widetilde{V_1}=\left\{(x,y,z)\in\mathbb{C}^3 : xy=z^2\right\}$. If the holomorphic invariant $\nu_{V_1}^{(1,0)}$ or $\nu_{V_2}^{(1,0)}$ in Theorem* 3.6 *is not equal to $\frac{1}{2}$, then the biholomorphic map $\Psi=(\psi_1,\psi_2,\psi_3)\colon V_1\to V_2$ must be one of the following forms:*

1) *$(\psi_1,\psi_2,\psi_3)=(a_{11}x,a_{22}y,a_{33}z)+$ higher order terms and ${a_{33}}^2=a_{11}a_{22}$, where $a_{11}a_{22}a_{33}\neq 0$.*
2) *$(\psi_1,\psi_2,\psi_3)=(a_{12}y,a_{21}x,a_{33}z)+$ higher order terms and ${a_{33}}^2=a_{12}a_{21}$, where $a_{11}a_{22}a_{33}\neq 0$.*

The following lemma generalizes Corollary 3.8 in [Ya].

Lemma 3.8. *Let $V_i, i = 1, 2$, be two complete Reinhardt domains in $\widetilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$, where $n > 1$. Then the biholomorphic map $\Psi = (\psi_1, \psi_2, \psi_3)\colon V_1 \to V_2$ must be one of the following forms:*

1) $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z) +$ *higher order terms and* ${a_{33}}^{n+1} = a_{11}a_{22}$, *where* $a_{11}a_{22}a_{33} \neq 0$.
2) $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z) +$ *higher order terms and* ${a_{33}}^{n+1} = a_{12}a_{21}$, *where* $a_{11}a_{22}a_{33} \neq 0$.

Proof. In view of Proposition 3.4, the constant a_{ij} satisfies the equations (3.11) to (3.17).

Case 1: $a_{11} \neq 0$

$$(3.11) \Rightarrow a_{21} = 0. \tag{3.24}$$

$$(3.14), (3.24) \Rightarrow a_{23} = 0. \tag{3.25}$$

$$(3.14), (3.25) \Rightarrow a_{13}a_{22} = 0. \tag{3.26}$$

$$(3.24), (3.16) \Rightarrow a_{11}a_{22} - a_{33}^{n+1} = 0. \tag{3.27}$$

Since $a_{21} = a_{23} = 0$, we have $a_{22} \neq 0$ because of (3.17). It follows from (3.12) and (3.26) that $a_{12} = a_{13} = 0$. Since $a_{11} \neq 0$ and $a_{22} \neq 0$, $a_{33} \neq 0$ from (3.27). Notice that a_{ij} satisfies (3.22) and (3.23), so $a_{31} = 0$ and $a_{32} = 0$. We thus arrive at case (1) of the lemma.

Case 2: $a_{11} = 0$

$$(3.16) \Rightarrow a_{12}a_{21} - a_{33}^{n+1} = 0. \tag{3.28}$$

$$(3.14) \Rightarrow a_{13}a_{21} = 0. \tag{3.29}$$

We claim that $a_{13} = 0$. If $a_{13} \neq 0$, then (3.13) and (3.29) would imply that $a_{21} = a_{23} = 0$. From (3.15), we have $a_{22} = 0$. This contradicts to (3.17). So we have shown $a_{13} = 0$. It follows from (3.17) that $a_{12} \neq 0$. So from (3.12), we get $a_{22} = 0$. From (3.15), we get $a_{23} = 0$. So from (3.17), $a_{21} \neq 0$. So $a_{33} \neq 0$ from (3.16). Then from (3.22) and (3.23), we can get $a_{31} = 0$ and $a_{32} = 0$. We thus arrive at case (2) of the lemma. q.e.d.

4. Biholomorphisms between two bounded complete Reinhardt domains in A_n-variety

In this section, we shall show that our Bergman function of order 1 can be used to determine the biholomorphisms between two bounded complete Reinhardt domains in A_n-variety.

Theorem 4.1. *Let V_i, $i = 1, 2$, be two bounded complete Reinhardt domains in A_n-variety $\widetilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. If $n = 1$ and $\nu_{V_1}^{(1,0)}$ or $\nu_{V_2}^{(1,0)}$ the holomorphic invariant defined in the previous section* *is not equal to* $\frac{1}{2}$ *or* $n > 1$, *then the biholomorphism* Ψ *from* V_1 *to* V_2 *must be one of the following forms:*

1) $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z)$ *and* $a_{33}^{n+1} = a_{11}a_{22}$, *where*

$$|a_{11}| = \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}, \tag{4.1}$$

$$|a_{22}| = \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}, \tag{4.2}$$

$$|a_{33}| = \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}. \tag{4.3}$$

2) $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z)$ *and* $a_{33}^{n+1} = a_{12}a_{21}$ *where*

$$|a_{12}| = \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}, \tag{4.4}$$

$$|a_{21}| = \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}, \tag{4.5}$$

$$|a_{33}| = \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}. \tag{4.6}$$

Moreover, if V_1 *is biholomorphic to* V_2, *then*

$$\frac{\|\phi_{11}\|_{M_1}^{n+1}}{\|\phi_{00}\|_{M_1}^{n-1}\|\phi_{10}\|_{M_1}\|\phi_{n,n+1}\|_{M_1}} = \frac{\|\phi_{11}\|_{M_2}^{n+1}}{\|\phi_{00}\|_{M_2}^{n-1}\|\phi_{10}\|_{M_2}\|\phi_{n,n+1}\|_{M_2}}. \tag{4.7}$$

Proof. In view of Corollary 3.7 and Lemma 3.8, we know that $\Psi = (\psi_1, \psi_2, \psi_3)$ must be one of the following forms:

1) $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z)+$higher order terms and ${a_{33}}^{n+1} = a_{11}a_{22}$.
2) $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z)+$higher order terms and ${a_{33}}^{n+1} = a_{12}a_{21}$.

We only need to get rid of the higher order terms in the statement of Corollary 3.7 and Lemma 3.8. Recall that the Bergman function is of the form

$$B_V^{(1)}(x, y) = \frac{\Theta_V^{(1)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \Theta_V^{(1)}\right)}.$$

Since

$$B_{V_1}(x, y, z) = B_{V_2}(\Psi(x, y, z)),$$

$$\|\phi_{00}\|^2_{M_1}\Theta^{(1)}_{V_1} = \|\phi_{00}\|^2_{M_2}\Theta^{(1)}_{V_2}. \tag{4.8}$$

Putting equations (1) and (2) above in (4.8) and comparing the 3rd order terms in (4.8), we see that the 2nd order terms of (ψ_1, ψ_2, ψ_3) are zero. Repeating this argument, we see that (ψ_1, ψ_2, ψ_3) has only linear terms.

Except for (4.7), the rest of the theorem follows from Proposition 3.5, Corollary 3.7 and Lemma 3.8. To see (4.7), we have two cases.

In case (1), $a_{33}^{n+1} = a_{11}a_{22}$ and (4.1), (4.2), (4.3) imply

$$\frac{\|\phi_{11}\|^{n+1}_{M_2}}{\|\phi_{00}\|^{n+1}_{M_2}} \cdot \frac{\|\phi_{00}\|^{n+1}_{M_1}}{\|\phi_{11}\|^{n+1}_{M_1}} = \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}} \cdot \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}.$$

So (4.7) holds.

In case (2), $a_{33}^{n+1} = a_{12}a_{21}$ and (4.4), (4.5), (4.6) imply

$$\frac{\|\phi_{11}\|^{n+1}_{M_2}}{\|\phi_{00}\|^{n+1}_{M_2}} \cdot \frac{\|\phi_{00}\|^{n+1}_{M_1}}{\|\phi_{11}\|^{n+1}_{M_1}} = \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}.$$

So (4.7) holds. q.e.d.

The following Corollary 4.2 corrects some misprints in Theorem 5.1 of [Ya] and generalizes the case to A_n type.

Corollary 4.2. *Let V be a bounded complete Reinhardt domain in A_n-variety $\widetilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. Let $\nu_V^{(1,0)}$ be the CR invariant defined in the Theorem 3.6. Then the automorphism group of V for $n = 1$ and $\nu_V^{(1,0)} \neq \frac{1}{2}$ or $n > 1$ consists of biholomorphic map $\Psi = (\psi_1, \psi_2, \psi_3)$ of the following forms:*

1) $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z)$, *where* $|a_{11}| = |a_{22}| = |a_{33}| = 1$.
2) $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z)$,

$$\text{where } |a_{11}| = \tfrac{\|\phi_{10}\|_M}{\|\phi_{n,n+1}\|_M},\ |a_{12}| = \tfrac{\|\phi_{n,n+1}\|_M}{\|\phi_{10}\|_M},\ |a_{21}| = 1.$$

Now we are going to deal with the biholomorphism between complete Reinhardt domains with $\nu_V^{(1,0)} = \frac{1}{2}$.

Theorem 4.3. *Let V_i, $i = 1, 2$, be two bounded complete Reinhardt domains in A_1-variety $\widetilde{V_1} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2\}$. If $\nu_{V_1}^{(1,0)}$ or $\nu_{V_2}^{(1,0)}$ the holomorphic invariant defined in the previous section is equal to $\frac{1}{2}$, then the biholomorphism Ψ from V_1 to V_2 must be one of the following forms:*

1) $$(\psi_1, \psi_2, \psi_3) = \Bigg(e^{i\theta_1}\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ e^{i\theta_2}\frac{\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}}y,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\Bigg).$$

2) $$(\psi_1,\psi_2,\psi_3)=\Bigg(e^{i\theta_1}\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}}y,\ e^{i\theta_2}\frac{\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \pm\, e^{i\frac{\theta_1+\theta_2}{2}}\frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\Bigg).$$

3) $$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=\begin{pmatrix}\alpha r e^{-i\theta}a_{31} & \frac{1}{r}e^{-i\theta}a_{32} & \pm a_{13}\\ \frac{1}{\alpha r}e^{i\theta}a_{31} & re^{i\theta}a_{32} & \pm\frac{1}{\alpha}e^{2i\theta}a_{13}\\ a_{31} & a_{32} & \pm\frac{\alpha r^2+1}{2\alpha r}e^{i\theta}a_{13}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix},$$

where

$$\alpha=\frac{-\|\phi_{10}\|_{M_2}}{\|\phi_{12}\|_{M_2}} \tag{4.9}$$

$$a_{31}=\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}\ \frac{r\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}\,(r^2\|\phi_{10}\|_{M_2}+\|\phi_{12}\|_{M_2})}\ e^{i\theta_{31}} \tag{4.10}$$

$$a_{32}=\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}}\ \frac{r\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}\,(\|\phi_{12}\|_{M_2}+r^2\|\phi_{10}\|_{M_2})}\ e^{i\theta_{32}} \tag{4.11}$$

$$a_{13}=\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}\ \frac{2r\|\phi_{10}\|_{M_2}\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}\,(r^2\|\phi_{10}\|_{M_2}+\|\phi_{12}\|_{M_2})}\ e^{i\left(\frac{\pi}{2}+\frac{\theta_{31}}{2}+\frac{\theta_{32}}{2}-\theta\right)} \tag{4.12}$$

$0<r<\infty$, $0\le\theta_{31},\theta_{32},\ \theta<2\pi$.

Proof. By going through the proof of Theorem 3.7 in [Ya], we see that statements (1) and (2) in the theorem follow directly from Theorem 4.1. The remaining case that we need to deal with is the case 1 (b) in the proof of Theorem 3.7 in [Ya]. Let us summarize the situation.

$$\nu_{V_2}^{(1,0)}=\frac{\|\phi_{11}\|^2_{M_2}}{\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}=\frac{1}{2}=\nu_{V_1}^{(1,0)}=\frac{\|\phi_{11}\|^2_{M_1}}{\|\phi_{10}\|_{M_1}\|\phi_{12}\|_{M_1}} \tag{4.13}$$

$$a_{11}=r_1a_{31},\quad a_{21}=\frac{1}{r_1}a_{31} \tag{4.14}$$

$$a_{22}=r_2a_{32},\quad a_{12}=\frac{1}{r_2}a_{32} \tag{4.15}$$

$$a_{23}=\frac{r_2}{r_1}a_{13},\quad a_{33}=\left(\frac{r_2}{2}+\frac{1}{2r_1}\right)a_{13} \tag{4.16}$$

$$\alpha=\frac{r_1}{\overline{r}_2}=\frac{-2\|\phi_{11}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}=\frac{-\|\phi_{10}\|_{M_2}}{\|\phi_{12}\|_{M_2}} \tag{4.17}$$

$$r_2=r\ e^{i\theta} \tag{4.18}$$

$$r_1 = \alpha r e^{-i\theta} = -\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{12}\|_{M_2}}\ r e^{-i\theta} \tag{4.19}$$

Putting (4.14) into (3.18), we get

$$|a_{31}|^2\|\phi_{00}\|_{M_2}^2\left[\frac{|r_1|^2}{\|\phi_{10}\|_{M_2}^2}+\frac{1}{\|\phi_{11}\|_{M_2}^2}+\frac{1}{|r_1|^2\|\phi_{12}\|_{M_2}^2}\right]=\frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{10}\|_{M_1}^2}.$$

By applying (4.13) in the above equation we get

$$|a_{31}|=\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}\ \frac{|r_1|\ \|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}\left(|r_1|^2\|\phi_{12}\|_{M_2}+\|\phi_{10}\|_{M_2}\right)}.$$

In view of (4.19), we have

$$|a_{31}|=\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}\ \frac{r\ \|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}\left(\|\phi_{10}\|_{M_2}r^2+\|\phi_{12}\|_{M_2}\right)}. \tag{4.20}$$

Putting (4.15) into (3.19), we get

$$|a_{32}|^2\left[\frac{\|\phi_{00}\|_{M_2}^2}{|r_2|^2\|\phi_{10}\|_{M_2}^2}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}+\frac{|r_2|^2\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\right]=\frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{12}\|_{M_1}^2}.$$

By applying (4.13) in the above equation, we get

$$|a_{32}|^2\|\phi_{00}\|_{M_2}^2\left[\frac{\|\phi_{12}\|_{M_2}^2+2|r_2|^2\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}+|r_2|^4\|\phi_{10}\|_{M_2}^2}{|r_2|^2\|\phi_{10}\|_{M_2}^2\|\phi_{12}\|_{M_2}^2}\right]$$
$$=\frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{12}\|_{M_1}^2}.$$

In view of (4.18), we have

$$|a_{32}|=\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}}\ \frac{r\ \|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}\left(\|\phi_{12}\|_{M_2}+r^2\|\phi_{10}\|_{M_2}\right)}. \tag{4.21}$$

Putting (4.16) into (3.20), we get

$$|a_{13}|^2\left[\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}+\frac{(r_1r_2+1)^2}{4|r_1|^2}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\left|\frac{r_2}{r_1}\right|^2\right]$$
$$=\frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{11}\|_{M_1}^2}.$$

By applying (4.13) in the above equation, we get

$$|a_{13}|^2=\frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{11}\|_{M_1}^2}\cdot$$
$$\frac{|r_1|^2\|\phi_{10}\|_{M_2}^2\|\phi_{12}\|_{M_2}^2}{\|\phi_{00}\|_{M_2}^2\Big[|r_1|^2\|\phi_{12}\|_{M_2}^2+\frac{(1+r_1r_2)^2}{2}\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}+|r_2|^2\|\phi_{10}\|_{M_2}^2\Big]}.$$

In view of (4.18) and (4.19), we have

$$|a_{13}| = \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} \frac{2r\ \|\phi_{10}\|_{M_2}\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}\,(r^2\|\phi_{10}\|_{M_2}+\|\phi_{12}\|_{M_2})}. \tag{4.22}$$

Putting (4.14), (4.15) (4.16) into (3.7), we get

$$a_{13}^2 = a_{31}a_{32}\frac{4r_1}{r_2}. \tag{4.23}$$

Putting (4.20), (4.21)and (4.22) into (4.23), we get

$$\theta_{13} = \frac{\pi}{2} + \frac{\theta_{31}}{2} + \frac{\theta_{31}}{2} - \theta. \tag{4.24}$$

We have shown that

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}\alpha r e^{-i\theta}a_{31} & \frac{1}{r}e^{-i\theta}a_{32} & \pm a_{13}\\ \frac{1}{\alpha r}e^{i\theta}a_{31} & re^{i\theta}a_{32} & \pm\frac{1}{\alpha}e^{2i\theta}a_{13}\\ a_{31} & a_{32} & \pm\frac{\alpha r^2+1}{2\alpha r}e^{i\theta}a_{13}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$
$$+\text{ higher order terms}, \tag{4.25}$$

where α, a_{31}, a_{32}, a_{13}, are described in (4.8)–(4.12). It remains to prove that all the higher order terms vanish. In view of Theorem 2.8, we have

$$B_{V_1}(x,y,z) = B_{V_2}(\psi(x,y,z)). \tag{4.26}$$

Putting (4.25) into (4.26) and comparing the 3$^{\text{rd}}$ order terms in (4.26), we see easily that the 2$^{\text{nd}}$ order terms of (ψ_1,ψ_2,ψ_3) are zero. By repeating this argument, we see that (ψ_1,ψ_2,ψ_3) has only linear terms. q.e.d.

5. Continuous numerical invariants of bounded complete Reinhardt domains

In [Ya], we have succeeded in constructing the continuous numerical invariant of domains in A_1-variety

$$\nu_V^{(1,0)} = \frac{\|\phi_{11}\|_M^2}{\|\phi_{10}\|_M\|\phi_{12}\|_M}$$

from 1st order Bergman function. In this section, we shall construct infinitely many continuous numerical invariants of domains in A_n-variety from Bergman function.

Lemma 5.1. *Let V_i, $i=1,2$, be two bounded complete Reinhardt domains in A_n-variety $\widetilde{V_n} = \left\{(x,y,z)\in\mathbb{C}^3 : xy = z^{n+1}\right\}$ and M_i is a resolution of V_i. If $(\psi_1,\psi_2,\psi_3) = (a_{11}x, a_{22}y, a_{33}z)$ and $a_{33}^{n+1} = a_{11}a_{22}$, where*

$$|a_{11}| = \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}},\ |a_{22}| = \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}},$$

$$|a_{33}| = \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}},$$

is a biholomorphism from V_1 to V_2, then the following equations hold:

$$\frac{\|\phi_{10}\|_{M_1}^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|_{M_1}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}\|\phi_{00}\|_{M_1}^{\alpha-\frac{n-1}{n+1}\beta-1}} = \frac{\|\phi_{10}\|_{M_2}^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|_{M_2}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2}\|\phi_{00}\|_{M_2}^{\alpha-\frac{n-1}{n+1}\beta-1}}, \qquad \alpha \geqslant \frac{n}{n+1}\beta.$$

Proof. From (4.8) $\|\phi_{00}\|_{M_1}^2 \Theta_{V_1}^{(1)} = \|\phi_{00}\|_{M_2}^2 \Theta_{V_2}^{(1)}$, we have

$$\|\phi_{00}\|_{M_1}^2 \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant 1}} \frac{|x|^{2\alpha-\frac{2n\beta}{n+1}}|y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}^2} = \|\phi_{00}\|_{M_2}^2 \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant 1}} \frac{|\psi_1|^{2\alpha-\frac{2n\beta}{n+1}}|\psi_2|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2}^2}.$$

Comparing the coefficients of $|x|^{2\alpha-\frac{2n\beta}{n+1}}|y|^{\frac{2\beta}{n+1}}$ each side, we get

$$\frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{\alpha\beta}\|_{M_1}^2} = \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{\alpha\beta}\|_{M_2}^2} \cdot \frac{\|\phi_{00}\|_{M_1}^{2\alpha-\frac{2n\beta}{n+1}}}{\|\phi_{10}\|_{M_1}^{2\alpha-\frac{2n\beta}{n+1}}} \cdot \frac{\|\phi_{10}\|_{M_2}^{2\alpha-\frac{2n\beta}{n+1}}}{\|\phi_{00}\|_{M_2}^{2\alpha-\frac{2n\beta}{n+1}}} \cdot$$

$$\frac{\|\phi_{00}\|_{M_1}^{\frac{2\beta}{n+1}}}{\|\phi_{n,n+1}\|_{M_1}^{\frac{2\beta}{n+1}}} \cdot \frac{\|\phi_{n,n+1}\|_{M_2}^{\frac{2\beta}{n+1}}}{\|\phi_{00}\|_{M_2}^{\frac{2\beta}{n+1}}}.$$

Simplifying this equation, we get

$$\frac{\|\phi_{10}\|_{M_1}^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|_{M_1}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}\|\phi_{00}\|_{M_1}^{\alpha-\frac{n-1}{n+1}\beta-1}} = \frac{\|\phi_{10}\|_{M_2}^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|_{M_2}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2}\|\phi_{00}\|_{M_2}^{\alpha-\frac{n-1}{n+1}\beta-1}}, \qquad \alpha \geqslant \frac{n}{n+1}\beta.$$

q.e.d.

Lemma 5.2. *Let V_i, $i = 1, 2$, be two bounded complete Reinhardt domains in A_n-variety $\widetilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ and M_i is a resolution of V_i. If $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z)$ and $a_{33}^{n+1} = a_{12}a_{21}$, where*

$$|a_{12}| = \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}, \; |a_{21}| = \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}},$$

$$|a_{33}| = \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}},$$

is a biholomorphism from V_1 to V_2, then the following equations hold:

$$\frac{\|\phi_{10}\|_{M_1}^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|_{M_1}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}\|\phi_{00}\|_{M_1}^{\alpha-\frac{n-1}{n+1}\beta-1}}=\frac{\|\phi_{10}\|_{M_2}^{\frac{\beta}{n+1}}\|\phi_{n,n+1}\|_{M_2}^{\alpha-\frac{n}{n+1}\beta}}{\|\phi_{n\alpha-(n-1)\beta,(n+1)\alpha-n\beta}\|_{M_2}\|\phi_{00}\|_{M_2}^{\alpha-\frac{n-1}{n+1}\beta-1}},$$

where

$$\alpha\geqslant\frac{n}{n+1}\beta.$$

Proof. From (4.8), $\|\phi_{00}\|_{M_1}^2\Theta_{V_1}^{(1)}=\|\phi_{00}\|_{M_2}^2\Theta_{V_2}^{(1)}$, we have

$$\|\phi_{00}\|_{M_1}^2\sum_{\substack{\alpha\geqslant\frac{n}{n+1}\beta\\ \alpha\geqslant1}}\frac{|x|^{2\alpha-\frac{2n\beta}{n+1}}|y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}^2}=\|\phi_{00}\|_{M_2}^2\sum_{\substack{\alpha\geqslant\frac{n}{n+1}\beta\\ \alpha\geqslant1}}\frac{|\psi_1|^{2\alpha-\frac{2n\beta}{n+1}}|\psi_2|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2}^2}.$$

Comparing the coefficients of $|x|^{2\alpha-\frac{2n\beta}{n+1}}|y|^{\frac{2\beta}{n+1}}$ each side, we get

$$\frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{\alpha\beta}\|_{M_1}^2}=\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n\alpha-(n-1)\beta,(n+1)\alpha-n\beta}\|_{M_2}^2}\cdot\frac{\|\phi_{00}\|_{M_1}^{2\alpha-\frac{2n\beta}{n+1}}}{\|\phi_{10}\|_{M_1}^{2\alpha-\frac{2n\beta}{n+1}}}\cdot\frac{\|\phi_{n,n+1}\|_{M_2}^{2\alpha-\frac{2n\beta}{n+1}}}{\|\phi_{00}\|_{M_2}^{2\alpha-\frac{2n\beta}{n+1}}}\cdot$$

$$\frac{\|\phi_{00}\|_{M_1}^{\frac{2\beta}{n+1}}}{\|\phi_{n,n+1}\|_{M_1}^{\frac{2\beta}{n+1}}}\cdot\frac{\|\phi_{10}\|_{M_2}^{\frac{2\beta}{n+1}}}{\|\phi_{00}\|_{M_2}^{\frac{2\beta}{n+1}}}.$$

Simplifying this equation, we get

$$\frac{\|\phi_{10}\|_{M_1}^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|_{M_1}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}\|\phi_{00}\|_{M_1}^{\alpha-\frac{n-1}{n+1}\beta-1}}=\frac{\|\phi_{10}\|_{M_2}^{\frac{\beta}{n+1}}\|\phi_{n,n+1}\|_{M_2}^{\alpha-\frac{n}{n+1}\beta}}{\|\phi_{n\alpha-(n-1)\beta,(n+1)\alpha-n\beta}\|_{M_2}\|\phi_{00}\|_{M_2}^{\alpha-\frac{n-1}{n+1}\beta-1}},$$

where $\alpha\geqslant\frac{n}{n+1}\beta$. q.e.d.

Theorem 5.3. ***(Theorem D)***

Proof of Theorem D. ($\Leftarrow$) In view of Theorem 2.10, there exists a biholomorphic map $\Phi(z_1,z_2)=(az_1,bz_2)$ or (az_2,bz_1). Observe that the fiber of the quotient map π is of the form

$$\left\{(z_1,z_2),(\delta z_1,\delta^n z_2),(\delta^2 z_1,\delta^{2n}z_2),\cdots,(\delta^n z_1,\delta^{n^2}z_2)\right\},$$

where δ is a primitive $(n+1)$-th root of unit. And Φ sends one fiber to another fiber. Hence Φ descends to a biholomorphic map $\Psi:V_1\to V_2$ given by

$$\Psi(x,y,z)=(a^{n+1}x,b^{n+1}y,abz)$$

or

$$\Psi(x,y,z) = (a^{n+1}y, b^{n+1}x, abz).$$

($\Rightarrow$) Suppose Ψ is a biholomorphic map from V_1 to V_2. Observe that W_1 is a simply connected domain in $\mathbb{C}^2$. It follows that $W_1\backslash\{0\}$ is also simply connected. Observe also that $\pi_i : W_i\backslash\{0\} \to V_i\backslash\{0\}$, $i = 1,2$, are $n+1$-fold covering maps. The holomorphic map $\Psi \circ \pi_1 : W_1\backslash\{0\} \to V_2\backslash\{0\}$ can be uniquely lifted to a holomorphic map $\Phi : W_1\backslash\{0\} \to W_2\backslash\{0\}$. Similarly the holomorphic map $\Psi^{-1} \circ \pi_2 : W_2\backslash\{0\} \to V_1\backslash\{0\}$ can be uniquely lifted to a holomorphic map $\widetilde{\Phi} : W_2\backslash\{0\} \to W_1\backslash\{0\}$. Thus we have the following commutative diagram.

$$\begin{tikzcd}
W_1\backslash\{0\} \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & W_2\backslash\{0\} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\pi_2"'] & W_1\backslash\{0\} \arrow[d, "\pi_1"'] \\
V_1\backslash\{0\} \arrow[r, "\Psi"] & V_2\backslash\{0\} \arrow[r, "\Psi^{-1}"] & V_1\backslash\{0\}
\end{tikzcd}$$

By the unique lifting property, we have $\widetilde{\Phi} \circ \Phi = 1_{W_1\backslash\{0\}}$ and $\Phi \circ \widetilde{\Phi} = 1_{W_2\backslash\{0\}}$. By Hartog theorem, Φ extends to a biholomorphism from W_1 to W_2. q.e.d.

Corollary 5.4. *Let $\pi : \mathbb{C}^2 \to \widetilde{V_n} = \{(x,y,z) \in \mathbb{C}^3\colon\ xy = z^{n+1}\}$ with $\pi(z_1,z_2) = (z_1^{n+1}, z_2^{n+1}, z_1z_2)$. Let $\mathcal{V} = \{V\colon V$ a bounded complete Reinhardt domain in A_n-variety$\}$ and $\mathcal{W} = \{W = \pi^{-1}(V) : V \in \mathcal{V}\}$. Then the moduli space of $\mathcal{V}$ is equal to the moduli space of $\mathcal{W}$.*

Proof. It is a direct consequence of the Theorem 5.3. q.e.d.

Using Lemma 5.1, 5.2 and Theorem 5.3, we can get a lot of biholomorphic invariants. In order to simplify the notation, we let

$$g^{(\alpha,\beta)} = \frac{\|\phi_{10}\|^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|\,\|\phi_{00}\|^{\alpha-\frac{n-1}{n+1}\beta-1}}.$$

Theorem 5.5. ***(Theorem A)***

Proof of Theorem A. According to the Theorem 5.3, there is a biholomorphism Ψ either of the form in the Lemma 5.1 or in the Lemma 5.2. So either

1) $g_{M_1}^{(\alpha,\beta)} = g_{M_2}^{(\alpha,\beta)}$

   for any α and β satisfying $\quad \alpha \geqslant \frac{n}{n+1}\beta, \alpha \geqslant 1, \beta \geqslant 0$, or

2) $g_{M_1}^{(\alpha,\beta)} = g_{M_2}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}$

   for any α and β satisfying $\quad \alpha \geqslant \frac{n}{n+1}\beta, \alpha \geqslant 1, \beta \geqslant 0$.

For the case 1), we have

$$g_{M_1}^{(\alpha,\,\beta)} = g_{M_2}^{(\alpha,\,\beta)} \text{ and } g_{M_1}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)} = g_{M_2}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}$$

for any α and β satisfying

$$\alpha \geqslant \frac{n}{n+1}\beta, \alpha \geqslant 1, \beta \geqslant 0.$$

So it is easy to see that

$$\xi_{V_1}^{(\alpha,\beta)} = \xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)} = \zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)} = \eta_{V_2}^{(\alpha,p,q)},$$

$$\omega_{V_1}^{(\alpha_1,\,\alpha_2,\,p_1,\,p_2)} = \omega_{V_2}^{(\alpha_1,\,\alpha_2,\,p_1,\,p_2)}.$$

For the case 2), we have

$$g_{M_1}^{(\alpha,\,\beta)} = g_{M_2}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)},\ g_{M_1}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)} = g_{M_2}^{(\alpha,\,\beta)}$$

for any α and β satisfying

$$\alpha \geqslant \frac{n}{n+1}\beta, \alpha \geqslant 1, \beta \geqslant 0.$$

So it is easy to see

$$\xi_{V_1}^{(\alpha,\beta)} = \xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)} = \zeta_{V_2}^{(\alpha,\beta)}.$$

Moreover, we notice that

$$g_{M_1}^{(\alpha,\,p)} = g_{M_2}^{(n\alpha-(n-1)p,(n+1)\alpha-np)},\ g_{M_1}^{(n\alpha-(n-1)p,(n+1)\alpha-np)} = g_{M_2}^{(\alpha,\,p)},$$

$$g_{M_1}^{(\alpha,\,q)} = g_{M_2}^{(n\alpha-(n-1)q,(n+1)\alpha-nq)},\ g_{M_1}^{(n\alpha-(n-1)q,(n+1)\alpha-nq)} = g_{M_2}^{(\alpha,\,q)},$$

for

$$\alpha \geqslant 1, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q.$$

So

$$\eta_{V_1}^{(\alpha,p,q)} = \eta_{V_2}^{(\alpha,p,q)}.$$

For the same reason,

$$g_{M_1}^{(\alpha_1,\,p_1)} = g_{M_2}^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_1-np_1)},$$

$$g_{M_1}^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_1-np_1)} = g_{M_2}^{(\alpha_1,\,p_1)},$$

$$g_{M_1}^{(\alpha_2,\,p_2)} = g_{M_2}^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)},$$

$$g_{M_1}^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)} = g_{M_2}^{(\alpha_2,\,p_2)},$$

where

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

implies

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)} = \omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)}.$$

Therefore $\xi^{(\alpha,\beta)}$, $\zeta^{(\alpha,\beta)}$, $\eta^{(\alpha,p,q)}$ and $\omega^{(\alpha_1,\alpha_2,p_1,p_2)}$ are all invariants.

q.e.d.

Corollary 5.6. *Let V_i, $i = 1, 2$, be two bounded complete Reinhardt domains in A_n-variety $\widetilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. If V_1 is a biholomorphic to V_2, then*

$$\nu^{(\alpha,\beta)} := \frac{\|\phi_{\alpha\alpha}\|^2}{\|\phi_{\alpha\beta}\| \cdot \|\phi_{n\alpha-(n-1)\beta,(n+1)\alpha-n\beta}\|} \qquad \alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta$$

are all invariants, i.e.

$$\nu_{V_1}^{(\alpha,\beta)} = \nu_{V_2}^{(\alpha,\beta)}.$$

Proof. Since

$$\nu^{(\alpha,\beta)} = \frac{\sqrt{\xi^{(\alpha,\beta)} \cdot \xi^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}}}{\sqrt{\xi^{(\alpha,\alpha)}}},$$

$\nu^{(\alpha,\beta)}$s are all invariants. q.e.d.

Remark: The fundamental invariant ν_X mentioned in [Ya] is $\nu^{(1,0)}$.

The following corollary is an immediate consequence of Theorem 5.3 and Theorem 5.5.

Corollary 5.7. *Let $\pi : \mathbb{C}^2 \to \widetilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ with $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$. Let $\mathcal{V} = \{V : V$ a bounded complete Reindardt domain in A_n-variety $\widetilde{V}\}$ and $\mathcal{W} = \{W : W = \pi^{-1}(V) \colon V \in \mathcal{V}\}$. Then*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}\omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

are holomorphic invariants of $\mathcal{W}$.

We have seen that by the Theorem 5.5, we can get a lot of invariants from Bergman functions. However it is natural to ask whether these invariants are sufficient to recover the Bergman function up to automorphism of A_n-variety. The answer is positive. For proving this, we need the following lemma.

Lemma 5.8. *If $x_i, x_i' \in \mathbb{R}$, where $1 \leq i \leq n,\ i, n \in \mathbb{N}$ satisfying*

$$x_j x_{n-j+1} = x_j' x_{n-j+1}', \tag{5.1}$$

$$x_j + x_{n-j+1} = x_j' + x_{n-j+1}', \tag{5.2}$$

$$(x_k - x_{n-k+1})(x_l - x_{n-l+1}) = (x_k' - x_{n-k+1}')(x_l' - x_{n-l+1}'), \tag{5.3}$$

for all $1 \leq j, k, l \leq n,\ j, k, l \in \mathbb{N}$, then $x_i = x_i'$ for all $1 \leq i \leq n$ or $x_i = x_{n-i+1}'$ for all $1 \leq i \leq n$.

Proof. If $n = 2$, the conditions are just $x_1 x_2 = x_1' x_2',\ x_1 + x_2 = x_1' + x_2'$. So we get

$$\begin{cases} x_1 = x_1' \\ x_2 = x_2' \end{cases} \text{or} \begin{cases} x_1 = x_2' \\ x_2 = x_1' \end{cases}.$$

If $n \geqslant 3$, it is easy to get

$$\begin{cases} x_i = x_i' \\ x_{n-i+1} = x_{n-i+1}' \end{cases} \text{or} \begin{cases} x_i = x_{n-i+1}' \\ x_{n-i+1} = x_i' \end{cases}.$$

However, the equation (5.3) force the result to be $x_i = x_i'$ or $x_i = x_{n-i+1}'$ for all $1 \leq i \leq n$. q.e.d.

Theorem 5.9. *Let $V_i,\ i = 1, 2$, be two bounded complete Reinhardt domains in A_n-variety $\widetilde{V_n} = \left\{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\right\}$ and M_i is a resolution of V_i. If*

$$\xi_{V_1}^{(\alpha,\beta)} = \xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)} = \zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)} = \eta_{V_2}^{(\alpha,p,q)},$$
$$\omega_{V_1}^{(\alpha_1, \alpha_2, p_1, p_2)} = \omega_{V_2}^{(\alpha_1, \alpha_2, p_1, p_2)},$$

where

$$\alpha \geqslant 1,\ \alpha \geqslant \frac{n}{n+1}\beta,\ 0 \leqslant p, q \leqslant \left[\frac{n+1}{n}\alpha\right],\ p \neq q,$$
$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right],\ \alpha_i \geqslant 1,\ \alpha_1 \neq \alpha_2,\ i = 1, 2,$$

then there exists an automorphism $\Psi = (\psi_1, \psi_2, \psi_3)$ of A_n-variety $\widetilde{V_n} = \left\{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\right\}$ given by

$(\psi_1, \psi_2, \psi_3) =$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} x,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}} y, \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} z\right),$$

or

$(\psi_1, \psi_2, \psi_3) =$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}} y,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} x, \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} z\right).$$

such that

$$B_{V_1}(x,y,z)=B_{V_2}(\Psi(x,y,z)).$$

Proof. $\xi_{V_1}^{(\alpha,\beta)}=\xi_{V_2}^{(\alpha\beta)}$, $\zeta_{V_1}^{(\alpha,\beta)}=\zeta_{V_2}^{(\alpha,\beta)}$ and $\eta_{V_1}^{(\alpha,p,q)}=\eta_{V_2}^{(\alpha,p,q)}$, means

$$g_{M_1}^{(\alpha,\beta)}\cdot g_{M_1}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}=g_{M_2}^{(\alpha,\beta)}\cdot g_{M_2}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)},$$
$$g_{M_1}^{(\alpha,\beta)}+g_{M_1}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}=g_{M_2}^{(\alpha,\beta)}+g_{M_2}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)}$$

and

$$\begin{aligned}&(g_{M_1}^{(\alpha,p)}-g_{M_1}^{(n\alpha-(n-1)p,(n+1)\alpha-np)})\cdot(g_{M_1}^{(\alpha,q)}-g_{M_1}^{(n\alpha-(n-1)q,(n+1)\alpha-nq)})\\ =\ &(g_{M_2}^{(\alpha,p)}-g_{M_2}^{(n\alpha-(n-1)p,(n+1)\alpha-np)})\cdot(g_{M_2}^{(\alpha,q)}-g_{M_2}^{(n\alpha-(n-1)q,(n+1)\alpha-nq)})\end{aligned}$$

where

$$\alpha\geqslant 1,\ \alpha\geqslant\frac{n}{n+1}\beta,\ 0\leqslant p,\, q\leqslant\left[\frac{n+1}{n}\alpha\right],\ p\neq q.$$

For fixed $\alpha\geqslant 1$ by Lemma 5.8, we can get

$$g_{M_1}^{(\alpha,\beta)}=g_{M_2}^{(\alpha,\beta)},$$

for any $\beta\geqslant 0$ satisfying $\alpha\geqslant\frac{n}{n+1}\beta$ or

$$g_{M_1}^{(\alpha,\beta)}=g_{M_2}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)},$$

for any $\beta\geqslant 0$ satisfying $\alpha\geqslant\frac{n}{n+1}\beta$.

If for some different α_1, $\alpha_2\geqslant 1$, and p_1, p_2 satisfying $0\leqslant p_i\leqslant\left[\frac{n+1}{n}\alpha_i\right]$, $i=1$, 2, such that

$$g_{M_1}^{(\alpha_1,p_1)}=g_{M_2}^{(\alpha_1,p_1)},$$

$$g_{M_1}^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_1-np_1)}=g_{M_2}^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_2-np_1)},$$

$$g_{M_1}^{(\alpha_2,p_2)}=g_{M_2}^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)},$$

$$g_{M_1}^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)}=g_{M_2}^{(\alpha_2,p_2)},$$

then

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)}=\omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)}$$

forces that

$$g_{M_1}^{(\alpha_1,p_1)}=g_{M_1}^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_1-np_1)},$$
$$g_{M_2}^{(\alpha_1,p_1)}=g_{M_2}^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_2-np_1)},$$

or

$$g_{M_1}^{(\alpha_2,p_2)}=g_{M_1}^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)},$$

$$g_{M_2}^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)}=g_{M_2}^{(\alpha_2,\,p_2)}.$$

So we have

$$\begin{aligned} & g_{M_1}^{(\alpha_1,\,p_1)}=g_{M_1}^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_1-np_1)} \\ = \quad & g_{M_2}^{(\alpha_1,\,p_1)}=g_{M_2}^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_2-np_1)} \end{aligned}$$

or

$$\begin{aligned} & g_{M_2}^{(\alpha_2,\,p_2)}=g_{M_2}^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)} \\ = \quad & g_{M_2}^{(\alpha_2,\,p_2)}=g_{M_2}^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)}. \end{aligned}$$

So we only have following two cases:

1)

$$g_{M_1}^{(\alpha,\,\beta)}=g_{M_2}^{(\alpha,\,\beta)},$$

for any

$$\alpha\geqslant 1,\ \beta\geqslant 0,\ \alpha\geqslant\frac{n}{n+1}\beta$$

or

2)

$$g_{M_1}^{(\alpha,\,\beta)}=g_{M_2}^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)},$$

for any

$$\alpha\geqslant 1,\ \beta\geqslant 0,\ \alpha\geqslant\frac{n}{n+1}\beta.$$

Therefore if for the first case, we take

$$(\psi_1,\psi_2,\psi_3)=$$
$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\ \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right),$$

and if for the second case, we take

$$(\psi_1,\psi_2,\psi_3)=$$
$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right).$$

Then we can always get

$$\|\phi_{00}\|_{M_1}^2\sum_{\substack{\alpha\geqslant\frac{n}{n+1}\beta\\ \alpha\geqslant 1}}\frac{|x|^{2\alpha-\frac{2n\beta}{n+1}}|y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}^2}$$

$$=\|\phi_{00}\|_{M_2}^2\sum_{\substack{\alpha\geqslant\frac{n}{n+1}\beta\\ \alpha\geqslant 1}}\frac{|\psi_1|^{2\alpha-\frac{2n\beta}{n+1}}|\psi_2|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2}^2}$$

by comparing the coefficients of $|x|^{2\alpha-\frac{2n\beta}{n+1}}|y|^{\frac{2\beta}{n+1}}$, i.e. $\|\phi_{00}\|^2_{M_1}\Theta^{(1)}_{V_1} = \|\phi_{00}\|^2_{M_2}\Theta^{(1)}_{V_2}$. So $B^{(1)}_{V_1}(x,y,z) = B^{(1)}_{V_2}(\Psi(x,y,z)$, by (3.3).

q.e.d.

Theorem 5.10. ***(Theorem B)***

Proof of Theorem B. By the Fornaess Lemma (See Lemma 5.12 below), there exists a dense set in the boundary of M_i such that the Bergman kernel blows up at the points in this dense set. It follows that the Bergman function B_{V_i} is equal to 1 in a dense subset of ∂V_i. Recall that B_{V_i} is zero at the origin and $0 < B_V < 1$ on $V \setminus \{(0,0,0)\}$. In view of Theorem 5.9, $B_{V_1}(x,y,z) = B_{V_2}(\Psi(x,y,z))$, we see immediately that Ψ preserves the level sets of Bergman functions and hence sends a dense subset of ∂V_1 to a dense subset of ∂V_2. By continuity, Ψ sends ∂V_1 to ∂V_2. q.e.d.

Lemma 5.11. *(Henkin [Hen], Ramirez [Ra]) Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^2$. Let p be a point in the boundary of D. Then there exists an L^2 holomorphic function on D which blows up only at p.*

Proof. There exists, [Hen] [Ra], a holomorphic function f defined on a neighborhood of $\overline{D}$ such that $f(p) = 0, \Re f \leq 0$ on $\overline{D}$ and moreover $|f(q) - f(p)| \geq |q-p|^2$ on $\overline{D}$. We can then set $F(z) = \frac{1}{f^{\frac{3}{4}}}$. Then

$$\int_D |F|^2 \leq \int_D \frac{1}{|f|^{\frac{3}{2}}} \leq \int_D \frac{1}{|z-p|^3} < \infty.$$

q.e.d.

Remark. To find a function f as in the proof of the Lemma all we need is that p is a strictly pseudoconvex boundary point and that $\overline{D}$ has a Stein neighborhood basis.

Lemma 5.12. *(Fornaess) Let D be a bounded complete Reinhardt pseudoconvex domain with real analytic boundary in $\mathbb{C}^2$. Let $E = \{p \in \partial D;\ \exists g \in H^2(D)$ which blows up only at $p\}$. Then E is dense in the boundary of D, and the Bergman kernel of D blows up at points in E.*

Proof. We note that since ∂D has a real analytic boundary it follows that strictly pseudoconvex boundary points are dense. Moreover $\overline{D}$ has a Stein neighborhood basis [Di-Fo]. Therefore the lemma follows from the remark and the previous Lemma. q.e.d.

From Theorem 5.10 and Corollary 5.4, we can get the following two corollaries easily.

Corollary 5.13. *The moduli space of bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domains in A_n-variety $\widetilde{V_n} = \{(x,y,z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is given by the image of the map $\Phi : \{V : V$ a bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domain in $\widetilde{V_n}\} \to \mathbb{R}^\infty$, where the component function of Φ are the invariant functions*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}\omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1,\ \alpha \geqslant \frac{n}{n+1}\beta,\ 0 \leqslant p,\, q \leqslant \left[\frac{n+1}{n}\alpha\right],\ p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right],\ \alpha_i \geqslant 1,\ \alpha_1 \neq \alpha_2,\ i = 1,2.$$

Corollary 5.14. *Let $\mathcal{W}_P = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt pseudoconvex C^ω-smooth domain in A_n-variety$\}$ and $\mathcal{W}_{SP} = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt strictly pseudoconvex domain in A_n-variety$\}$. Then the moduli space of $\mathcal{W}_{\mathcal{P}}$ (respectively $\mathcal{W}_{SP}$) is given by the image of the map $\widetilde{\Phi_P} : \mathcal{W}_P \to \mathbb{R}^\infty$ (respectively $\widetilde{\Phi_{SP}} : \mathcal{W}_{SP} \to \mathbb{R}^\infty$), where the component functions of $\widetilde{\Phi_P}$ (respectively $\widetilde{\Phi_{SP}}$) are the invariant functions*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1,\ \alpha \geqslant \frac{n}{n+1}\beta,\ 0 \leqslant p,\, q \leqslant \left[\frac{n+1}{n}\alpha\right],\ p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right],\ \alpha_i \geqslant 1,\ \alpha_1 \neq \alpha_2,\ i = 1,2.$$

In particular, the moduli space of $\mathcal{W}_P$ (respectively $\mathcal{W}_{SP}$) is as same as the moduli space of bounded complete Reinhardt pseudoconvex C^ω-smooth domains (respectively bounded complete Reinhardt strictly pseudoconvex domains) in A_n-variety $\widetilde{V_n} = \{(x,y,z) \in \mathbb{C}^3 : xy = z^{n+1}\}$.

6. Explicit computation of new invariant

Let a, b, c be positive real number and d be an integer greater than or equal to 1. We shall follow the notations in our previous section. Let $V^{(d)}_{(a,b,c)} = \{(x,y,z) \in \mathbb{C}^3 \colon xy = z^2,\ a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}$.

Recall that $(x,y,z) = (u_0, u_0{v_0}^2, u_0v_0)$. Then $M^{(d)}_{(a,b,c)}$ be the resolution of $V^{(d)}_{(a,b,c)}$ with local coordinate chart $W_0 = \{(u_0,v_0)\colon a|u_0|^{2d} + b|u_0|^{2d}|v_0|^{4d} + c|u_0|^{2d}|v_0|^{2d} < \varepsilon_0\}$. Next write $u_0 = re^{i\theta}$, and $v_0 = \rho e^{i\phi}$.

In the following paragraphs, we denote

$$V_1 = V^{(d)}_{(a_1,b_1,c_1)},\ V_2 = V^{(d)}_{(a_2,b_2,c_2)},\ M_1 = M^{(d)}_{(a_1,b_1,c_1)},\ M_2 = M^{(d)}_{(a_2,b_2,c_2)}.$$

First let us consider the case $d = 1$ and fix ε_0.

$$\|\phi_{\alpha\beta}\|^2_{M_{(a,b,c)}} = \int_{M_{(a,b,c)}} \phi_{\alpha\beta} \wedge \overline{\phi_{\alpha\beta}}$$

$$= \int_{W_0} |{u_0}^{\alpha}|^2 |{v_0}^{\beta}|^2 \, du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0} = 16\pi^2 \iint\limits_D r^{2\alpha+1} \rho^{2\beta+1} \, dr \, d\rho$$

where $D = \big\{(r,\rho)\colon r \geqslant 0, \rho \geqslant 0, ar^2 + br^2\rho^4 + cr^2\rho^2 < \varepsilon_0\big\}$. Then

$$\|\phi_{00}\|^2 = 16\pi^2 \int_0^\infty \int_0^{\frac{\sqrt{\varepsilon_0}}{\sqrt{a+c\rho^2+b\rho^4}}} r\rho \, dr \, d\rho$$

$$= 8\pi^2 \int_0^\infty \frac{\varepsilon_o \rho}{a + c\rho^2 + b\rho^4} \, d\rho = 4\varepsilon_0 \pi^2 \int_0^\infty \frac{1}{a + c\rho + b\rho^2} \, d\rho,$$

$$\|\phi_{1\beta}\|^2 = 16\pi^2 \int_0^\infty \int^{\frac{\sqrt{\varepsilon_0}}{\sqrt{a+c\rho^2+b\rho^4}}} r^3 \rho^{2\beta+1} \, dr \, d\rho$$

$$= 4\varepsilon_0^2 \pi^2 \int_0^\infty \frac{\rho^{2\beta+1}}{(a + c\rho^2 + b\rho^4)^2} \, d\rho = 2\varepsilon_0^2 \pi^2 \int_0^\infty \frac{\rho^\beta}{(a + c\rho + b\rho^2)^2} \, d\rho.$$

By calculation, we can get the following results:

If $c^2 - 4ab < 0$,

(6.1)
$$\|\phi_{00}\|^2 = 4\varepsilon_0 \pi^2 \left(\frac{\pi}{\sqrt{4ab-c^2}} - \frac{2}{\sqrt{4ab-c^2}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right)$$

(6.2)
$$\|\phi_{10}\|^2 = 2\varepsilon_0 \pi^2 \left(-\frac{c}{(4ab-c^2)a} + \frac{2\pi b}{(4ab-c^2)^{\frac{3}{2}}} - \frac{4b}{(4ab-c^2)^{\frac{3}{2}}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right)$$

(6.3)
$$\|\phi_{11}\|^2 = 2\varepsilon_0 \pi^2 \left(\frac{2}{4ab-c^2} - \frac{\pi c}{(4ab-c^2)^{\frac{3}{2}}} + \frac{2c}{(4ab-c^2)^{\frac{3}{2}}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right)$$

(6.4)
$$\|\phi_{12}\|^2 = 2\varepsilon_0 \pi^2 \left(-\frac{c}{(4ab-c^2)b} + \frac{2\pi a}{(4ab-c^2)^{\frac{3}{2}}} - \frac{4a}{(4ab-c^2)^{\frac{3}{2}}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right)$$

If $c^2 - 4ab > 0$,

(6.5)
$$\|\phi_{00}\|^2 = 4\varepsilon_0 \pi^2 \frac{1}{\sqrt{c^2 - 4ab}} \cdot \ln \frac{c + \sqrt{c^2 - 4ab}}{c - \sqrt{c^2 - 4ab}}$$

(6.6)
$$\|\phi_{10}\|^2 = 2\varepsilon_0^2 \pi^2 \frac{c\sqrt{c^2 - 4ab} - 2ab \ln \frac{c+\sqrt{c^2-4ab}}{c-\sqrt{c^2-4ab}}}{(c^2 - 4ab)^{\frac{3}{2}} \cdot a}$$

$$\|\phi_{11}\|^2 = 2\varepsilon_0^2\pi^2 \frac{-2\sqrt{c^2-4ab} + c\ln\frac{c+\sqrt{c^2-4ab}}{c-\sqrt{c^2-4ab}}}{(c^2-4ab)^{\frac{3}{2}}} \tag{6.7}$$

$$\|\phi_{12}\|^2 = 2\varepsilon_0^2\pi^2 \frac{c\sqrt{c^2-4ab} - 2ab\ln\frac{c+\sqrt{c^2-4ab}}{c-\sqrt{c^2-4ab}}}{(c^2-4ab)^{\frac{3}{2}}b} \tag{6.8}$$

If $c^2 - 4ab = 0$,

$$\|\phi_{00}\|^2 = 4\varepsilon_0\pi^2 \cdot \frac{2}{c} \tag{6.9}$$

$$\|\phi_{10}\|^2 = 2\varepsilon_0^2\pi^2 \cdot \frac{8b}{3c^3} \tag{6.10}$$

$$\|\phi_{11}\|^2 = 2\varepsilon_0^2\pi^2 \cdot \frac{2}{3c^2} \tag{6.11}$$

$$\|\phi_{12}\|^2 = 2\varepsilon_0^2\pi^2 \cdot \frac{2}{3bc} \tag{6.12}$$

Remark: $\|\phi_{12}\|^2 = \frac{a}{b}\|\phi_{10}\|^2$ for all the three cases above.

Lemma 6.1. *If $x > 0$, then*

$$\arctan x > \frac{-x + x\sqrt{9+8x^2}}{2(1+x^2)}.$$

Proof. Let

$$f(x) = \arctan x - \frac{-x + x\sqrt{9+8x^2}}{2(1+x^2)}.$$

Then

$$f'(x) = \frac{(x^2+3)\sqrt{9+8x^2} - 7x^2 - 9}{2(1+x^2)^2\sqrt{9+8x}}$$

$$(x^2+3)\sqrt{9+8x^2} > 7x^2 + 9 \Longleftrightarrow x^6 + x^4 > 0.$$

So $f'(x) > 0$, $f(x) > f(0) = 0$, i.e., $\arctan x > \frac{-x+x\sqrt{9+8x^2}}{2(1+x^2)}$.

q.e.d.

Lemma 6.2. *If $0 < x < 1$,*

$$\ln\frac{1+x}{1-x} > \frac{-x + x\sqrt{9-8x^2}}{1-x^2}.$$

Proof. Let

$$f(x) = \ln\frac{1+x}{1-x} - \frac{-x + x\sqrt{9-8x^2}}{1-x^2}.$$

Then

$$f'(x) = \frac{(3-x^2)\sqrt{9-8x^2} - (9-7x^2)}{\sqrt{9-8x^2}(1-x^2)^2}.$$

$$(3-x^2)\sqrt{9-8x^2} > 9 - 7x^2 \Longleftrightarrow x^4 + x^6 > 0.$$

So $f'(x)>0$, $f(x)>f(0)=0$, i.e., $\ln\frac{1+x}{1-x} > \frac{-x+x\sqrt{9-8x^2}}{1-x^2}$.

q.e.d.

Proposition 6.3. *If $c^2-4ab<0$, let $\frac{\sqrt{4ab-c^2}}{c}=x$, then*

$$f(x)=\frac{\|\phi_{11}\|^2}{\|\phi_{10}\|\|\phi_{12}\|}=\frac{x-\arctan x}{\sqrt{1+x^2}\left(-\frac{x}{1+x^2}+\arctan x\right)}$$

which is a strict increasing function in terms of x. In particular $\frac{1}{2}<\nu_V^{(1,0)}<\frac{2}{\pi}$.

Proof.

$$\frac{\|\phi_{11}\|^2}{\|\phi_{10}\|\|\phi_{12}\|}$$

$$=\frac{\frac{2}{4ab-c^2}-\frac{\pi c}{(4ab-c^2)^{\frac{3}{2}}}+\frac{2c}{(4ab-c^2)^{\frac{3}{2}}}\arctan\frac{c}{\sqrt{4ab-c^2}}}{\sqrt{\frac{a}{b}}\left(-\frac{c}{(4ab-c^2)a}+\frac{2\pi\cdot b}{(4ab-c^2)^{\frac{3}{2}}}-\frac{4b}{(4ab-c^2)^{\frac{3}{2}}}\arctan\frac{c}{\sqrt{4ab-c^2}}\right)}.$$

Notice that $\frac{ab}{c^2}=\frac{x^2+1}{4}$, so by calculation,

$$f(x)=\frac{x-\arctan x}{\sqrt{1+x^2}\left(-\frac{x}{1+x^2}+\arctan x\right)}$$

and

$$f'(x)=\frac{x\left(-2x^2+x\arctan x+(x^2+1)\arctan^2 x\right)}{\sqrt{1-x^2}\,(x-(1+x^2)\cdot\arctan x)^2}.$$

By Lemma 6.1,

$$\begin{aligned}&-2x^2+x\arctan x+(x^2+1)\arctan^2 x>\\&\qquad -2x^2+\frac{-x^2+x^2\sqrt{9+8x^2}}{2(1+x^2)}+\frac{(-x+x\sqrt{9+8x^2})^2}{4(1+x^2)}=0\end{aligned}$$

So $f'(x)>0$.

In particular,

$$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{x-\arctan x}{\sqrt{1+x^2}\left(-\frac{x}{1+x^2}+\arctan x\right)}=\frac{1}{2}.$$

So $\nu^{(1,0)}>\frac{1}{2}$. Observe that $\lim\limits_{x\to\infty}f(x)=\dfrac{2}{\pi}$, so $\nu^{(1,0)}<\frac{2}{\pi}$.

q.e.d.

Proposition 6.4. *If $c^2-4ab>0$, let $\frac{\sqrt{c^2-4ab}}{c}=x$, then*

$$f(x)=\frac{\|\phi_{11}\|^2}{\|\phi_{10}\|\|\phi_{12}\|}=\frac{-2x+\ln\frac{1+x}{1-x}}{\sqrt{1-x^2}\left(\frac{2x}{1-x^2}-\ln\frac{1+x}{1-x}\right)},$$

which is a strictly decreasing function in term of x. In particular, $0<\nu^{(1,0)}<\frac{1}{2}$.

Proof.

$$\frac{\|\phi_{11}\|^2}{\|\phi_{10}\|\|\phi_{12}\|}=\frac{\left(-2\sqrt{c^2-4ab}+c\ln\frac{c+\sqrt{c^2-4ab}}{c-\sqrt{c^2-4ab}}\right)\Big/\left(c^2-4ab\right)^{\frac{3}{2}}}{\sqrt{\frac{a}{b}}\left(c\sqrt{c^2-4ab}-2ab\ln\frac{c+\sqrt{c^2-4ab}}{c-\sqrt{c^2-4ab}}\right)\Big/a\left(c^2-4ab\right)^{\frac{3}{2}}}.$$

Notice that $\frac{ab}{c^2}=\frac{1-x^2}{4}$, so by calculation,

$$f(x)=\frac{-2x+\ln\frac{1+x}{1-x}}{\sqrt{1-x^2}\left(\frac{2x}{1-x^2}-\ln\frac{1+x}{1-x}\right)}$$

and

$$f'(x)=\frac{x\left(8x^2-2x\ln\frac{1+x}{1-x}+(x^2-1)\ln^2\frac{1+x}{1-x}\right)}{\sqrt{1-x^2}\left(2x+(x^2-1)\ln\frac{1+x}{1-x}\right)^2}.$$

By Lemma 6.2,

$$\begin{aligned}(1-x^2)\ln^2\frac{1+x}{1-x}+2x\ln\frac{1+x}{1-x}-8x^2 &>\\ \frac{(-x+x\sqrt{9-8x^2})^2}{1-x^2}+2x\frac{(-x+x\sqrt{9-8x^2})}{1-x^2}-8x^2&=0.\end{aligned}$$

So $f'(x)<0$.

In particular,

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{-x+\ln\frac{1+x}{1-x}}{\frac{2x}{\sqrt{1-x^2}}-\sqrt{1-x^2}\ln\frac{1+x}{1-x}}=\frac{1}{2}.$$

So $\nu^{(1,0)}<\frac{1}{2}$. Observe that $\lim_{x\to 1^-}f(x)=0$. So $\nu^{(1,0)}>0$. q.e.d.

Proposition 6.5. *Let*

$$V_i=\left\{(x,y,z)\in\mathbb{C}^3\colon xy=z^2, a_i|x|^2+b_i|y|^2+c_i|z|^2<\varepsilon_0\right\},$$

$i=1,2$. *Then V_1 is biholomorphic to V_2 if and only if $\frac{a_1b_1}{c_1^2}=\frac{a_2b_2}{c_2^2}$.*

Proof. ($\Rightarrow$) Since $\nu^{(1,0)}$ is a biholomorphic invariant, we can get the result from the Proposition 6.3 and 6.4.

($\Leftarrow$) Let $\Psi=\left(\sqrt{\frac{b_2}{b_1}}\cdot\frac{c_1}{c_2}x,\sqrt{\frac{a_2}{a_1}}\cdot\frac{c_1}{c_2}y,\sqrt{\frac{c_1}{c_2}}z\right)$.

Then

$$\begin{aligned}a_2|\psi_1|^2+b_2|\psi_2|^2+c_2|\psi_3|^2&=a_2\cdot\frac{b_2{c_1}^2}{b_1{c_2}^2}|x|^2+b_2\frac{a_2{c_1}^2}{a_1{c_2}^2}|y|^2+c_2\cdot\frac{c_1}{c_2}|z|^2\\&=a_1|x|^2+b_1|y|^2+c_1|z|^2,\end{aligned}$$

i.e. Ψ maps the boundary of V_1 to the boundary of V_2. So ψ is a biholomorphic map from V_1 to V_2. q.e.d.

Theorem 6.6. *Let*

$$V^{(1)}_{(a,b,c)} = \left\{(x,y,z)\colon xy = z^2, a|x|^2 + b|y|^2 + c|z|^2 < \varepsilon_0\right\}.$$

Let $\sim$ denote the biholomorphic equivalence. Then the map

$$\varphi\colon \left\{V^{(1)}_{(a,b,c)}\right\} \to \mathbb{R}_+ \,, V^{(1)}_{(a,b,c)} \mapsto \nu^{(1,0)}$$

is injective up to a biholomorphism. More precisely the induced map $\widetilde{\varphi}\colon \left\{V^{(1)}_{(a,b,c)}\right\}/\!\sim\ \to \mathbb{R}_+$ is one-to-one map from $\left\{V^{(1)}_{(a,b,c)}\right\}/\!\sim$ onto $\left(0, \frac{2}{\pi}\right)$.

Proof. The theorem follows from Proposition 6.3, 6.4 and 6.5 directly.
q.e.d.

In view of Proposition 6.5, Theorem 4.1 and Theorem 4.3, we can simplify the forms of biholomorphic map from V_1 to V_2.

Theorem 6.7. *Let*

$$V_i = \left\{(x,y,z) \in \mathbb{C}^3\colon xy = z^2, a_i|x|^2 + b_i|y|^2 + c_i|z|^2 < \varepsilon_0\right\},$$

$i = 1, 2$.

If $\nu^{(1,0)}_{V_1} \neq \frac{1}{2}$ or $\nu^{(1,0)}_{V_2} \neq \frac{1}{2}$, then the biholomorphism Ψ from V_1 to V_2 must be one of the following forms:
form (1):

$$\Psi = \left(e^{i\theta_1}\sqrt{\frac{b_2}{b_1}\cdot\frac{c_1}{c_2}}x,\ e^{i\theta_2}\sqrt{\frac{a_2}{a_1}\cdot\frac{c_1}{c_2}}y,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\sqrt{\frac{c_1}{c_2}}z\right),$$

form (2):

$$\Psi = \left(e^{i\theta_1}\sqrt{\frac{b_1}{a_2}}y,\ e^{i\theta_2}\sqrt{\frac{a_1}{b_2}}x,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\sqrt{\frac{c_1}{c_2}}z\right).$$

If $\nu^{(1,0)}_{V_1} = \nu^{(1,0)}_{V_2} = \frac{1}{2}$, then the biholomorphism Ψ from V_1 to V_2 must be one of the following forms:
form (1):

$$\Psi = \left(e^{i\theta_1}\sqrt{\frac{b_2}{b_1}\cdot\frac{c_1}{c_2}}x,\ e^{i\theta_2}\sqrt{\frac{b_1}{b_2}}y,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\sqrt{\frac{c_1}{c_2}}z\right),$$

form (2):

$$\Psi = \left(e^{i\theta_1}\frac{2\sqrt{b_1b_2}}{c_2}y,\ e^{i\theta_2}\frac{c_1}{2\sqrt{b_1b_2}}x,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\sqrt{\frac{c_1}{c_2}}z\right),$$

and form (3):

$$\Psi =$$

$$\Psi =$$

$$\begin{pmatrix} \frac{-2c_1b_2\sqrt{b_2}r^2}{\sqrt{b_1}c_2(2b_2r^2+c_2)}e^{i(\theta_{31}-\theta)}, & \frac{2\sqrt{b_1b_2}}{2b_2r^2+c_2}e^{i(\theta_{32}-\theta)}, & \pm\frac{4\sqrt{c_1}b_2r}{\sqrt{c_2}(2b_2r^2+c_2)}e^{i\theta_{13}} \\ -\frac{c_1c_2}{2\sqrt{b_1b_2}(2b_2r^2+c_2)}e^{i(\theta_{31}+\theta)}, & \frac{2\sqrt{b_1b_2}r^2}{2b_2r^2+c_2}e^{i(\theta_{32}+\theta)}, & \pm\frac{-2\sqrt{c_1c_2}r}{2b_2r^2+c_2}e^{i(\theta_{13}+2\theta)} \\ \frac{c_1\sqrt{b_2}r}{\sqrt{b_1}(2b_2r^2+c_2)}e^{i\theta_{31}}, & \frac{2\sqrt{b_1b_2}r}{2b_2r^2+c_2}e^{i\theta_{32}}, & \pm\frac{\sqrt{c_1}\left(2b_2r^2-c_2\right)}{\sqrt{c_2}(2b_2r^2+c_2)}e^{i(\theta_{13}+\theta)} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Proof. Since $\frac{a_1b_1}{c_1^2} = \frac{a_2b_2}{c_2^2}$, $\frac{\sqrt{4a_1b_1-c_1^2}}{c_1^2} = \frac{\sqrt{4a_2b_2-c_2^2}}{c_2^2}$. If $c_1^2 - 4a_1b_1 < 0$, then $c_2^2 - 4a_2b_2 < 0$.

Using (6.1), (6.2), (6.3) and (6.4), we can get

$$\begin{aligned}
\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} &= \frac{\|\phi_{10}\|_{M_2} \cdot \sqrt{\frac{c_2^3}{b_2}}}{\|\phi_{00}\|_{M_2} \cdot \sqrt{c_2}} \cdot \frac{\|\phi_{00}\|_{M_1} \cdot \sqrt{c_1}}{\|\phi_{10}\|_{M_1} \cdot \sqrt{\frac{c_1^3}{b_1}}} \cdot \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} \\
&= \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} \\
\frac{\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} &= \frac{\|\phi_{12}\|_{M_2} \cdot \sqrt{\frac{c_2^3}{a_2}}}{\|\phi_{00}\|_{M_2} \cdot \sqrt{c_2}} \cdot \frac{\|\phi_{00}\|_{M_1} \cdot \sqrt{c_1}}{\|\phi_{12}\|_{M_1} \cdot \sqrt{\frac{c_1^3}{a_1}}} \cdot \sqrt{\frac{a_2}{a_1}} \cdot \frac{c_1}{c_2} \\
&= \sqrt{\frac{a_2}{a_1}} \cdot \frac{c_1}{c_2} \\
\frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} &= \frac{\|\phi_{11}\|_{M_2} \cdot c_2}{\|\phi_{00}\|_{M_2} \cdot \sqrt{c_2}} \cdot \frac{\|\phi_{00}\|_{M_1} \cdot \sqrt{c_1}}{\|\phi_{11}\|_{M_1} \cdot c_1} \cdot \sqrt{\frac{c_1}{c_2}} \\
&= \sqrt{\frac{c_1}{c_2}} \\
\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} &= \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\sqrt{\frac{a_1}{b_1}}\|\phi_{10}\|_{M_1}} \\
&= \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} \cdot \sqrt{\frac{b_1}{a_1}} = \sqrt{\frac{b_2}{a_1}} \cdot \frac{c_1}{c_2} = \sqrt{\frac{b_1}{a_2}} \\
\frac{\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} &= \frac{\sqrt{\frac{a_2}{b_2}}\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} \\
&= \sqrt{\frac{a_2}{b_2}} \cdot \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} = \sqrt{\frac{a_2}{b_1}} \cdot \frac{c_1}{c_2} = \sqrt{\frac{a_1}{b_2}}.
\end{aligned}$$

If $c_1^2 - 4a_1b_1 > 0$, then $c_2^2 - 4a_2b_2 > 0$.

Using (6.5), (6.6), (6.7) and (6.8), we can get the same results.

So form (1):

$$\Psi = \left(e^{i\theta_1}\sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2}x,\ e^{i\theta_2}\sqrt{\frac{a_2}{a_1}} \cdot \frac{c_1}{c_2}y,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\sqrt{\frac{c_1}{c_2}}z \right), \tag{6.13}$$

and form (2):

$$\Psi = \left(e^{i\theta_1}\sqrt{\frac{b_1}{a_2}}y,\ e^{i\theta_2}\sqrt{\frac{a_1}{b_2}}x,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\sqrt{\frac{c_1}{c_2}}z \right). \tag{6.14}$$

If $c_1^2 - 4a_1b_1 = 0$, then $c_2^2 - 4a_2b_2 = 0$. Using (6.9), (6.10), (6.11) and (6.12), we can get the same result. But in this case $\nu_{V_1}^{(1,0)} = \nu_{V_2}^{(1,0)} = \frac{1}{2}$, the form (3) will appear.

Form (3)

$$\Psi = \begin{pmatrix} \alpha r e^{-i\theta}a_{31} & \frac{1}{r}e^{-i\theta}a_{32} & \pm a_{13} \\ \frac{1}{\alpha r}e^{i\theta}a_{31} & re^{i\theta}a_{32} & \pm\frac{1}{\alpha}e^{2i\theta}a_{13} \\ a_{31} & a_{32} & \pm\frac{\alpha r^2+1}{2\alpha r}e^{i\theta}a_{13} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\alpha = \frac{-\|\phi_{10}\|_{M_2}}{\|\phi_{12}\|_{M_2}} = -\frac{2b_2}{c_2}$.

$$\begin{aligned} a_{31} &= \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} \cdot \frac{r\|\phi_{10}\|_{M_2} \cdot \|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2} \cdot (r^2\|\phi_{10}\|_{M_2} + \|\phi_{12}\|_{M_2})} e^{i\theta_{31}} \\ &= \frac{\sqrt{\frac{2}{c_1}}}{\sqrt{\frac{8b_1}{3c_1^3}}} \cdot \frac{r\sqrt{\frac{8b_2}{3c_2^3}} \cdot \sqrt{\frac{2}{3b_2c_2}}}{\sqrt{\frac{2}{c_2}} \cdot \left(r^2\sqrt{\frac{8b_2}{3c_2^3}} + \sqrt{\frac{2}{3b_2c_2}}\right)} e^{i\theta_{31}} \\ &= \frac{c_1\sqrt{b_2}r}{\sqrt{b_1}\left(2b_2r^2 + c_2\right)} e^{i\theta_{31}} \\ &= \frac{2\sqrt{a_1b_1} \cdot \sqrt{b_2}r}{\sqrt{b_1}\left(2b_2r^2 + 2\sqrt{a_2b_2}\right)} e^{i\theta_{31}} \\ &= \frac{\sqrt{a_1} \cdot r}{\sqrt{b_2}r^2 + \sqrt{a_2}} e^{i\theta_{31}} \end{aligned}$$

$$\begin{aligned} a_{32} &= \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} \cdot \frac{r\|\phi_{10}\|_{M_2} \cdot \|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}\left(\|\phi_{12}\|_{M_2} + r^2\|\phi_{00}\|_{M_2}\right)} e^{i\theta_{32}} \\ &= \frac{\sqrt{\frac{2}{c_1}}}{\sqrt{\frac{2}{3b_1c_1}}} \cdot \frac{r\sqrt{\frac{8b_2}{3c_2^3}} \cdot \sqrt{\frac{2}{3b_2c_2}}}{\sqrt{\frac{2}{c_2}}\left(r^2\sqrt{\frac{8b_2}{3c_2^3}} + \sqrt{\frac{2}{3b_2c_2}}\right)} e^{i\theta_{32}} \\ &= \frac{2\sqrt{b_1b_2}r}{2b_2r^2 + c_2} e^{i\theta_{32}} \\ &= \frac{2\sqrt{b_1b_2}r}{2b_2r^2 + 2\sqrt{a_2b_2}} e^{i\theta_{32}} \end{aligned}$$

$$= \frac{\sqrt{b_1}r}{\sqrt{b_2}r^2+\sqrt{a_2}}e^{i\theta_{32}}$$

$$\begin{aligned} a_{13} &= \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} \cdot \frac{2r\|\phi_{10}\|_{M_2}\cdot\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}\left(r^2\|\phi_{10}\|_{M_2}+\|\phi_{12}\|_{M_2}\right)}e^{i\theta_{13}} \\ &= \frac{\sqrt{\frac{2}{c_1}}}{\sqrt{\frac{2}{3c_1^2}}} \cdot \frac{2r\sqrt{\frac{8b_2}{3c_2^3}}\cdot\sqrt{\frac{2}{3c_2^3}}}{\sqrt{\frac{2}{c_2}}\left(r^2\sqrt{\frac{8b_2}{3c_2^3}}+\sqrt{\frac{2}{3b_2c_2}}\right)}e^{i\theta_{13}} \\ &= \frac{4\sqrt{c_1}b_2r}{\sqrt{c_2}\left(2b_2r^2+c_2\right)}e^{i\theta_{13}} \\ &= \frac{4\sqrt{2}\sqrt[4]{a_1b_1}b_2r}{\sqrt{2}\sqrt[4]{a_2b_2}\left(2b_2r^2+2\sqrt{a_2b_2}\right)}e^{i\theta_{13}} \\ &= \frac{2\sqrt[4]{a_1b_1}\sqrt{b_2}r}{\sqrt[4]{a_2b_2}\left(\sqrt{b_2}r^2+\sqrt{a_2}\right)}e^{i\theta_{13}}. \end{aligned}$$

So, form (1):

$$\Psi = \left(e^{i\theta_1}\sqrt{\frac{b_2}{b_1}}\cdot\frac{c_1}{c_2}x,\ e^{i\theta_2}\sqrt{\frac{b_1}{b_2}}y,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\sqrt{\frac{c_1}{c_2}}z\right), \tag{6.15}$$

form (2):

$$\Psi = \left(e^{i\theta_1}\frac{2\sqrt{b_1b_2}}{c_2}y,\ e^{i\theta_2}\frac{c_1}{2\sqrt{b_1b_2}}x,\ \pm e^{i\frac{\theta_1+\theta_2}{2}}\sqrt{\frac{c_1}{c_2}}z\right) \tag{6.16}$$

and form (3):

$$\Psi = \tag{6.17}$$

$$\begin{pmatrix} \frac{-2c_1b_2\sqrt{b_2}r^2}{\sqrt{b_1}c_2(2b_2r^2+c_2)}e^{i(\theta_{31}-\theta)}, & \frac{2\sqrt{b_1b_2}}{2b_2r^2+c_2}e^{i(\theta_{32}-\theta)}, & \pm\frac{4\sqrt{c_1}b_2r}{\sqrt{c_2}(2b_2r^2+c_2)}e^{i\theta_{13}} \\ -\frac{c_1c_2}{2\sqrt{b_1b_2}(2b_2r^2+c_2)}e^{i(\theta_{31}+\theta)}, & \frac{2\sqrt{b_1b_2}r^2}{2b_2r^2+c_2}e^{i(\theta_{32}+\theta)}, & \pm\frac{-2\sqrt{c_1c_2}r}{2b_2r^2+c_2}e^{i(\theta_{13}+2\theta)} \\ \frac{c_1\sqrt{b_2}r}{\sqrt{b_1}(2b_2r^2+c_2)}e^{i\theta_{31}}, & \frac{2\sqrt{b_1b_2}r}{2b_2r^2+c_2}e^{i\theta_{32}}, & \pm\frac{\sqrt{c_1}\left(2b_2r^2-c_2\right)}{\sqrt{c_2}(2b_2r^2+c_2)}e^{i(\theta_{13}+\theta)} \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

q.e.d.

Next, let's consider the case $d \geqslant 2$ for

$$V^{(d)}_{(a,b,c)} = \left\{(x,y,z)\in\mathbb{C}^3\colon xy=z^2,\ a|x|^{2d}+b|y|^{2d}+c|z|^{2d}<\varepsilon_0\right\}.$$

From the Corollary 5.6, we know

$$\nu^{(2d-1,d-1)} = \frac{\|\phi_{2d-1,2d-1}\|^2}{\|\phi_{2d-1,d-1}\|\cdot\|\phi_{2d-1,3d-1}\|}$$

is an invariant.

Recall that

$$\|\phi_{\alpha\beta}\|^2 = 16\pi^2 \iint\limits_D r^{2\alpha+1}\rho^{2\beta+1}\, dr\, d\rho$$

where $D = \{(r,\rho)\colon r \geqslant 0, \rho \geqslant 0, ar^{2d} + br^{2d}\rho^{4d} + cr^{2d}\rho^{2d} < \varepsilon_0\}$. So

$$\begin{aligned}
\|\phi_{2d-1,2d-1}\|^2 &= 16\pi^2 \int_0^\infty \int_0^{\frac{\sqrt[2d]{\varepsilon_0}}{\sqrt[2d]{a+c\rho^{2d}+b\rho^{4d}}}} r^{4d-1}\rho^{4d-1}\, dr\, d\rho \\
&= \frac{4\varepsilon_0^2\pi^2}{d} \int_0^\infty \frac{\rho^{4d-1}}{(a+c\rho^{2d}+b\rho^{4d})^2}\, d\rho \\
&= \frac{2\varepsilon_0^2\pi^2}{d^2} \int_0^\infty \frac{\rho}{(a+c\rho+b\rho^2)^2}\, d\rho \\
\|\phi_{2d-1,d-1}\|^2 &= 16\pi^2 \int_0^\infty \int_0^{\frac{\sqrt[2d]{\varepsilon_0}}{\sqrt[2d]{a+c\rho^{2d}+b\rho^{4d}}}} r^{4d-1}\rho^{2d-1}\, dr\, d\rho \\
&= \frac{4\varepsilon_0^2\pi^2}{d} \int_0^\infty \frac{\rho^{2d-1}}{(a+c\rho^{2d}+b\rho^{4d})^2}\, d\rho \\
&= \frac{2\varepsilon_0^2\pi^2}{d^2} \int_0^\infty \frac{1}{(a+c\rho+b\rho^2)^2}\, d\rho \\
\|\phi_{2d-1,3d-1}\|^2 &= 16\pi^2 \int_0^\infty \int_0^{\frac{\sqrt[2d]{\varepsilon_0}}{\sqrt[2d]{a+c\rho^{2d}+b\rho^{4d}}}} r^{4d-1}\rho^{6d-1}\, dr\, d\rho \\
&= \frac{4\varepsilon_0^2\pi^2}{d} \int_0^\infty \frac{\rho^{6d-1}}{(a+c\rho^{2d}+b\rho^{4d})^2}\, d\rho \\
&= \frac{2\varepsilon_0^2\pi^2}{d^2} \int_0^\infty \frac{\rho^2}{(a+c\rho+b\rho^2)^2}\, d\rho.
\end{aligned}$$

Notice that the value of the invariant

$$\frac{\|\phi_{2d-1,2d-1}\|^2}{\|\phi_{2d-1,d-1}\| \cdot \|\phi_{2d-1,3d-1}\|}$$

is equal to

$$\frac{\|\phi_{11}\|^2}{\|\phi_{10}\| \cdot \|\phi_{12}\|}$$

as the case $d = 1$.

Then we can get the following proposition.

Proposition 6.8. *Let*

$$V_1 = \left\{(x,y,z) \in \mathbb{C}^3\colon xy = z^2, a_1|x|^{2d} + b_1|y|^{2d} + c_1|z|^{2d} < \varepsilon_0\right\},$$

and

$$V_2 = \left\{(x,y,z) \in \mathbb{C}^3\colon xy = z^2, a_2|x|^{2d} + b_2|y|^{2d} + c_2|z|^{2d} < \varepsilon_0\right\}.$$

Then V_1 is biholomorphic to V_2 if and only if $\frac{a_1b_1}{c_1^2} = \frac{a_2b_2}{c_2^2}$.

Proof. ($\Rightarrow$) From the discussion above, we get

$$\frac{\|\phi_{2d-1,2d-1}\|^2_{M_1}}{\|\phi_{2d-1,d-1}\|_{M_1}\cdot\|\phi_{2d-1,3d-1}\|_{M_1}} = \frac{\|\phi_{2d-1,2d-1}\|^2_{M_2}}{\|\phi_{2d-1,d-1}\|_{M_2}\cdot\|\phi_{2d-1,3d-1}\|_{M_2}}.$$

And using the Proposition 6.3 and the Proposition 6.4, we can get

$$\frac{a_1b_1}{c_1^2} = \frac{a_2b_2}{c_2^2}.$$

($\Leftarrow$) Just take a biholomorphic map $\Psi = \left(\sqrt[2d]{\frac{a_1}{a_2}}x,\ \sqrt[2d]{\frac{b_1}{b_2}}y,\ \sqrt[d]{\frac{c_1}{c_2}}z\right)$.

q.e.d.

Combining the Theorem 6.6 and the Proposition 6.8, we can get the following theorem.

Theorem 6.9. *Let d be a fixed positive integer and*

$$V^{(d)}_{(a,b,c)} = \left\{(x,y,z)\in\mathbb{C}^3\colon xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\right\}.$$

Let $\sim$ denote the biholomorphic equivalence. Then the map

$$\varphi\colon \left\{V^{(d)}_{(a,b,c)}\right\} \to \mathbb{R}_+,\ V^{(d)}_{(a,b,c)} \mapsto \nu^{(2d-1,d-1)}$$

is injective up to a biholomorphism. More precisely the induced map $\widetilde{\varphi}\colon \left\{V^{(d)}_{(a,b,c)}\right\}/\sim\ \to \mathbb{R}_+$ is one-to-one map from $\left\{V^{(d)}_{(a,b,c)}\right\}/\sim$ onto $\left(0,\frac{2}{\pi}\right)$. So the moduli space of $\left\{V^{(d)}_{(a,b,c)}\right\}$ is of dimension 1 and independent on d.

Let

$$W^{(d)}_{(a,b,c)} = \left\{(x,y)\in\mathbb{C}^2\colon a|x|^{4d} + b|y|^{4d} + c|xy|^{2d} < \varepsilon_0\right\},$$

then this is a special Reinhardt domain in $\mathbb{C}^2$. Using the Corollary 5.4, we can get the moduli space of $W^{(d)}_{(a,b,c)}$ which coincides with the moduli space of $V^{(d)}_{(a,b,c)}$.

Last, as an application to our theory, we compute explicitly the invariant $\nu^{(3,1)}$ for two domains $V^{(1)}_{(1,1,1)}$ and $V^{(2)}_{(1,1,1)}$ in A_1-variety.

We have known that for the domain $V^{(2)}_{(1,1,1)}$,

$$\nu^{(3,1)} = \frac{\|\phi_{3,3}\|^2}{\|\phi_{3,1}\|\cdot\|\phi_{3,5}\|} = \frac{\int_0^\infty \frac{\rho}{(1+\rho+\rho^2)^2}\,d\rho}{\left(\int_0^\infty \frac{1}{1+\rho+\rho^2)^2}\,d\rho\cdot\int_0^\infty \frac{\rho^2}{(1+\rho+\rho^2)^2}\,d\rho\right)^{\frac{1}{2}}}$$
$$= \frac{2(9-\sqrt{3}\pi)}{4\sqrt{3}\pi - 9}.$$

However, for the domain $V^{(1)}_{(1,1,1)}$,

$$\begin{aligned}\|\phi_{3,3}\|^2 &= 16\pi^2\int_0^\infty\int_0^{\frac{\sqrt{\varepsilon_0}}{\sqrt{1+\rho^2+\rho^4}}} r^7\rho^7\,dr\,d\rho\\ &= 2\varepsilon_0^4\pi^2\int_0^\infty \frac{\rho^7}{(1+\rho^2+\rho^4)^4}\,d\rho = \varepsilon_0^4\pi^2\cdot\left(\frac{1}{3}-\frac{14\pi}{81\sqrt{3}}\right),\\ \|\phi_{3,1}\|^2 &= 16\pi^2\int_0^\infty\int_0^{\frac{\sqrt{\varepsilon_0}}{\sqrt{1+\rho^2+\rho^4}}} r^7\rho^3\,dr\,d\rho\\ &= 2\varepsilon_0^4\pi^2\int_0^\infty \frac{\rho^3}{(1+\rho^2+\rho^4)^4}\,d\rho = \varepsilon_0^4\pi^2\cdot\left(\frac{1}{2}-\frac{20\pi}{81\sqrt{3}}\right),\\ \|\phi_{3,5}\|^2 &= 16\pi^2\int_0^\infty\int_0^{\frac{\sqrt{\varepsilon_0}}{\sqrt{1+\rho^2+\rho^4}}} r^7\rho^{11}\,dr\,d\rho\\ &= 2\varepsilon_0^4\pi^2\int_0^\infty \frac{\rho^{11}}{(1+\rho^2+\rho^4)^4}\,d\rho = \varepsilon_0^4\pi^2\cdot\left(\frac{1}{2}-\frac{20\pi}{81\sqrt{3}}\right).\end{aligned}$$

Then

$$\nu^{(3,1)} = \frac{\|\phi_{3,3}\|^2}{\|\phi_{3,1}\|\cdot\|\phi_{3,5}\|} = \frac{\frac{1}{3}-\frac{14\pi}{81\sqrt{3}}}{\frac{1}{2}-\frac{20\pi}{81\sqrt{3}}} = \frac{162-28\sqrt{3}\pi}{243-40\sqrt{3}\pi}.$$

So the invariant $\nu^{(3,1)}$ is different for the domains $V^{(1)}_{(1,1,1)}$ and $V^{(2)}_{(1,1,1)}$, i.e. $V^{(1)}_{(1,1,1)}$ is not biholomorphic to $V^{(2)}_{(1,1,1)}$. And by the Theorem 5.4, the domain $W^{(1)}_{(1,1,1)}$ in $\mathbb{C}^2$ is not biholomorphic to the domain $W^{(2)}_{(1,1,1)}$ in $\mathbb{C}^2$.

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