HYPERSURFACE WEIGHTED DUAL GRAPHS OF NORMAL SINGULARITIES OF SURFACES

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Introduction. Let p be a normal singularity of the 2-dimensional Stein space V. Let $\pi: M \to V$ be a resolution of V such that the irreducible components A_i, $1 \leqslant i \leqslant n$, of $A = \pi^{-1}(p)$ are nonsingular and have only normal crossings. Associated to A is a weighted dual graph Γ (e.g. see [10] or [14]) which, along with the genera of the A_i, fully describes the topology and differentiable structure of A and the topological and differentiable nature of the embedding of A in M.

One of the important questions in normal two dimensional singularities is "the classification of all weighted dual graphs for hypersurface singularities." It is known that in the weighted dual graphs for hypersurface singularities, the K' cycle (see Definition 0.9) exists. M. Artin has studied the rational singularities [those for which $R^1\pi_*(\mathcal{O}) = 0$]. Double points are hypersurface singularities. He has shown that if p is a rational double point, then the graph associated to p is one of the graphs A_k, $k \geqslant 1$; D_k, $k \geqslant 4$; E_6; E_7; E_8 which arise in the classification of Lie groups. In [26], Wagreich introduces a definition for p to be weakly elliptic. He proved that for double points, $Z \cdot Z \geqslant -2$, where Z is the fundamental cycle. Using this fact, he listed a lot of the possible weighted dual graphs of elliptic double points [26] (34 possible cases). I was kindly informed by Laufer, and Wagreich himself that the list is incomplete. In this work, we will give a complete list (131 cases) of all weighted dual graphs for weakly elliptic double points (cf. Theorem 2.9). Moreover, for each of these weighted dual graphs, a typical defining equation is given. The defining equations have been found by means of an unpublished technique of Laufer. Rational singularities have $H^1(M, \mathcal{O}) = 0$. The hypersurface rational singularities are actually double points. For $H^1(M, \mathcal{O}) = \mathbb{C}$, Laufer was able to list all weighted dual graphs of hypersurface singularities. In this paper, we list all possible weighted dual graphs of hypersurface singularities with $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$. As a consequence of this classification, the following theorem is proved.

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THEOREM. *Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only singular point. Suppose $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$ and p is a hypersurface singularity. Let E be the minimally elliptic cycle. If $H^1(A, \mathbb{Z}) = 0$, then p is an almost minimally elliptic singularity.* (*For definition see* [29].)

An example in [28] shows that the above theorem is sharp. Our main tool is the previous result [28] that $-K' =$ summation of the elliptic sequence, the complete list of minimally elliptic hypersurface singularities by Laufer [18] and Theorem 2.7 of Wagreich [26].

We begin by recalling some theorems and definitions in Section 0. In Section 1, we get a lower estimate on the dimension of Zariski tangent space of general two dimensional normal singularity in terms of the fundamental cycle Z,

$$\dim m/m^2 \geqslant \chi(Z) - Z \cdot Z + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z)),$$

which will give us a necessary condition on hypersurface weighted dual graphs. This kind of estimate is sharp in the sense that equality holds for certain singularities. In case of maximally elliptic singularities, we know that $\dim H^1(M, \mathcal{O}(-Z)) = \dim H^1(M, \mathcal{O}(-2Z))$. In particular, for maximally elliptic singularities, $Z \cdot Z \geqslant -3$. This enables us to list all the possible maximally elliptic hypersurface singularities. However, the list is too long to be included. In Section 2, we give a topological classification of elliptic double points. In Section 3, we list all possible hypersurface weighted dual graphs for those singularities with $H^1(M, \mathcal{O}) = \mathbb{C}^2$.

I gratefully acknolwedge the encouragement and help of Professor Henry B. Laufer during the investigation of these results, especially for showing me his unpublished technique in finding a defining equation for the weighted dual graph of double points. I would also like to thank Professor Kuga, Professor Siu and Professor Wagreich for their encouragement and discussion of mathematics.

0. Preliminaries. Let $\pi: M \to V$ be a resolution of normal two dimensional Stein space V. We assume that p is the only singularity of V. Let $\pi^{-1}(p) = A = \bigcup A_i$, $1 \leqslant i \leqslant n$, be the decomposition of the exceptional set A into irreducible components. Suppose π is the minimal good resolution. Let Γ be the associated weighted dual graph. The vertices of Γ correspond to the A_i. The edge of Γ connecting the vertices corresponding to A_i and A_j, $i \neq j$, corresponds to the points of $A_i \cap A_j$. Finally, associated to each A_i is its genus, g_i, as a Riemann surface, and its weight, $A_i \cdot A_i$, the topological self-intersection number. Γ will denote the graph along with the genera and the weights.

Definition 0.1. $\deg A_i = \Sigma A_j \cdot A_i$, $j \neq i$.

A cycle (or divisorial cycle) D on A is an integral combination of the A_i. $D = \Sigma d_i A_i$, $1 \leqslant i \leqslant n$ with d_i an integer. In this paper, "cycle" will always mean a cycle on A. There is a natural partial ordering, denoted by $<$, between cycles defined by comparing the coefficients. We shall only be considering cycles $D \geqslant 0$. We let $\operatorname{supp} D = |D| = \cup A_i$, $d_i \neq 0$, denote the support of D.

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Let $\mathcal{O}(-D)$ be the sheaf of germs of holomorphic functions on M which vanish to order d_i on A_i. Let $\mathcal{O}_D$ denote $\mathcal{O}/\mathcal{O}(-D)$. We use "dim" to denote dimension over $\mathbb{C}$. Then

$$\chi(D) = \dim H^0(M, \mathcal{O}_D) - \dim H^1(M, \mathcal{O}_D). \tag{0.1}$$

Some authors work instead with the arithmetic genus $P_a(D) = 1 - \chi(D)$. The Riemann-Roch theorem [24, p. 75] says

$$\chi(D) = -\tfrac{1}{2}(D \cdot D + D \cdot K). \tag{0.2}$$

In (0.2), K is the canonical divisor on M. $D \cdot K$ may be defined as follows. Let ω be a meromorphic 2-form on M, i.e., a meromorphic section of K. Let (ω) be the divisor of ω. Then $D \cdot K = D \cdot (\omega)$, and this number is independent of the choice of ω. In fact, let g_i be the geometric genus of A_i, i.e., the genus of the desingularization of A_i. Then [24, p. 75]

$$A_i \cdot K = -A_i \cdot A_i + 2g_i - 2 + 2\delta_i, \tag{0.3}$$

where δ_i is the "number" of nodes and cusps on A_i. Each singular point on A_i other than a node or cusp counts as at least two nodes. Fortunately, such more complicated singularities will not occur in this paper.

The minimal resolution of V is characterized by there being no A_i which is a nonsingular rational curve with $A_i \cdot A_i = -1$ [5, p. 364]. The intersection matrix $(A_i \cdot A_j)$ is negative definite [14], so by (0.3) we see the following.

PROPOSITION 0.2. *π is the minimal resolution of V if and only if $A_i \cdot K \geqslant 0$ for all A_i.*

It follows immediately from (1.2) that if B and C are cycles, then

$$\chi(B + C) = \chi(B) + \chi(C) - B \cdot C. \tag{0.4}$$

Associated to π is a unique fundamental cycle Z [1, pp. 131–132] such that $Z>0$, $A_i \cdot Z \leqslant 0$ (all A_i), and such that Z is minimal with respect to those two properties. Z may be computed from the intersection matrix as follows [15, p. 607] via what is called a computation sequence (in the sense of Laufer) for Z:

$$\begin{aligned} Z_0 &= 0, \qquad Z_1 = A_{i_1}, \\ Z_2 &= Z_1 + A_{i_2}, \ldots, \\ Z_j &= Z_{j-1} + A_{i_j}, \ldots, \\ Z_l &= Z_{l-1} + A_{i_l} = Z, \end{aligned}$$

where A_{i_1} is arbitrary and $A_{i_j} \cdot Z_{j-1} > 0$, $1 < j \leqslant l$.

Since M is two dimensional and not compact,

$$H^2(M, \mathcal{F}) = 0 \tag{0.5}$$

for any coherent analytic sheaf $\mathcal{F}$ on M [25].

Wagreich [26] defined the singularity p to be elliptic if $\chi(D) \geqslant 0$ for all cycles $D>0$ and $\chi(F)=0$ for some cycles $F>0$. He proved that this definition is independent of the resolution. It is easy to see that under this hypothesis, $\chi(Z)=0$.

Definition 0.3. A cycle $E>0$ is *minimally elliptic* if $\chi(E)=0$ and $\chi(D)>0$ for all cycles D such that $0<D<E$.

THEOREM 0.4 (Laufer). *Let Γ be a weighted dual graph including genera for the vertices, associated to a minimal resolution with nonsingular A_i and normal crossings. Suppose that $\chi(Z)=0$. Then, generically (in the sense of Laufer [18]), $H^1(M, \mathcal{O}) = \mathbb{C}$. Consequently $\chi(D) \geqslant 0$ for any cycle $D \geqslant 0$. Let E be the minimally elliptic cycle, $E \leqslant Z$. If there exists $A_i \subseteq |E|$ with $A_i \cdot Z < 0$, then $H^1(M, \mathcal{O}) = \mathbb{C}$ for all p associated to Γ.*

Henceforth, we will adopt the following definition.

Definition 0.5. p is said to be *weakly elliptic* if $\chi(Z)=0$.

Definition 0.6. Let $\pi: M \to V$ be the minimal resolution of V. p is *minimally elliptic* if p is elliptic and every connected proper subvariety of A is the exceptional set for a rational singularity.

The following definitions and theorems can be found in [28] and [29].

LEMMA 0.7. *Let $\pi: M \to V$ be a resolution of normal two dimensional space V with p as its only weakly elliptic singularity. Let $\pi^{-1}(p) = A = \bigcup A_i$,*

$1 \leqslant i \leqslant n$, be the decomposition of the exceptional set A into irreducible components. Let E be the minimally elliptic cycle on A. If supp*E consists of more than one irreducible component, then all A_i, $1 \leqslant i \leqslant n$, are rational curves.* $\operatorname{supp} E = A_1$ *if and only if A_1 is a nonsingular elliptic curve or A_1 is a singular rational curve with a node or cusp singularity. In this case, all A_i, $2 \leqslant i \leqslant n$, are nonsingular rational curves.*

PROPOSITION 0.8. *Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singular point. In the computation sequence for the fundamental cycle Z, we have $A_{i_j} \cdot Z_{j-1} = 1$ for all $1 < j \leqslant l$ except possibly one $1 < k \leqslant l$ such that $A_{i_k} \cdot Z_{k-1} = 2$. In this case, A_{i_k} is in $|E|$.*

Definition 0.9. Let K be the canonical divisor on M. We define the negative cycle $K' = \sum k_i A_i$ on A, where $k_i \in \mathbb{Z}$, the set of integers, to be a cycle such that $A_i \cdot K' = A_i \cdot K$ for all $A_i \subseteq A$. (K' does not always exist.)

Definition 0.10. Let A be the exceptional set of the minimal good resolution $\pi: M \to V$, where V is a normal two dimensional Stein space with p as its only weakly elliptic singularity. If $E \cdot Z < 0$, we say that the *elliptic sequence* is $\{Z\}$ and the *length of elliptic sequence* is equal to one. Suppose $E \cdot Z = 0$. Let B_1 be the maximal connected subvariety of A such that $B_1 \supseteq \operatorname{supp} E$ and $A_i \cdot Z = 0$ for all $A_i \subseteq B_1$. Since A is an exceptional set, $Z \cdot Z < 0$. So B_1 is properly contained in A. Let Z_{B_1} be the fundamental cycle on B_1. Suppose $Z_{B_1} \cdot E = 0$. Let B_2 be the maximal connected subvariety of B_1 such that $B_2 \supseteq |E|$ and $A_i \cdot Z_{B_1} = 0$ for all $A_i \subseteq B_2$. By the same argument as above, B_2 is properly contained in B_1. Continuing this process, we finally obtain B_m with $Z_{B_m} \cdot E < 0$. We call $\{Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_m}\}$ the *elliptic sequence*, and the *length of the elliptic sequence* is $m+1$.

THEOREM 0.11. *Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singularity. Suppose p is not a minimally elliptic singularity. If $E \cdot Z < 0$ and $|E| \neq A$, then K' does not exist. If K' exists, then the elliptic sequence is of the following form:*

$$Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_l}, Z_{B_{l+1}} = Z_E, \qquad l \geqslant 0.$$

Moreover $-K' = \sum_{i=0}^{l} Z_{B_i} + E$.

Definition 0.12. Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singularity.

Suppose K' exists. If $\dim H^1(M, \mathcal{O}) =$ length of the elliptic sequence, then p is called a *maximally elliptic singularity*.

Definition 0.12. Let π, M, V, p be as in Theorem *0.11*. If for all $A_i \not\subseteq |E|$ and $A_i \cap |E| \neq \varnothing$, then $A_i \cdot Z < 0$. We call p an almost minimally elliptic singularity.

PROPOSITION 0.14. *Let p be the maximally elliptic singularity. Let $Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_l}, Z_E = Z_{B_{l+1}}$ be the elliptic sequence. Then for any $0 \leqslant h \leqslant l$, there exists $f \in H^0(M, \mathcal{O}(-\Sigma_{i=0}^{h} Z_{B_i}))$ such that $f \notin H^0(M, \mathcal{O}(-\Sigma_{i=0}^{h+1} Z_{B_i}))$. In fact, the vanishing order of f on A_j is precisely $\Sigma_{i=0}^{h} {}_{B_i}z_j$, where $Z_{B_i} = \Sigma_k {}_{B_i}z_k A_k$ and $A_j \subseteq B_{h+1}$.*

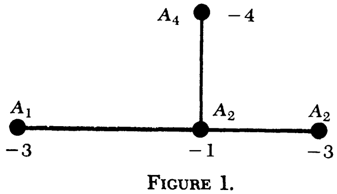
THEOREM 0.15. *Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_l}, Z_E$ be the elliptic sequence. Then the multiplicity of ${}_V\mathcal{O}_p \geqslant -\Sigma_{i=0}^{l} Z_{B_i}^2$. Moreover if $Z_E \cdot Z_E \leqslant -2$, then the equality holds.*

THEOREM 0.16. *Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space V with p as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$, $H^1(|E|, \mathbb{Z}) = 0$, and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0}, Z_{B_1}, \ldots, Z_{B_l}, Z_E = Z_{B_{l+1}}$ be the elliptic sequence. Let D be the subvariety of B_l consisting of those irreducible components $A_i \subseteq B_l$ such that $A_i \cap |E| \neq \varnothing$. If $Z/D = Z_{B_l}/D$, then $l = 0$, i.e., p is an almost minimally elliptic singularity.*

Notation and Terminology.

${}_V\mathcal{O}$ = the sheaf of germs of holomorphic functions on V.
${}_V\mathcal{O}_p$ = the stalk of the sheaf ${}_V\mathcal{O}$ over p.
E = minimally elliptic cycle.
Z = fundamental cycle.
m = maximal ideal of ${}_V\mathcal{O}_p$.
$|D|$ = support of the divisor D.

Convention of weighted dual graphs: vertices without specifying genera are of genus zero. We record the multiplicity z_i of A_i in the fundamental cycle $Z = \Sigma z_i A_i$ by placing that integer in the corresponding position of the vertex. See e.g. Figure 1.



$$Z = \begin{smallmatrix} & 1 & \\ 1 & 3 & 1 \end{smallmatrix} = A_1 + 3A_2 + A_3 + A_4$$

FIGURE 1.

Let $D=\Sigma d_i A_i$ be a positive cycle. Let $B \subseteq |D|$. Then $D|_B = \Sigma f_i A_i$ is a positive cycle, where $f_i = d_i$ if $A_i \subseteq B$ and $f_i = 0$ if $A_i \nsubseteq B$.

1. Lower Estimate of the Dimension of Zariski Tangent Space and Upper Estimate of Multiplicities of Hypersurface Singularities.

THEOREM 1.1. *Let $\pi: M \to V$ be a resolution of normal two dimensional Stein space V with p as its only singular point. Let Z be the fundamental cycle. Then*

$$\dim m/m^2 \geqslant \chi(Z) - Z\cdot Z + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z)).$$

If p is weakly elliptic, then $\dim m/m^2 \geqslant -Z\cdot Z + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z))$. Suppose π is the minimal good resolution and p is a maximally elliptic singularity. Then $\dim m/m^2 \geqslant -Z\cdot Z$. Moreover, if $Z_E \cdot Z_E \leqslant -3$, then $\dim m^n/m^{n+1} = -nZ\cdot Z$ for all $n \geqslant 1$.

Proof. It is true that $H^0(A, \mathcal{O}(-Z)) = \operatorname{dir\,lim} H^0(U, \mathcal{O}(-Z))$, U a neighborhood of A. Since Z is minimal, $m \cong H^0(A, \mathcal{O}(-Z))$. Since $m^2 \subseteq H^0(A, \mathcal{O}(-2Z))$, we have $\dim m/m^2 \geqslant \dim H^0(A, \mathcal{O}(-Z))/H^0(A, \mathcal{O}(-2Z))$. The cohomology exact sequence

$$\begin{aligned} 0 &\to H^0(A, \mathcal{O}(-2Z)) \to H^0(A, \mathcal{O}(-Z)) \\ &\to H^0(A, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \to H^1(A, \mathcal{O}(-2Z)) \\ &\to H^1(A, \mathcal{O}(-Z)) \to H^1(A, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \to 0 \end{aligned}$$

says that

$$\begin{aligned} &\dim H^0(A, \mathcal{O}(-Z))/H^0(A, \mathcal{O}(-2Z)) \\ &\quad = \dim H^0(A, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) - \dim H^1(A, \mathcal{O}(-2Z)) \\ &\qquad + \dim H^1(A, \mathcal{O}(-Z)) - \dim H^1(A, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \\ &\quad = \dim H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) - \dim H^1(M, \mathcal{O}(-2Z)) \\ &\qquad + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \end{aligned}$$

by Lemma 3.1 of [15].

Look at the following cohomology exact sequence:

$$\begin{aligned} 0 &\to H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \to H^0(M, \mathcal{O}_{2Z}) \\ &\to H^0(M, \mathcal{O}_Z) \to H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \\ &\to H^1(M, \mathcal{O}_{2Z}) \to H^1(M, \mathcal{O}_Z) \to 0. \end{aligned}$$

Since $H^0(M,\mathcal{O}_Z)\cong\mathbb{C}$, and $H^0(M,\mathcal{O}_{2Z})\to H^0(M,\mathcal{O}_Z)$ is not a zero map, we have two short exact sequences

$$0\to H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-2Z))\to H^0(M,\mathcal{O}_{2Z})\to H^0(M,\mathcal{O}_Z)\to 0,$$
$$0\to H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-2Z))\to H^1(M,\mathcal{O}_{2Z})\to H^1(M,\mathcal{O}_Z)\to 0.$$

Hence,

$$\begin{aligned}
\dim m/m^2 & \\
&\geqslant \dim H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-2Z))-\dim H^1(M,\mathcal{O}(-2Z))\\
&\quad+\dim H^1(M,\mathcal{O}(-Z))-\dim H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-2Z))\\
&=\dim H^0(M,\mathcal{O}_{2Z})-\dim H^0(M,\mathcal{O}_Z)-\dim H^1(M,\mathcal{O}(-2Z))\\
&\quad+\dim H^1(M,\mathcal{O}(-Z))-\dim H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-2Z))\\
&=\chi(2Z)-\chi(Z)+\dim H^1(M,\mathcal{O}_{2Z})-\dim H^1(M,\mathcal{O}_Z)\\
&\quad-\dim H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-2Z))+\dim H^1(M,\mathcal{O}(-Z))\\
&\quad-\dim H^1(M,\mathcal{O}(-2Z))\\
&=\chi(2Z)-\chi(Z)+\dim H^1(M,\mathcal{O}(-Z))-\dim H^1(M,\mathcal{O}(-2Z))\\
&=\chi(Z)-Z\cdot Z+\dim H^1(M,\mathcal{O}(-Z))-\dim H^1(M,\mathcal{O}(-2Z)).
\end{aligned}$$

If p is weakly elliptic, then $\chi(Z)=0$. So $\dim m\ /m^2\geqslant -Z\cdot Z+\dim H^1(M,\mathcal{O}(-Z))-\dim H^1(M,\mathcal{O}(-2Z))$.

Suppose π is the minimal good resolution and p is a maximally elliptic singular point. We claim that $H^1(M,\mathcal{O}(-nZ))\cong\mathbb{C}^{l+1}$, where $l+2$ is the length of elliptic sequence $Z_{B_0}, Z_{B_1},\ldots, Z_{B_l}, Z_E=Z_{B_{l+1}}$. Choose a computation sequence for Z of the following form: $Z_0=0,\ldots, Z_k=E,\ldots, Z_{r_0}=Z_E,\ldots, Z_{r_1}=Z_{B_l},\ldots, Z_{r_l}=Z_{B_1},\ldots, Z_{r_{l+1}}=Z_{B_0}=Z$. Consider the following sheaf of exact sequences:

$$\begin{aligned}
&0\to\mathcal{O}(-nZ-Z_1)/\mathcal{O}(-nZ-B-Z_E)\to\mathcal{O}(-nZ)/\mathcal{O}(-nZ-B-Z_E)\\
&\quad\to\mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z_1)\to 0,\\
&0\to\mathcal{O}(-nZ-Z_k)/\mathcal{O}(-nZ-B-Z_E)\to\mathcal{O}(-nZ-Z_{k-1})/\mathcal{O}(-nZ-B-Z_E)\\
&\quad\to\mathcal{O}(-nZ-Z_{k-1})/\mathcal{O}(-nZ-Z_k)\to 0,\\
&0\to\mathcal{O}(-nZ-Z_{B_1})/\mathcal{O}(-nZ-B-Z_E)\to\mathcal{O}(-nZ-Z_{r_l-1})/\mathcal{O}(-nZ-B-Z_E)\\
&\quad\to\mathcal{O}(-nZ-Z_{r_l-1})/\mathcal{O}(-nZ-Z_{B_1})\to 0,\\
&0\to\mathcal{O}(-nZ-G_h-Z_1)/\mathcal{O}(-nZ-B-Z_E)\to\mathcal{O}(-nZ-G_h)/\mathcal{O}(-nZ-B-Z_E)\\
&\quad\to\mathcal{O}(-nZ-G_h)/\mathcal{O}(-nZ-G_h-Z_1)\to 0,\\
&0\to\mathcal{O}(-nZ-G_h-Z_k)/\mathcal{O}(-nZ-B-Z_E)\to\mathcal{O}(-nZ-G_h-Z_{k-1})/\mathcal{O}(-nZ-B-Z_E)
\end{aligned}$$

$$\to \mathcal{O}(-nZ-G_h-Z_{k-1})/\mathcal{O}(-nZ-G_h-Z_k)\to 0,$$

$$0\to \mathcal{O}(-nZ-G_{h+1})/\mathcal{O}(-nZ-B-Z_E)\to \mathcal{O}(-nZ-G_h-Z_{r_{l-h}-1})/\mathcal{O}(-nZ-B-Z_E)$$
$$\to \mathcal{O}(-nZ-G_h-Z_{r_{l-h}-1})/\mathcal{O}(-nZ-G_{h+1})\to 0,$$

$$0\to \mathcal{O}(-nZ-B-Z_1)/\mathcal{O}(-nZ-B-Z_E)\to \mathcal{O}(-nZ-B)/\mathcal{O}(-nZ-B-Z_E)$$
$$\to \mathcal{O}(-nZ-B)/\mathcal{O}(-nZ-B-Z_1)\to 0,$$

$$0\to \mathcal{O}(-nZ-B-Z_k)/\mathcal{O}(-nZ-B-Z_E)\to \mathcal{O}(-nZ-B-Z_{k-1})/\mathcal{O}(-nZ-B-Z_E)$$
$$\to \mathcal{O}(-nZ-B-Z_{k-1})/\mathcal{O}(-nZ-B-Z_k)\to 0,$$

$$0\to \mathcal{O}(-nZ-B-Z_{r_0-1})/\mathcal{O}(-nZ-B-Z_E)\to \mathcal{O}(-nZ-B-Z_{r_0-2})/\mathcal{O}(-nZ-B-Z_E)$$
$$\to \mathcal{O}(-nZ-B-Z_{r_0-2})/\mathcal{O}(-nZ-B-Z_{r_0-1})\to 0,$$

where

$$B=\sum_{i=1}^{l} Z_{B_i},\qquad G_h=\sum_{i=1}^{h} Z_{B_i}\quad\text{and}\quad \sum_{i=1}^{0} Z_{B_i}=0.$$

We claim that $H^0(M,\mathcal{O}(-nZ-G_h-Z_{j-1})/\mathcal{O}(-nZ-B-Z_E))\to H^0(M,\mathcal{O}(-nZ-G_h-Z_{j-1})/\mathcal{O}(-nZ-G_h-Z_j))$ is surjective for all $-1\leqslant h\leqslant l-1$ and $0\leqslant j\leqslant r_{h+1}$. The Chern class of the line bundle associated to $\mathcal{O}(-nZ-G_h-Z_{j-1})/\mathcal{O}(-nZ-G_h-Z_j)$ is $-A_{i_j}\cdot(nZ+G_h+Z_{j-1})=-A_{i_j}\cdot Z_{j-1}$, which is <0 for $j>1$ and 0 for $j=1$. For $j>1$, the claim is trivially true because $H^0(M,\mathcal{O}(-nZ-G_h-Z_{j-1})/\mathcal{O}(-nZ-G_h-Z_j))=0$. For $j=1$, by Proposition 0.14, we know that there exists $f\in H^0(M,\mathcal{O}(-nZ-G_h))$ such that the image of f in $H^0(M,\mathcal{O}(-nZ-G_h)/\mathcal{O}(-nZ-G_h-Z_1))$ is nonzero. Therefore, $H^0(M,\mathcal{O}(-nZ-G_h)/\mathcal{O}(-nZ-G_h-Z_1))\cong\mathbb{C}$, and $H^0(M,\mathcal{O}(-nZ-G_h))\to H^0(M,\mathcal{O}(-nZ-G_h)/\mathcal{O}(-nZ-G_h-Z_1))$ is surjective. Now the usual cohomology exact sequence argument will show that $H^1(M,\mathcal{O}(-nZ)/\mathcal{O}(-nZ-B-Z_E))\cong\mathbb{C}^{l+1}$. By Proposition 2.7 of [28], $H^1(M,\mathcal{O}(-nZ-B-Z_E))=0$. So the exact sequence

$$H^1(M,\mathcal{O}(-nZ-B-Z_E))\to H^1(M,\mathcal{O}(-nZ))$$
$$\to H^1(M,\mathcal{O}(-nZ)/\mathcal{O}(-nZ-B-Z_E))\to 0$$

shows that $H^1(M,\mathcal{O}(-nZ))\cong\mathbb{C}^{l+1}$.

$$\begin{aligned}\dim m/m^2 &\geqslant -Z\cdot Z+\dim H^1(M,\mathcal{O}(-Z))-\dim H^1(M,\mathcal{O}(-2Z))\\ &= -Z\cdot Z.\end{aligned}$$

If $Z_E\cdot Z_E\leqslant -3$, then $-Z\cdot Z\geqslant 3$. In this case, all the inequalities above are actually equalities. In particular, $m^2=H^0(A,\mathcal{O}(-2Z))$. By Theorem 3.15 of

[28], we have $m^n = H^0(A, \mathcal{O}(-nZ))$, $n \geqslant 1$. Hence

$$\begin{aligned}\dim m^n/m^{n+1} &= \dim H^0(A,\mathcal{O}(-nZ))/H^0(A,\mathcal{O}(-(n+1)Z)) \\ &= \dim H^0(A,\mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z)) \\ &\quad -\dim H^1(A,\mathcal{O}(-nZ-Z)) + \dim H^1(A,\mathcal{O}(-nZ)) \\ &\quad -\dim H^1(A,\mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z)) \\ &= \dim H^0(M,\mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z)) \\ &\quad -\dim H^1(M,\mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z)) - (l+1) + (l+1) \\ &= \dim H^0(M,\mathcal{O}_{nZ+Z}) - \dim H^0(M,\mathcal{O}_{nZ}) - \dim H^1(M,\mathcal{O}_{nZ+Z}) \\ + \dim H^1(M,\mathcal{O}_{nZ}) &= \chi((n+1)Z) - \chi(nZ) = -nZ\cdot Z \qquad \text{Q.E.D.}\end{aligned}$$

COROLLARY 1.2. *Let $\pi : M \to V$ be the minimal good resolution of normal two dimensional Stein space V with p as its only maximally elliptic singularity. Suppose p is a hypersurface singularity. Then $Z\cdot Z \geqslant -3$.*

The following theorem of Laufer and Lipman, gives an upper estimate of multiplicity in terms of $\dim H^1(M, \mathcal{O})$.

THEOREM 1.3. *Let $V = \{f(x,y,z)=0\}$ have an isolated singularity at $(0,0,0)$. Let n be the multiplicity of V. Then $\dim H^1(M,\mathcal{O}) \geqslant (n-1)(n-2)/2$, where M is a resolving manifold of V.*

Proof. The proof is a refinement of the proof of [18, Theorem 3.14].

2. Topological Classification of Weakly Elliptic Double Points. In 1964, M. Artin gave a complete topological classification of rational double points. In 1970, Wagreich proved that for double points, $Z\cdot Z \geqslant -2$. Using this fact, he listed a lot of the possible weighted dual graphs of weakly elliptic double points. Using the fact that $-K'$ is the summation of an elliptic sequence and a combinatorial argument, we list all possible weighted dual graphs for weakly elliptic double points. Moreover, all these weighted dual graphs actually arise from weakly elliptic double points, because we can find a defining equation for each of them. The defining equations have been found by an unpublished technique of Laufer.

PROPOSITION 2.1. *Let Γ be a weighted dual graph including genera for the vertices, associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_0}, \ldots, Z_{B_l}, Z_E$ be the elliptic sequence. Then $Z\cdot Z \leqslant Z_{B_1}\cdot Z_{B_1} \leqslant \cdots \leqslant Z_{B_l}\cdot Z_{B_l} \leqslant Z_E\cdot Z_E$. If $Z_{B_i}\cdot Z_{B_i} = Z_{B_{i+1}}\cdot Z_{B_{i+1}}$, then $A_j\cdot A_j = -2$ for all $A_j \subseteq B_i$, $A_j \nsubseteq B_{i+1}$, $0 \leqslant i \leqslant l$.*

Proof. For $0 \leqslant i \leqslant l$, let $A_j \subseteq B_i$ and $A_j \nsubseteq B_{i+1}$. If $A_j \cap B_{i+1} = \varnothing$, then $A_j \cdot (Z_{B_i} + Z_{B_{i+1}}) = A_j \cdot Z_{B_i} \leqslant 0$. If $A_j \cap B_{i+1} \neq \varnothing$, then $A_j \cdot Z_{B_i} < 0$ by the definition of elliptic sequence. Since $A_j \cdot Z_{B_{i+1}} = 1$ in this case, $A_j \cdot (Z_{B_i} + Z_{B_{i+1}}) \leqslant 0$. We observe that $Z_{B_i} \geqslant Z_{B_{i+1}}$, i.e., $Z_{B_i} - Z_{B_{i+1}}$ is a positive cycle. It follows that $(Z_{B_i} - Z_{B_{i+1}}) \cdot (Z_{B_i} + Z_{B_{i+1}}) \leqslant 0$. Hence $Z_{B_i} \cdot Z_{B_i} \leqslant Z_{B_{i+1}} \cdot Z_{B_{i+1}}$.

Suppose that $Z_{B_i} \cdot Z_{B_i} = Z_{B_{i+1}} \cdot Z_{B_{i+1}}$. We want to prove $A_j \cdot A_j = -2$ for all $A_j \subseteq B_i$ and $A_j \nsubseteq B_{i+1}$. Since $(Z_{B_i} - Z_{B_{i+1}}) \cdot (Z_{B_i} + Z_{B_{i+1}}) = Z_{B_{i+1}}^2 - Z_{B_{i+1}}^2 = 0$, we have $A_j \cdot (Z_{B_i} + Z_{B_{i+1}}) = 0$. Recall that $K' = -\sum_{i=0}^{l} Z_{B_i} - E$. Then

$$\begin{aligned} 0 \leqslant A_j \cdot K' &= -A_j \cdot \left(\sum_{i=0}^{l} Z_{B_i} + E \right) \\ &= -A_j \left(Z_{B_i} + Z_{B_{i+1}} + \cdots + Z_{B_l} + E \right) \\ &= -A_j \cdot \left(Z_{B_{i+2}} + \cdots + Z_{B_l} + E \right) \\ &\leqslant 0 \qquad \text{since} \quad A_j \nsubseteq B_{i+2} \end{aligned}$$

Therefore $0 = A_j \cdot K' = 2g_j - 2 - A_j \cdot A_j = -2 - A_j \cdot A_j$ and $A_j \cdot A_j = -2$. Q.E.D.

PROPOSITION 2.2. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Supose K' exists. Let $Z = Z_{B_0}, \ldots, Z_{B_l}, Z_E$ be the elliptic sequence. If $Z \cdot Z = -1$, then there exists a unique $A_1 \nsubseteq B_1$, $A_1 \cdot A_1 = -2$, such that $Z = Z_{B_1} + A_1$ and $A_1 \cap B_1 \neq \varnothing$. Moreover, if $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$, then $A_2 \cdot Z_{B_1} = -1$ and $z_2 = 1$.*

Proof. By the definition of the elliptic sequence and $Z \cdot Z = -1$, there exists a unique $A_1 \nsubseteq B_1$ such that $A_1 \cap B_1 \neq \varnothing$ and $z_1 = 1$. By Proposition 2.1, we know that $A_1 \cdot A_1 = -2$. Since $z_1 = 1$, $A_1 \cdot Z < 0$ and $A_1 \cdot A_1 = -2$, we conclude that $z_2 = 1$ and A_1 cannot intersect any $A_0 \nsubseteq B_1$ with $A_0 \neq A_1$. Hence $A_2 \cdot Z_{B_1} = -1$. Otherwise $A_2 \cdot Z_{B_1} = 0$ would imply that $z_2 \geqslant 2$. So $Z = Z_{B_1} + A_1$.

COROLLARY 2.3. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists and $Z = Z_{B_0}, \ldots, Z_{B_l} = Z_E$ is the elliptic sequence. If $Z \cdot Z = -1$, then Γ must be one of the forms in Table 1 below.*

We now explain the notation we shall use in Tables 1, 2, 3 and 4. We shall employ Laufer's notion of unweighted dual graphs which he needed to describe minimally elliptic singularities (see [18]). The special cases of Proposition 2.4 of [28], where it is not true that the A_i in the support of the minimally elliptic cycle are nonsingular rational curves with normal crossings, are described and named individually. In the other dual graphs of $|E|$, $*$ denotes a vertex which will have $z_{*E}=1$ and 0 denotes a vertex will have z_{0E} equal to 2 or 3 and $A_0 \cdot A_0 = -3$. The remaining vertices, each denoted by ●, will all have weight -2. Each vertex is a nonsingular rational curve unless otherwise specified.

LIST I.

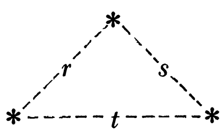
$\cdots r \cdots$

denotes

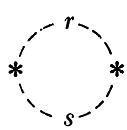
with r vertices and $r+1$ edges. The case $r=0$ is included.

El $\quad *$ The vertex A_* is a nonsingular elliptic curve.

No



or

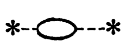


or $\quad * \quad r \quad r \geqslant 1$ with each A_* a nonsingular rational curve, or

$*$ with A_* a rational curve with a node singularity ($r=0$).

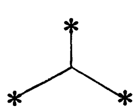
Cu $\quad *$ A_* is a rational curve with a cusp singularity.

Ta



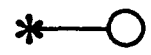
The vertices are nonsingular rational curves which meet tangentially to first order.

Tr



The vertices are nonsingular rational curves which meet transversely at the same point.

$A_{*,0}$

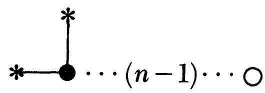


$z_{0E}=2$

$A_{1,*,0}$

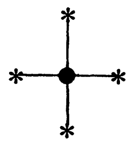
$z_{0E}=3$

$A_{n,**,0}$, $n \geqslant 1$

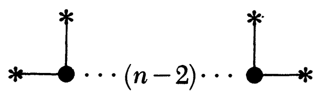


$z_{0E}=2$

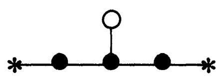
$A_{1,****}$



$A_{n,****}$, $n \geqslant 2$

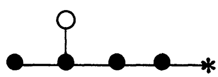


$A'_{3,**,0}$



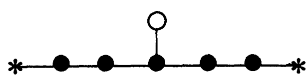
$z_{0E}=2$

$A_{4,*,0}$



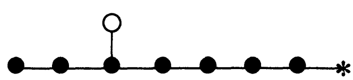
$z_{0E}=3$

$A'_{5,**,0}$



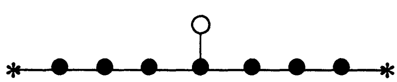
$z_0E=2$

$A_{7,*,0}$



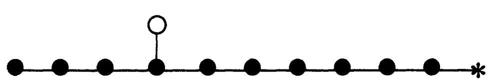
$z_{0E}=3$

$A'_{7,**,0}$



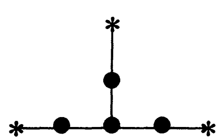
$z_{0E}=2$

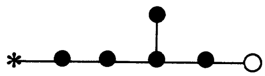
$A_{10,*,0}$

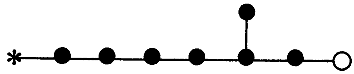


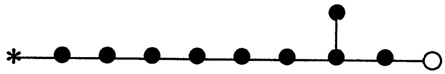
$z_{0E}=3$

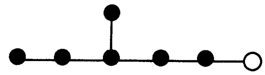
$D_{4,***}$

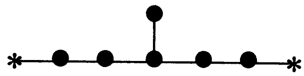


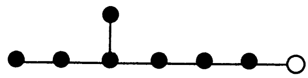
$D_{5,*,0}$  $z_{0E}=2$

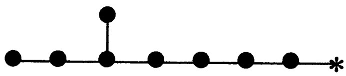
$D_{7,*,0}$  $z_{0E}=2$

$D_{9,*,0}$  $z_{0E}=2$

$E_{6,0}$  $z_{0E}=3$

$E_{6,**}$ 

$E_{7,0}$  $z_{0E}=2$

$E_{8,*}$ 

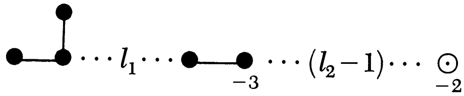
In order to describe weakly elliptic double points or hypersurface singularities with geometric genus equal to two thoroughly, we need to introduce some special notions of weighted dual graphs, each of which consists of one special vertex ⊙ which we call the end component of the corresponding weighted dual graph. We make a convention that an edge with at most one vertex attached to it will be omitted.

LIST II.

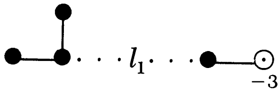
A_l, $l \geqslant 1$: $\cdots(l-1)\cdots \odot$, $A_0:=$empty graph

(by our convention A_1 is $\underset{-2}{\odot}$).

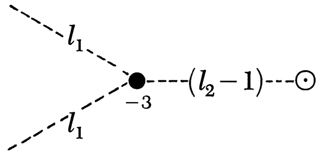
$DA(l_1+4, -3, l_2)$, $l_1 \geqslant 0$, $l_2 \geqslant 1$:



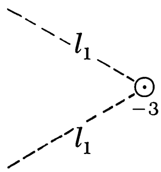
$l_1 \geqslant 0$, $l_2 = 0$



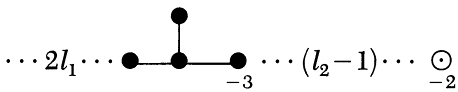
$AAA(l_1, l_1, -3, l_2)$, $l_1 \geqslant 0$, $l_2 \geqslant 1$:



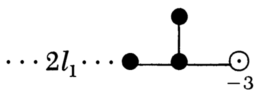
$l_1 \geqslant 0$, $l_2 = 0$:



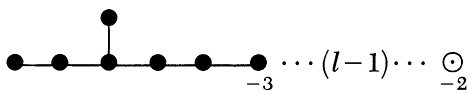
$A'A(2l_1 + 3, -3, l_2)$, $l_1 \geqslant 0$, $l_2 \geqslant 1$:



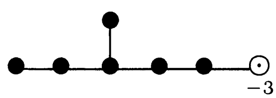
$l_1 \geqslant 0$, $l_2 = 0$:



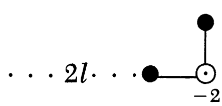
$EA(6, -3, l)$, $l \geqslant 1$:



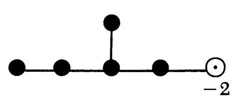
$l = 0$



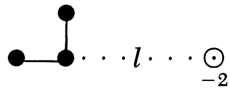
A'_{2l+3}, $l \geqslant 0$:



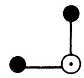
E_6:



$D(l+4)$, $l \geqslant 0$:



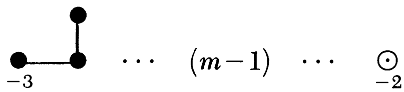
$l = -1$:

D_3 

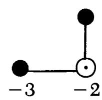
F_{-4} 

F_{-3}: 

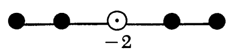
$A'(m+2, -3)$, $m \geqslant 1$:



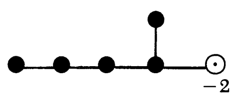
$m = 0$:



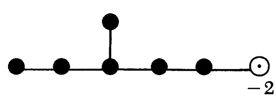
A_5'':



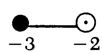
D_6:



E_7:



AF:



In Tables 1, 2, 3 and 4 below, the underlying part of a weighted dual graph $\underline{\Gamma_E}$ corresponds to the weighted dual graph of a minimally elliptic singularity. The weighted dual graphs Γ_E are described by giving values for the $A_*\cdot A_*$ in the graphs in List I. These $A_*\cdot A_*$ are listed from left to right. The union of subgraphs Γ_0 with the A_0 identified is indicated by $+$. Thus, for example $\Gamma_E = A_{n,*,*,0} + A'_{3,**,0}$, with $n=2$ and weights $A_*\cdot A_*$ given by $-3, -2, -2, -2$ denotes the weighted dual graph shown in Figure 2.

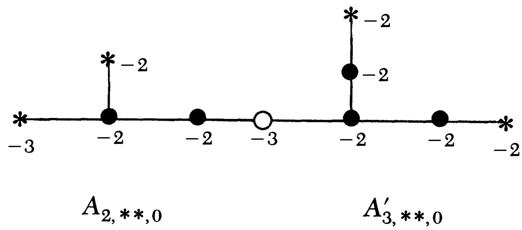


FIGURE 2.

The weighted dual graphs denoted by N_0 may have either three, two or one A_*. These correspond respectively to three, two and one given value for A_*. The weighted dual graphs in Tables 1, 2, 3 and 4 are described by attaching (also indicated by $+$) the end components of graphs in List II to a weighted dual graph of a minimally elliptic singularity. Except in cases (98) and (117) of Table 4, all the end components of graphs in List II are attached to the A_* components in Γ_E whose self-intersection numbers (in column 2) are undefined.

Table 1 The Weighted Dual Graphs for Weakly Elliptic Singularities with $Z\cdot Z = -1$.[a]

	Dual Graph	$A_*\cdot A_*$	Equation
(1)	$\underline{El} + A_l$	$\underline{-1}$	$z^2 = y^3 + x^{6+6l}$
(2)	$\underline{N_0} + A_l$	$\left\{\begin{matrix} -1, r=0 \\ -3, r\geqslant 1\end{matrix}\right\}$	$z^2 = (y + x^{2+2l})(y^2 + x^{r+5+4l})$
(3)	$\underline{Cu} + A_l$	$\underline{-1}$	$z^2 = y^3 + x^{7+6l}$
(4)	$\underline{Ta} + A_l$	$-2, \underline{-3}$	$z^2 = y^3 + x^{5+4l}y$
(5)	$\underline{Tr} + A_l$	$-2, -2, \underline{-3}$	$z^2 = y^3 + x^{8+6l}$
(6)	$A_{1,****} + A_l$	$-2, -2, -2, \underline{-3}$	$z^2 = y^3 + x^{9+6l}$
(7)	$A_{n,****} + A_l$, $n \geqslant 2$	$-2, -2, -2, \underline{-3}$	$z^2 = (y + x^{3+2l})(y^2 + x^{n+5+4l})$
(8)	$D_{4,***} + A_l$	$-2, -2, \underline{-3}$	$z^2 = y^3 + x^{10+6l}$
(9)	$E_{6,**} + A_l$	$-2, \underline{-3}$	$z^2 = y^3 + x^{7+4l}y$
(10)	$E_{8,*} + A_l$	$\underline{-3}$	$z^2 = y^3 + x^{11+6l}$

[a] $l \geqslant 0$.

The singly underlined A_* component is attached to one graph from List II. The doubly underlined A_* component is attached to two graphs from List II. The triply underlined component is attached to three graphs from List III. The attaching order is from left to right: the graphs from List II and the underlined self-intersection numbers $A_* \cdot A_*$ are listed from left to right.

Example 1. In Table 4, (56), $\underline{A_{n,***}} + A_1 + A_1 + A_1$ $(n \geqslant 2)$ and weights $A_* \cdot A_*$ given by $-2, \underline{-3}, \underline{-3}, \underline{-3}$ denote the weighted dual graph shown in Figure 3.

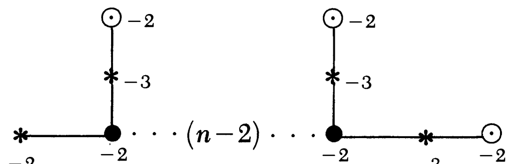


FIGURE 3.

Example 2. In Table 4, (58), $\underline{N_0} + A_1 + A_1 + A_1$ $(r \geqslant 0,\ s \geqslant 0)$ and weights $A_* \cdot A_*$ given by $\underline{-3}, \underline{\underline{-4}}$ denotes the weighted dual graph shown in Figure 4.

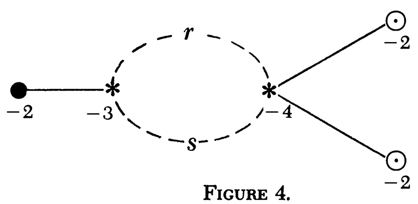


FIGURE 4.

Example 3. In Table 4, (73), $\underline{D_{4,***}} + A_1 + A_1 + A_1$ and weights $A_* \cdot A_*$ given by $-2, -2, \underline{\underline{\underline{-5}}}$ denote the weighted dual graph shown in Figure 5.

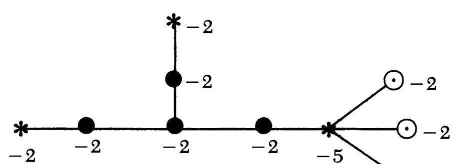


FIGURE 5.

Example 4. In Table 4, (116), $\underline{A_{*,0} + A_{*,0} + A_{*,0} + A_{2,**,0}} + A_1 + A_1$ and weights $A_* \cdot A_*$ given by $\underline{-3}, -2, -2, -2, \underline{-2}$ denote the weighted dual graph shown in Figure 6.

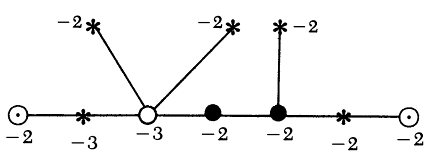


FIGURE 6.

Example 5. In Table 2, (20), $\overline{E_{6,**}+AAA}(l_1,l_1,-3,3)$, and weights $A_*\cdot A_*$ given by $-2, \underline{-3}$ denote the weighted dual graph shown in Figure 7.

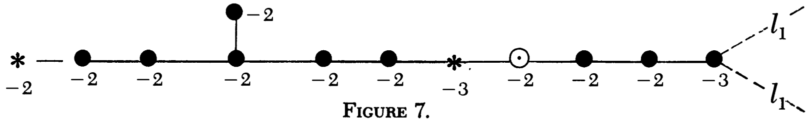
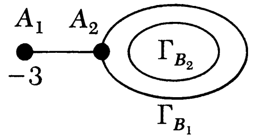


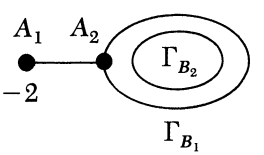
FIGURE 7.

PROPOSITION 2.4. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z=Z_{B_0},\ldots,Z_{B_l}, Z_E$ be the elliptic sequence. If $Z\cdot Z=-2$, $Z_{B_1}\cdot Z_{B_1}=-1$, then Γ must be one of the following forms*:

(1) $A_2\subseteq B_1, A_2\not\subseteq B_2, Z=A+Z_{B_1}, A_2\cdot Z_{B_1}=-1, z_2=1$:



(2) $Z=2A_1+D$, D is a positive cycle, $|D|=B_1$; $z_2=3$, $A_2\cdot Z_{B_1}=0$, $A_2\subseteq B_1$, $A_2\not\subseteq B_2$:



where Γ_{B_i} is the graph of B_i.

Proof. By the definition of elliptic sequence and the fact that $Z\cdot Z=-2$ we have the following two cases.

(I) There exist $A_1,A_2\not\subseteq B_1$, $A_1\cap B_1\neq\emptyset\neq A_2\cap B_1$ and $A_1\neq A_2$. In this case, $A_1\cdot Z=-1=A_2\cdot Z$ and $z_1=1=z_2$. For $i=1,2$, we have $0\geqslant A_i\cdot(-K')=A_i(\Sigma_{i=0}^{l}Z_{B_i}+E)\geqslant A_i\cdot(Z+Z_{B_1})=0$. So $0=-A_i\cdot K'=2+A_i\cdot A_i$ and hence $A_i\cdot A_i=-2$, $i=1,2$. Let $A_3,A_4\subseteq B_1$ such that $A_1\cdot A_3=1$, $A_2\cdot A_4=1$. Since $A_1\cdot A_1=A_2\cdot A_2=-2$, $z_1=z_2=1$, and $A_1\cdot Z=A_2\cdot Z=-1$, there is no $A_j\not\subseteq B_1$, $A_2\neq A_j\neq A_1$, such that $A_j\cdot A_1>0$ or $A_j\cdot A_2>0$, i.e., $A=A_1\cup A_2\cup B_1$. Moreover, we know that $z_3=z_4=1$. Hence $A_3\cdot Z_{B_1}<0$ and $A_4\cdot Z_{B_1}<0$. It follows that $A_3=A_4$ and $A_3\cdot Z_{B_1}=-1$, since $Z_{B_1}\cdot Z_{B_1}=-1$. As $z_3=1$, $Z=A_1+A_2+Z_{B_1}$, we have $A_3\cdot Z_{B_1}=1>0$, which is a contradiction. This case cannot occur.

(II) There exists a unique $A_1\not\subseteq B_1$ such that $A_1\cap B_1\neq\emptyset$. In this case, we have either (A) $A_1\cdot Z=-2$ and $z_1=1$, (B) $A_1\cdot Z=-1$ and $z_1=2$, or (C) $A_1\cdot Z=-1$ and $z_1=1$.

In (A), $0 \geqslant A_1 \cdot (-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i}) \geqslant A_1 \cdot (Z + Z_{B_1}) = -2 + 1 = -1$. So either $A_1 \cdot (K') = 0$ or $A_1 \cdot (-K') = -1$. If $A_1 \cdot (-K') = 0$, then $A_1 \cdot A_1 + 2 = 0$, i.e., $A_1 \cdot A_1 = -2$. It follows that $A_1 \cdot Z \geqslant -1$. This is a contradiction. If $A_1 \cdot (-K') = -1$, then $A_1 \cdot A_1 = -3$. Let $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$. Then $A_2 \not\subseteq B_2$. Since $A_1 \cdot Z = -2$, $A_1 \cdot A_1 = -3$ and $z_1 = 1$, there is no $A_i \not\subseteq B_1$, $A_i \neq A_1$, such that $A_i \cdot A_1 > 0$, i.e., $A = A_1 \cup B_1$. Moreover, we have $z_2 = 1$ and hence $Z = A_1 + Z_{B_1}$. So $A_2 \cdot Z_{B_1} = -1$ and we are in (1).

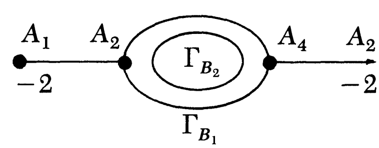
In (B), $0 \geqslant A_1 \cdot (-K') = A_1(\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_1(Z + Z_{B_1}) = 0$. Then $A_1 \cdot K' = 0$ and $A_1 \cdot Z_{B_i} = 0$, $2 \leqslant i \leqslant l+1$. Let $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$. We have $A_1 \cdot A_1 = -2$ and $A_2 \not\subseteq B_2$. For any $A_i \not\subseteq B_1$, $A_i \neq A_1$, we have $A_i \cdot Z = 0 = A_i \cdot Z_{B_1}$. It follows that $2 + A_i \cdot A_i = A_i(-K') = A_i \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) = 0$, i.e., $A_i \cdot A_i = -2$. We claim that $z_2 > 1$. For if $z_2 = 1$, then $\mathrm{supp}(Z - Z_{B_1})$ consists of those $A_i \not\subseteq B_1$. Consequently $Z^2 - Z_{B_1}^2 = (Z - Z_{B_1}) \cdot (Z + Z_{B_1}) = 0$. However, $Z^2 - Z_{B_1}^2 = -2 + 1 = -1$. This leads to a contradiction. Since $z_1 = 2$, $z_2 > 1$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, it is clear that $1 \leqslant \deg A_1 \leqslant 2$. If $\deg A_1 = 2$, then there exists a unique $A_3 \not\subseteq B_1$ such that $A_3 \cdot A_1 = 1$, $z_3 = 1$ and $z_2 = 2$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \not\subseteq B_1$, $A_i \neq A_1$. Since $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of rational double point. Because $z_3 = 1$, it is easy to see that this case cannot occur. We conclude that $\deg A_1 = 1$, i.e., $A = A_1 \cup B_1$. Since $z_1 = 2$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, we have $z_2 = 3$. Then we are in (2).

In (C), $0 \geqslant A_1 \cdot (-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = 0$. Then $A_1 \cdot K' = 0$ and $A_1 \cdot Z_{B_i} = 0$, $2 \leqslant i \leqslant l+1$. Let $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$. We have $A_1 \cdot A_1 = -2$. Since $z_1 = 1$ and $A_1 \cdot Z = -1$, we have $A = A_1 \cup B_1$ and $z_2 = 1$. So $Z = A_1 + Z_{B_1}$. But then $Z \cdot Z = (A_1 + Z_{B_1})(A_1 + Z_{B_1}) = A_1(A_1 + Z_{B_1}) = -1$, which is absurd.

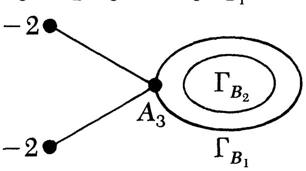
Q.E.D.

PROPOSITION 2.5. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_0}, \ldots, Z_{B_l}, Z_E$ be the elliptic sequence. If $Z \cdot Z = -2 = Z_{B_1} \cdot Z_{B_1}$, then Γ must be one of the following forms*:

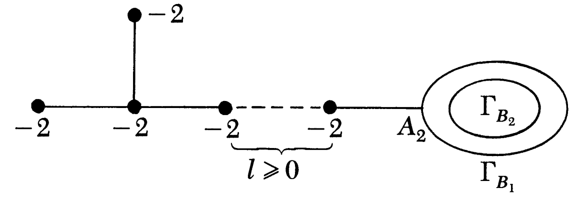
(1) $Z = A_1 + Z_{B_1} + A_2$, $A_3, A_4 \subseteq B_1$, $A_3, A_4 \not\subseteq B_2$; $z_3 = 1 = z_4$, $A_3 \cdot Z_{B_1} = -1 = A_4 \cdot Z_{B_1}$:



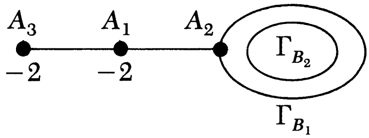
(2) $Z = \frac{1}{1} Z_{B_1}$, $A_3 \subseteq B_1$, $A_3 \not\subseteq B_2$; $z_3 = 1$, $A_3 \cdot Z_{B_1} = -2$:



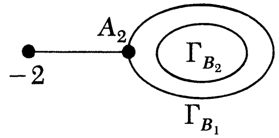
(3) $Z=1\overset{1}{2}2\ldots 2Z_{B_2}, A_2\subseteq B_1, A_2\not\subseteq B_2; z_2=1, A_2\cdot Z_{B_1}=-2$:



(4) $Z=12D$, $|D|=B_1$, D is a positive cycle, $z_2=2$, $A_2\cdot Z_{B_1}=0$, $A_2\subseteq B_1$, $A_2\not\subseteq B_2$:



(5) $Z=2D$, D is a positive cycle, $|D|=B_1$, $z_2=3$, $A_2\cdot Z_{B_1}=0$, $A_2\subseteq B_1$, $A_2\not\subseteq B_2$:



where Γ_{B_i} *is the graph of* B_i.

Proof. We firstly recall that by Proposition 2.1, $A_i\cdot A_i=-2$ for all $A_i\not\subseteq B_1$. By the definition of elliptic sequence and the fact that $Z\cdot Z=-2$, we have the following cases.

(I) There exist $A_1,A_2\not\subseteq B_1$, $A_1\neq A_2$, such that $A_1\cap B_1\neq\varnothing\neq A_2\cap B_1$. In this case $A_1\cdot Z=-1=A_2\cdot Z$ and $z_1=z_2=1$. Let $A_3,A_4\subseteq B_1$ such that $A_1\cdot A_3=1=A_2\cdot A_4$. Since $z_1=z_2=1$ and $A_1\cdot Z=-1=A_2\cdot Z$, there is no $A_i\not\subseteq B_1$, $A_1\neq A_i\neq A_2$, such that $A_i\cdot A_1>0$ or $A_i\cdot A_2>0$, i.e., $A=A_1\cup A_2\cup B_1$. Moreover, $z_3=1=z_4$ and $Z=A_1+A_2+Z_{B_1}$. If $A_3\neq A_4$, then $A_3\cdot Z_{B_1}=-1=A_4\cdot Z_{B_1}$ and $A_3,A_4\not\subseteq B_2$. We are in (1). If $A_3=A_4$, then $A_3\cdot Z_{B_1}=-2$ and $A_3\not\subseteq B_2$, and we are in (2).

(II) There exists a unique $A_1\not\subseteq B_1$ such that $A_1\cap B_1\neq\varnothing$. Since $Z\cdot Z=-2=Z_{B_1}\cdot Z_{B_1}$, $(Z-Z_{B_1})\cdot(Z+Z_{B_1})=0$. It follows that $A_i(Z+Z_{B_1})=0$ for all $A_i\not\subseteq B_1$. In particular, if $A_i\cap B_1=\varnothing$, then $A_i\cdot Z=0$. So we have either (A) $A_1\cdot Z=-2$ and $z_1=1$, or (B) $A_1\cdot Z=-1$ and $z_1=2$.

In (A) $A_1\cdot A_1$ must be less than -2. But this is impossible because $A_1\cdot A_1=-2$.

In (B) let $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$. We claim that $A_2 \not\subseteq B_2$. Otherwise $0 \geqslant A_1(-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant -1 + 2 = 1$. This is absurd. The proof breaks up into four subcases.

(B1) There exist $A_3, A_4 \not\subseteq B_1$, $A_3 \neq A_4$, such that $A_3 \cdot A_1 = 1 = A_4 \cdot A_1$ and $z_3 = z_4 = 1 = z_2$. It follows that $A = A_1 \cup A_3 \cup A_4 \cup B_1$ and $Z = 2A_1 + A_3 + A_4 + Z_{B_1}$. We are in (3).

(B2) There exists $A_3 \not\subseteq B_1$ such that $A_1 \cdot A_3 = 1$ and $z_3 = 2$. Because $A_i \cdot Z = 0$ for $A_i \not\subseteq B_1$, $A_i \neq A_1$, it is easy to see that we are in (3).

(B3) There exists $A_3 \not\subseteq B_1$ such that $A_1 \cdot A_3 = 1$, $z_3 = 1$ and $z_2 = 2$. Since $z_1 = 2$, $z_3 = 1$ and $A_3 \cdot Z = 0$, it follows that there is no $A_i \not\subseteq B_1$, $A_1 \neq A_i \neq A_3$ such that $A_i \cdot A_3 = 1$, i.e., $A = A_1 \cup A_3 \cup B_1$ and $Z = 2A_1 + A_3 + D$, where D is a positive cycle with support B_1. We claim that $A_2 \cdot Z_{B_1} = 0$. Otherwise $Z = A_1 + A_3 + Z_{B_1}$ and hence $A_1 \cdot Z = 0$. This leads to a contradiction. We are in (4).

(B4) $z_2 = 3$. Then $A = A_1 \cup B_1$ and $Z = 2A_1 + D$, where D is a positive cycle with support B_1. We claim that $A_2 \cdot Z_{B_1} = 0$. Otherwise $Z = A_2 + Z_{B_1}$. This leads to a contradiction. We are in (5). Q.E.D.

Definition 2.6. Let $\pi : M \to V$ be the minimal good resolution of weakly elliptic singularity p. Let $Z_{B_0} = Z, \ldots, Z_{B_l} = Z_E$ be the elliptic sequence. The set of self-intersection numbers of the elliptic sequence is $\{Z_{B_0}^2, \ldots, Z_{B_l}^2\}$.

COROLLARY 2.7. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists and the set of self-intersection numbers of the elliptic sequence consists of -2 and -1. Then Γ must be one of the forms in Table 2.*

Table 2. The Weighted Dual Graphs for Weakly Elliptic Singularities with $Z \cdot Z = -2$ and $Z_E \cdot Z_E = -1$.[a]

	Dual Graph	$A_* \cdot A_*$	Equation
(1)	$\underline{\mathrm{Ta}} + A_1$	$\underline{-2}, -3$	$z^2 = x^5 + y^7$
(2)	$\underline{\mathrm{El}} + DA(l_1+4, -3, l_2)$	$\underline{-1}$	$z^2 = (y^2 + x^{3+l_1})(x^3 + y^{12+6l_2})$
(3)	$\underline{N_0} + DA(l_1+4, -3, l_2)$	$\left\{ \begin{array}{l} \underline{-1}, r=0 \\ \underline{-3}, r \geqslant 1 \end{array} \right\}$	$z^2 = (y^2 + x^{3+l_1})(x + y^{4+2l_2})$ $\times (x^2 + y^{9+4l_2+r})$
(4)	$\underline{\mathrm{Cu}} + DA(l_1+4, -3, l_2)$	$\underline{-1}$	$z^2 = (y^2 + x^{3+l_1})(x^3 + y^{13+6l_2})$
(5)	$\underline{\mathrm{Ta}} + DA(l_1+4, -3, l_2)$	$-2, \underline{-3}$	$z^2 = x(y^2 + x^{l_1+3})(x^2 + y^{9+4l_2})$
(6)	$\underline{\mathrm{Tr}} + DA(l_1+4, -3, l_2)$	$-2, -2, \underline{-3}$	$z^2 = (y^2 + x^{l_1+3})(x^3 + y^{14+6l_2})$
(7)	$\underline{A_{1,****}} + DA(l_1+48-3, l_2)$	$-2, -2, -2, \underline{-3}$	$z^2 = (y^2 + x^{l_1+3})(x^3 + y^{15+6l_2})$

Table 2. (cont.)

	Dual Graph	$A_*\cdot A_*$	Equation
(8)	$\underline{A_{n,****}}+DA(l_1+4,-3,l_2),\ n\geqslant 2$	$-2,-2,-2,\underline{-3}$	$z^2=(y^2+x^{l_1+3})(x+y^{5+2l_2})\times(x^2+y^{n+9+4l_2})$
(9)	$\underline{D_{4,***}}+DA(l_1+4,-3,l_2)$	$-2,-2,\underline{-3}$	$z^2=(y^2+x^{3+l_1})(x^3+y^{16+6l_2})$
(10)	$\underline{E_{6,**}}+DA(l_1+4,-3,l_2)$	$-2,\underline{-3}$	$z^2=x(y^2+x^{3+l_1})(y^2+x^{11+4l_2})$
(11)	$\underline{E_{8,*}}+DA(l_1+4,-3,l_2)$	$\underline{-3}$	$z^2=(y^2+x^{3+l_1})(x^3+y^{17+6l_2})$
(12)	$\underline{\mathrm{El}}+AAA(l_1,l_1,-3,l_2)$	$\underline{-1}$	$z^2=(x+y^{l_1+1})(x^3+y^{9+6l_2+3l_1})$
(13)	$\underline{N_0}+AAA(l_1,l_1,-3,l_2)$	$\begin{cases}\underline{-1}, & r=0\\ \underline{-3}, & r\geqslant 1\end{cases}$	$z^2=(x+y^{l_1+1})(x+y^{3+l_1+2l_2})\times(x^2+y^{7+r+2l_1+4l_2})$
(14)	$\underline{Cu}+AAA(l_1,l_1,-3,l_2)$	$\underline{-1}$	$z^2=(x+y^{l_1+1})(x^3+y^{10+6l_2+3l_1})$
(15)	$\underline{\mathrm{Ta}}+AAA(l_1,l_1,-3,l_2)$	$-2,\underline{-3}$	$z^2=x(x+y^{l_1+1})(x^2+y^{7+2l_1+4l_2})$
(16)	$\underline{\mathrm{Tr}}+AAA(l_1,l_1,-3,l_2)$	$-2,-2,\underline{-3}$	$z^2=(x+y^{l_1+1})(x^3+y^{11+3l_1+6l_2})$
(17)	$\underline{A_{1,****}}+AAA(l_1,l_1,-3,l_2)$	$-2,-2,-2,\underline{-3}$	$z^2=(x+y^{l_1+1})(x^3+y^{12+3l_1+6l_2})$
(18)	$\underline{A_{n,****}}+AAA(l_1,l_1,-3,l_2)$	$-2,-2,-2,\underline{-3}$	$z^2=(x+y^{l_1+1})(x+y^{4+l_1+2l_2})\times(x^2+y^{n+7+2l_1+4l_2})$
(19)	$\underline{D_{4,***}}+AAA(l_1,l_1,-3,l_2)$	$-2,-2,\underline{-3}$	$z^2=(x+y^{l_1+1})(x^3+y^{13+3l_1+6l_2})$
(20)	$\underline{E_{6,**}}+AAA(l_1,l_1,-3,l_2)$	$-2,\underline{-3}$	$z^2=x(x+y^{l_1+1})(x^2+y^{9+2l_1+4l_2})$
(21)	$\underline{E_{8,*}}+AAA(l_1,l_1,-3,l_2)$	$\underline{-3}$	$z^2=(x+y^{l_1+1})(x^3+y^{14+3l_1+6l_2})$
(22)	$\underline{\mathrm{El}}+A'A(2l_1+3,-3,l_2)$	$\underline{-1}$	$z^2=y(x+y^{l_1+1})(x^3+y^{12+3l_1+6l_2})$
(23)	$\underline{N_0}+A'A(2l_1+3,-3,l_2)$	$\begin{cases}\underline{-1}, & r=0\\ \underline{-3}, & r\geqslant 1\end{cases}$	$z^2=y(x+y^{l_1+1})(x+y^{4+l_1+2l_2})\times(x^2+y^{r+9+2l_1+4l_2})$
(24)	$\underline{\mathrm{Cu}}+A'A(2l_1+3,-3,l_2)$	$\underline{-1}$	$z^2=y(x+y^{l_1+1})(x^3+y^{13+3l_1+6l_2})$
(25)	$\underline{\mathrm{Ta}}+A'A(2l_1+3,-3,l_2)$	$-2,\underline{-3}$	$z^2=yx(x+y^{l_1+1})(x^2+y^{9+2l_1+4l_2})$
(26)	$\underline{\mathrm{Tr}}+A'A(2l_1+3,-3,l_2)$	$-2,-2,\underline{-3}$	$z^2=y(x+y^{l_1+1})(x^3+y^{14+3l_1+6l_2})$
(27)	$\underline{A_{1,****}}+A'A(2l_1+3,-3,l_2)$	$-2,-2,-2,\underline{-3}$	$z^2=y(x+y^{l_1+1})(x^3+y^{15+3l_1+6l_2})$
(28)	$\underline{A_{n,****}}+A'A(2l_1+3,-3,l_2)$	$-2,-2,-2,\underline{-3}$	$z^2=y(x+y^{l_1+1})(x+y^{5+l_1+2l_2})\times(x^2+y^{n+9+2l_1+4l_2})$
(29)	$\underline{D_{4,***}}+A'A(2l_1+3,-3,l_2)$	$-2,-2,\underline{-3}$	$z^2=y(x+y^{l_1+1})(x^3+y^{16+3l_1+6l_2})$
(30)	$\underline{E_{6,**}}+A'A(2l_1+3,-3,l_2)$	$-2,\underline{-3}$	$z^2=yx(x+y^{l_1+1})(x^2+y^{11+2l_1+4l_2})$
(31)	$\underline{E_{8,*}}+A'A(2l_1+3,-3,l_2)$	$\underline{-3}$	$z^2=y(x+y^{l_1+1})(x^3+y^{17+3l_1+6l_2})$

Table 2 (cont.)

	Dual Graph	$A_*\cdot A_*$	Equation
(32)	$\underline{\mathrm{El}}+\mathrm{EA}(6,-3,l)$	$\underline{-1}$	$z^2=(x^2+y^3)(x^3+y^{15+6l})$
(33)	$\underline{N_0}+EA(6,-3,l)$	$\left\{\begin{matrix}\underline{-1}, r=0\\ \underline{-3}, r\geqslant 1\end{matrix}\right\}$	$z^2=(x^2+y^3)(x+y^{5+2l})$ $\times(x^2+y^{r+11+4l})$
(34)	$\underline{\mathrm{Cu}}+EA(6,-3,l)$	$\underline{-1}$	$z^2=(x^2+y^3)(x^3+y^{16+6l})$
(35)	$\underline{\mathrm{Ta}}+EA(6,-3,l)$	$-2, \underline{-3}$	$z^2=x(x^2+y^3)(x^2+y^{11+4l})$
(36)	$\underline{\mathrm{Tr}}+EA(6,-3,l)$	$-2,-2, \underline{-3}$	$z^2=(x^2+y^3)(x^3+y^{17+6l})$
(37)	$\underline{A_{1,****}}+EA(6,-3,l)$	$-2,-2,-2, \underline{-3}$	$z^2=(x^2+y^2)(x^3+y^{18+6l})$
(38)	$\underline{A_{n,****}}+EA(6,-3,l)$	$-2,-2,-2, \underline{-3}$	$z^2=(x^2+y^3)(x+y^{6+2l})$ $\times(x^2+y^{n+11+4l})$
(39)	$\underline{D_{4,***}}+EA(6,-3,l)$	$-2,-2, \underline{-3}$	$z^2=(x^2+y^3)(x^3+y^{9+6l})$
(40)	$\underline{E_{6,**}}+EA(6,-3,l)$	$-2, \underline{-3}$	$z^2=x(x^2+y^3)(x^2+y^{13+4l})$
(41)	$\underline{E_{8,*}}+EA(6,-3,l)$	$\underline{-3}$	$z^2=(x^2+y^3)(x^3+y^{20+6l})$

[a] $l_1\geqslant 0$, $l_2\geqslant 0$, $l\geqslant 0$.

Corollary 2.8. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists and the set of self-intersection numbers of elliptic sequence consists of -2. Then Γ must be one of the forms in Table 3.*

Table 3. The Weighted Dual Graphs for Weakly Elliptic Singularities with $Z\cdot Z=-2=Z_E\cdot Z_E$.[a]

	Dual Graph	$A_*\cdot A_*$	Equation
(1)	$\underline{N_0}+A_l+A_l\ (r\geqslant 0, s\geqslant 0)$	$\underline{-3}, \underline{-3}$	$z^2=(x^2+y^{r+3+2l})$ $\times[(x+y^{l+1})^2+y^{s+3+2l}]$
(2)	$\underline{\mathrm{Ta}}+A_l+A_l$	$\underline{-3}, \underline{-3}$	$z^2=y^4+x^{5+4l}$
(3)	$\underline{\mathrm{Tr}}+A_l+A_l$	$-2, \underline{-3}, \underline{-3}$	$z^2=y^4+x^{4+3l}y$
(4)	$\underline{A_{1,****}}+A_l+A_l$	$-2,-2, \underline{-3}, \underline{-3}$	$z^2=y^4+x^{6+4l}$
(5)	$\underline{A_{n,****}}+A_l+A_l$	$-2,-2, \underline{-3}, \underline{-3}$	$z^2=(y^2+x^{3+2l})(y^2+x^{n+2l+2})$
(6)	$\underline{A_{n,****}}+A_l+A_l$	$-2, \underline{-3}, \underline{-3}, -2$	$z^2=(y^2+x^{3+2l})^2+x^ay^b$, $2a+(3+2l)b=11+n+8l$
(7)	$\underline{D_{4,***}}+A_l+A_l$	$-2, \underline{-3}, \underline{-3}$	$z^2=y^4+x^{5+3l}y$

[a]From (1) to (40) l is assumed to be $\geqslant 0$. From (51) to (60) l is assumed to be $\geqslant -1$.

Table 3. (cont.)

	Dual Graph	$A_* \cdot A_*$	Equation
(8)	$\underline{E_{8,**}} + A_l + A_l$	$\underline{-3}, \underline{-3}$	$z^2 = y^4 + x^{7+4l}$
(9)	$\underline{\mathrm{El}} + A_l + A_l$	$\underline{\underline{-2}}$	$z^2 = y^4 + x^{4l+4}$
(10)	$\underline{N_0} + A_l + A_l$	$\left\{\begin{matrix} \underline{\underline{-2}}, r=0 \\ \underline{\underline{-4}}, r \geq 1 \end{matrix}\right\}$	$z^2 = (y + x^{l+1})(y + 2x^{l+1}) \times (y^2 + x^{r+2l+3})$
(11)	$\underline{\mathrm{Cu}} + A_l + A_l$	$\underline{\underline{-2}}$	$z^2 = (y + x^{l+1})(y^3 + x^{4+3l})$
(12)	$\underline{\mathrm{Ta}} + A_l + A_l$	$-2, \underline{\underline{-4}}$	$z^2 = x(x + y^{l+1})(x^2 + y^{3+2l})$
(13)	$\underline{\mathrm{Tr}} + A_l + A_l$	$-2, -2, \underline{\underline{-4}}$	$z^2 = (y + x^{l+1})(y^3 + x^{5+3l})$
(14)	$A_{1,****} + A_l + A_l$	$-2, -2, -2, \underline{\underline{-4}}$	$z^2 = (y + x^{l+1})(y^3 + x^{6+3l})$
(15)	$\underline{A_{n,****}} + A_l + A_l$	$-2, -2, -2, \underline{\underline{-4}}$	$z^2 = (y + x^{l+1})(y + x^{l+2}) \times (y^2 + x^{n+3+2l})$
(16)	$\underline{D_{4,***}} + A_l + A_l$	$-2, -2, \underline{\underline{-4}}$	$z^2 = (y + x^{l+1})(y^3 + x^{7+3l})$
(17)	$\underline{E_{6,**}} + A_l + A_l$	$-2, \underline{\underline{-4}}$	$z^2 = y(y + x^{l+1})(y^2 + x^{5+2l})$
(18)	$\underline{E_{8,*}} + A_l + A_l$	$\underline{\underline{-4}}$	$z^2 = (y + x^{l+1})(y^3 + x^{8+3l})$
(19)	$\underline{\mathrm{El}} + A'_{2l+3}$	$\underline{-2}$	$z^2 = y(x + y^{l+1})(x^3 + y^{6+3l})$
(20)	$\underline{N_0} + A'_{2l+3}$	$\left\{\begin{matrix} \underline{-2}, r=0 \\ \underline{-4}, r \geq 1 \end{matrix}\right\}$	$z^2 = y(x + y^{l+1})(x + y^{l+2}) \times (x^2 + y^{r+5+2l})$
(21)	$\underline{\mathrm{Cu}} + A'_{2l+3}$	$\underline{-2}$	$z^2 = y(x + y^{l+1})(x^3 + y^{7+3l})$
(22)	$\underline{\mathrm{Ta}} + A'_{2l+3}$	$-2, \underline{-4}$	$z^2 = xy(x + y^{l+1})(x^2 + y^{5+2l})$
(23)	$\underline{\mathrm{Tr}} + A'_{2l+3}$	$-2, -2, \underline{-4}$	$z^2 = y(x + y^{l+1})(x^3 + y^{8+3l})$
(24)	$\underline{A_{1,****}} + A'_{2l+3}$	$-2, -2, -2, \underline{-4}$	$z^2 = y(x + y^{l+1})(x^3 + y^{9+3l})$
(25)	$\underline{A_{n,****}} + A'_{2l+3}$	$-2, -2, -2, \underline{-4}$	$z^2 = y(x + y^{l+1})(x + y^{l+3}) \times (x^2 + y^{n+5+2l})$
(26)	$\underline{D_{4,***}} + A'_{2l+3}$	$-2, -2, \underline{-4}$	$z^2 = y(x + y^{l+1})(x^3 + y^{10+3l})$
(27)	$\underline{E_{6,**}} + A'_{2l+3}$	$-2, \underline{-4}$	$z^2 = xy(x + y^{l+1})(x^2 + y^{7+2l})$
(28)	$\underline{E_{8,*}} + A'_{2l+3}$	$\underline{-4}$	$z^2 = y(x + y^{l+1})(x^3 + y^{11+3l})$
(29)	$\underline{\mathrm{Ta}} + A_{2l}$	$\underline{-2}, -4$	$z^2 = xy(x^3 + y^{2+3l})$
(30)	$\underline{N_0} + A_{2l}$ $(r=0, s=0)$	$\underline{-2}, -4$	$z^2 = x(y^2 + x^{2l+1})(y^2 + x^{2l+2})$
(31)	$\underline{\mathrm{Tr}} + A_{2l}$	$\underline{-2}, -3, -3$	$z^2 = y(x^4 + y^{3+4l})$

Table 3. (cont.)

	Dual Graph	$A_*\cdot A_*$	Equation
(32)	$\underline{N_0} + A_{2l}$ $(r=0, t=0, s\geqslant 0)$	$\underline{-2}, -3, -3$	$z^2=x(y^2+x^{2l+1})(y^2+x^{3+2l+s})$
(33)	$\underline{A_{*,0}+A_{*,0}+A_{*,0}+A_{*,0}+A_{*,0}} + A_{2l}$	$\underline{-2}, -2, -2, -2, -2$	$z^2=y(x+y^{l+1})(x+2y^{l+1})$ $\times(x+3y^{l+1})(x+4y^{l+1})$
(34)	$\underline{A_{*,0}+A_{*,0}+A_{*,0}+A_{n,**,0}} + A_{2l}$ $(n\geqslant 1)$	$\underline{-2}, -2, -2, -2, -2$	$z^2=y(x^2+y^{2+2l})(x^2+y^{n+2+2l})$
(35)	$\underline{A_{*,0}+A_{*,0}+A'_{3,**,0}} + A_{2l}$	$\underline{-2}, -2, -2, -2$	$z^2=y(x+y^{l+1})(x^3+y^{4+3l})$
(36)	$\underline{A_{*,0}+A_{*,0}+D_{5,*,0}} + A_{2l}$	$\underline{-2}, -2, -2$	$z^2=xy(x+y^{l+1})(x^2+y^{3+2l})$
(37)	$\underline{A_{*,0}+A_{*,0}+A_{7,0}} + A_{2l}$	$\underline{-2}, -2$	$z^2=y(x+y^{l+1})(x^3+y^{5+3l})$
(38)	$\underline{A_{*,0}+A_{n,**,0}+A_{m,**,0}} + A_{2l}$ $(n\geqslant 1, m\geqslant 1)$	$\underline{-2}, -2, -2, -2, -2$	$z^2=(x+y)(x^2+y^{2+n+2l})$ $\times[(x+y^{l+1})^2+y^{m+2l+2}]$
(39)	$\underline{A_{*,0}+A'_{5,**,0}} + A_{2l}$	$\underline{-2}, -2, -2$	$z^2=y(x^4+y^{4l+5})$
(40)	$\underline{A_{*,0}+D_{7,*,0}} + A_{2l}$	$\underline{-2}, -2$	$z^2=xy(x^3+y^{4+3l})$
(41)	$\underline{\text{El}} + E_6$	$\underline{-2}$	$z^2=(x^2+y^3)(x^3+y^9)$
(42)	$\underline{N_0} + E_6$	$\left\{\begin{matrix}\underline{-2}, r=0\\ \underline{-4}, r\geqslant 1\end{matrix}\right\}$	$z^2=(x^2+y^3)(x+y^3)(x^2+y^{r+7})$
(43)	$\underline{\text{Cu}} + E_6$	$\underline{-2}$	$z^2=(x^2+y^3)(x^3+y^{10})$
(44)	$\underline{\text{Ta}} + E_6$	$-2, \underline{-4}$	$z^2=x(x^2+y^3)(x^2+y^7)$
(45)	$\underline{\text{Tr}} + E_6$	$-2, -2, \underline{-4}$	$z^2=(x^2+y^3)(x^3+y^{11})$
(46)	$\underline{A_{1,***}} + E_6$	$-2, -2, -2, \underline{-4}$	$z^2=(x^2+y^3)(x^3+y^{12})$
(47)	$\underline{A_{n,****}} + E_6$	$-2, -2, -2, \underline{-4}$	$z^2=(x^2+y^3)(x+y^4)(x^2+y^{n+7})$
(48)	$\underline{D_{4,***}} + E_6$	$-2-2, \underline{-4}$	$z^2=(x^2+y^3)(x^3+y^{13})$
(49)	$\underline{E_{6,**}} + E_6$	$-2, \underline{-4}$	$z^2=x(x^2+y^3)(x^2+y^9)$
(50)	$\underline{E_{8,*}} + E_6$	$\underline{-4}$	$z^2=(x^2+y^3)(x^3+y^{14})$
(51)	$\underline{\text{El}} + D(l+4)$	$\underline{-2}$	$z^2=(y^2+x^{3+l})(x^3+y^6)$
(52)	$\underline{N_0} + D(l+4)$	$\left\{\begin{matrix}\underline{-2}, r=0\\ \underline{-4}, r\geqslant 1\end{matrix}\right\}$	$z^2=(y^2+x^{3+l})(x+y^2)(x^2+y^{r+5})$
(53)	$\underline{\text{Cu}} + D(l+4)$	$\underline{-2}$	$z^2=(y^2+x^{3+l})(x^3+y^7)$
(54)	$\underline{\text{Ta}} + D(l+4)$	$-2, \underline{-4}$	$z^2=x(y^2+x^{3+l})(x^2+y^5)$
(55)	$\underline{\text{Tr}} + D(l+4)$	$-2, -2, \underline{-4}$	$z^2=(y^2+x^{3+l})(x^3+y^8)$

Table 3. (cont.)

	Dual Graph	$A_* \cdot A_*$	Equation
(56)	$\underline{A_{1,****}} + D(l+4)$	$-2, -2, -2, \underline{-4}$	$z^2=(y^2+x^{3+l})(x^3+y^9)$
(57)	$\underline{A_{n,****}} + D(l+4)$	$-2, -2, -2, \underline{-4}$	$z^2=(y^2+x^{3+l})(x+y^3)(x^2+y^{n+5})$
(58)	$\underline{D_{4,***}} + D(l+4)$	$-2, -2, \underline{-4}$	$z^2=(y^2+x^{3+l})(x^3+y^{10})$
(59)	$\underline{E_{6,**}} + D(l+4)$	$-2, \underline{-4}$	$z^2=x(y^2+x^{3+l})(x^2+y^7)$
(60)	$\underline{E_{8,*}} + D(l+4)$	$\underline{-4}$	$z^2=(y^2+x^{3+l})(x^3+y^{11})$
(61)	$\underline{N_0} + A_1\ (r=1, s=1)$	$\underline{-2}, -4$	$z^2=(x^2+y^4)(x^3+y^4)$
(62)	$\underline{A_{1,****}} + A_1$	$\underline{-2}, -2, -2, -4$	$z^2=x(x^4+y^6)$
(63)	$\underline{A_{n,****}} + A_1$	$-2, -2, \underline{-2}, -4$	$z^2=x(y^3+x^2)^2+x^a y^b$, $3a+2b=n+14$
(64)	$\underline{N_0} + A_1\ (r=1, t=1, s \geqslant 0)$	$\underline{-2}, -3, -3$	$z^2=(x^2+y^{5+s})(x^3+y^4)$
(65)	$\underline{A_{2,****}} + A_1$	$\underline{-2}, -2, -3, -3$	$z^2=(x^2+y^3)(x^3+y^5)$
(66)	$\underline{D_{4,***}} + A_1$	$\underline{-2}, -3, -3$	$z^2=x^5+y^8$
(67)	$\underline{A_{*,0}+A_{*,0}+A_{*,0}+A_{2,**,0}} + A_1$	$-2, -2, -2, \underline{-2}, -2$	$z^2=(x^3+y^6)(x^2+y^3)$
(68)	$\underline{A_{*,0}+A_{*,0}+D_{5,*,0}} + A_1$	$-2, -2, \underline{-2}$	$z^2=(x^2+y^4)(x^3+y^5)$
(69)	$\underline{A_{*,0}+A_{n,**,0}+A_{2,**,0}} + A_1$ $(n \geqslant 1)$	$-2, -2, -2, -2, \underline{-2}$	$z^2=(x+y^2)(x^2+y^3)(x^2+y^{4+n})$
(70)	$\underline{A_{*,0}+D_{7,*,0}} + A_1$	$-2, \underline{-2}$	$z^2=x(x^4+y^7)$
(71)	$\underline{D_{9,*,0}} + A_1$	$\underline{-2}$	$z^2=x^5+y^9$
(72)	$\underline{A_{2,**,0}+A'_{3,**,0}} + A_1$	$\underline{-2}, -2, -2, -2$	$z^2=(x^2+y^3)(x^3+y^7)$
(73)	$\underline{A_{2,**,0}+D_{5,*,0}} + A_1$	$\underline{-2}, -2, -2$	$z^2=x(x^2+y^3)(x^2+y^5)$
(74)	$\underline{A_{n,**,0}+D_{5,*,0}} + A_1$ $(n \geqslant 1)$	$-2, -2, \underline{-2}$	$z^2=(x^2+y^{4+n})(x^3+y^5)$
(75)	$\underline{A_{2,**,0}+E_{7,0}} + A_1$	$\underline{-2}, -2$	$z^2=(x^2+y^3)(x^3+y^8)$
(76)	$\underline{A'_{7,**,0}}$	$-2, -2$	$z^2=x^5+y^6$
(77)	$\underline{D_{9,*,0}}$	-2	$z^2=x^5+xy^5$
(78)	$\underline{A_{n,**,0}+A'_{3,**,0}}\ (n \geqslant 1)$	$-2, -2, -2, -2$	$z^2=(y^2+x^{n+2})(x^3+y^4)$
(79)	$\underline{A_{n,**,0}+D_{5,*,0}}\ (n \geqslant 1)$	$-2, -2, -2$	$z^2=(y^2+x^{n+2})(x^3+xy^3)$
(80)	$\underline{A_{n,**,0}+E_{7,0}}\ (n \geqslant 1)$	$-2, -2$	$z^2=(y^2+x^{n+2})(x^3+y^5)$

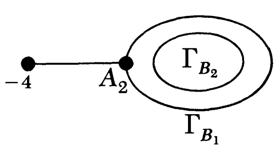
THEOREM 2.9. *Let $\pi: M \to V$ be the minimal good resolution of a normal two dimensional Stein space with p as its only weakly elliptic double point. Then the associated weighted dual graph is one of the form shown in Corollary 2.3, Corollary 2.7 and Corollary 2.8. Moreover any such weighted dual graph has a weakly elliptic double point structure.*

Theorem 2.9 gives a complete topological classification of weakly elliptic double points because of the following fact. Suppose $p_0 \in V_0$ and $p_1 \in V_1$ are isolated singularities of complex surfaces such that the graph of p_0 is the same as the graph of p_1. Then there are open neighborhoods $U_0 \ni p_0$ and $U_1 \ni p_1$ and a homeomorphism $h: U_0 \to U_1$, such that $h(p_0) = p_1$. For the proof, see Remark 3.9 of [26].

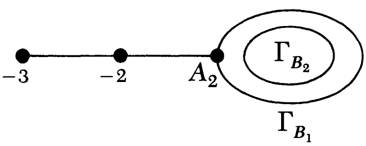
3. Topological Classification of Hypersurface Singularities with $h = \dim H^1(M, \mathcal{O}) = 2$. Rational singularities have $H^1(M, \mathcal{O}) \cong 0$. The hypersurface rational singularities are actually double points. For $H^1(M, \mathcal{O}) \cong \mathbb{C}$, Laufer was able to list all weighted dual graphs of hypersurface singularities. In this section, we are going to list all possible weighted dual graphs of hypersurface singularities with $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$.

PROPOSITION 3.1. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_0}, \ldots, Z_{B_l}, Z_E$ be the elliptic sequence. If $Z \cdot Z = -3$ and $Z_{B_1} \cdot Z_{B_1} = -1$, then Γ must be one of the following forms:*

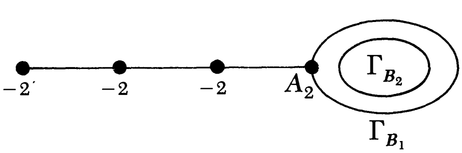
(1) $Z = 1Z_{B_1}$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$, $z_2 = 1$, $A_2 \cdot Z_{B_1} = -1$:



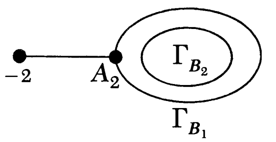
(2) $Z = 12D$, $|D| = B_1$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$, $z_2 = 2$, $A_2 \cdot Z_{B_1} = 0$:



(3) $Z = 123D$, $|D| = B_1$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$, $z_2 = 3$, $A_2 \cdot Z_{B_1} = 0$:



(4) $Z=3D$, $|D|=B_1$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$, $z_2=5$, $A_2 \cdot Z_{B_1}=0$:



where Γ_{B_i} *is the graph of* B_i.

Proof. Since $Z \cdot Z=-3$, by the definition of the elliptic sequence, we have the following three cases:

(I) There are only three distinct $A_1, A_2, A_3 \not\subseteq B_1$ such that $A_i \cap B_1 \neq \emptyset$, $1 \leqslant i \leqslant 3$.

(II) There are only $A_1, A_2 \not\subseteq B_1$ such that $A_1 \cap B_1 \neq \emptyset \neq A_2 \cap B_1$ and $A_1 \neq A_2$.

(III) There exists unique $A_1 \not\subseteq B_1$ such that $A_1 \cap B_1 \neq \emptyset$.

In the first case, we have $A_1 \cdot Z=-1=A_2 \cdot Z=A_3 \cdot Z$ and $z_1=z_2=z_3=1$. Since $0 \geqslant A_i \cdot (-K')=A_i \cdot (\Sigma_{i=0}^{l} Z_{B_i}+E) \geqslant A_i \cdot (Z+Z_{B_1})=0$ for $i=1,2,3$, we have $A_i \cap B_2=\emptyset$ and $A_i \cdot A_i=-2$ for $i=1,2,3$. Let $A_4, A_5, A_6 \subseteq B_1$ such that $A_1 \cdot A_4= A_2 \cdot A_5=A_3 \cdot A_6=1$. Then $z_4=z_5=z_6=1$. Hence $A_4 \cdot Z_{B_1}<0$, $A_5 \cdot Z_{B_1}<0$, $A_6 \cdot Z_{B_1}<0$. Since $Z_{B_1} \cdot Z_{B_1}=-1$, we have $A_4=A_5=A_6$ and $A_4 \cdot Z_{B_1}=-1$. However, $z_4=1$ will imply that $Z=A_1+A_2+A_3+Z_{B_1}$ and $A_4 \cdot Z=2>0$. This is a contradiction.

In the second case, there are two subcases.

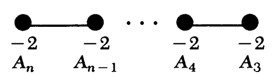
(IIA) $A_1 \cdot Z=-1=A_2 \cdot Z$. Since $0 \geqslant A_i \cdot (-K')=A_i \cdot (\Sigma_{i=0}^{l} Z_{B_i}+E) \geqslant A_i \cdot (Z+Z_{B_1}) =0$, we have $A_i \cap B_2=\emptyset$ and $A_i \cdot A_i=-2$ for $i=1,2$. We claim that there is no $A_3 \not\subseteq B_1$ such that $A_1 \neq A_3 \neq A_2$ and $A_3 \cdot Z<0$. Otherwise $A_3 \cdot Z=-1$ and $z_1=z_2 =z_3=1$. By our hypothesis, for any $A_i \not\subseteq B_1$, $A_1 \neq A_i \neq A_2$, we have $A_i \cap B_1=\emptyset$. Since A is connected, there exists $A_j \not\subseteq B_1$, $A_1 \neq A_j \neq A_2$ such that $A_j \cdot A_1=1$ or $A_j \cdot A_2=1$. It follows that either $A_1 \cdot Z \geqslant 0$ or $A_2 \cdot Z \geqslant 0$. This is a contradiction. Without loss of generality, we may assume that $z_1=1$, $z_2=2$. There is no $A_i \not\subseteq B_1$ such that $A_i \cdot A_1=1$. For $A_j \not\subseteq B_1$, $A_2 \neq A_j \neq A_1$, we have $A_j \cdot Z=0=A_j \cdot Z_{B_1}$. So $A_j \cdot (-K')=A_j \cdot (\Sigma_{i=0}^{l} Z_{B_i}+E)=0$ and $A_j \cdot A_j=-2$. Let $A_3, A_4 \subseteq B_1$ such that $A_1 \cdot A_3=1=A_2 \cdot A_4$. Then $z_3=1$ and $A_3 \cdot Z_{B_1}<0$. Since $Z_{B_1} \cdot Z_{B_1}=-1$, we have $A_3 \cdot Z_{B_1} =-1$. If $A_3=A_4$, then $Z=D+Z_{B_1}$ where $|D|$ consists of those A_i which are not in B_1. Hence $A_3 \cdot Z_{B_1}=-3$. This is a contradiction. We conclude that $A_3 \neq A_4$. z_4 cannot equal one: otherwise $A_4 \cdot Z_{B_1}=-2$, which is absurd. Therefore $z_4 \geqslant 2$. For any $A_i \subseteq B_1$, $A_i \neq A_3$, we have $A_i \cdot Z_{B_1}=0$. Since B_1 is connected, there exists $A_5 \subseteq B_1$ such that $z_5 \geqslant {}_{B_1}z_5+1$ and $A_5 \cdot A_3=1$. However, $z_3=1$ and $A_3 \cdot Z_{B_1}=-1$ will imply that $A_3 \cdot Z \geqslant 1$, which is absurd.

(IIB) $A_1 \cdot Z = -1$ *and* $A_2 \cdot Z = -2$. In this subcase, we have $z_1 = 1 = z_2$. Since $0 \geqslant A_1 \cdot (-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = 0$, we have $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 = \varnothing$. Also $0 \geqslant A_2 \cdot (-K') = A_2 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_2 \cdot (Z + Z_{B_1}) = -2 + 1 = -1$. Either $A_2 \cdot A_2 = -2$ and $A_2 \cap B_2 \neq \varnothing$ or $A_2 \cdot A_2 = -3$, $A_2 \cap B_2 = \varnothing$. $A_2 \cdot A_2 = -2$ and $A_2 \cap B_2 \neq \varnothing$ cannot occur: otherwise $A_2 \cdot Z \geqslant -1$, which contradicts to our assumption $A_2 \cdot Z = -2$. Therefore $A_2 \cdot A_2 = -3$ and $A_2 \cap B_2 = \varnothing$. Let $A_3, A_4 \subseteq B_1$ such that $A_1 \cdot A_3 = 1$, $A_2 \cdot A_4 = 1$. Then $A = A_1 \cup A_2 \cup B_1$ and $z_3 = 1 = z_4$. Moreover, $Z = A_1 + A_2 + Z_{B_1}$ and $A_3 \cdot Z_{B_1} < 0$ and $A_4 \cdot Z_{B_1} < 0$. If $A_3 \neq A_4$, then $Z_{B_1} \cdot Z_{B_1} \leqslant -2$, which is absurd. If $A_3 = A_4$, then $A_3 \cdot Z_{B_1} = -1$, since $Z_{B_1} \cdot Z_{B_1} = -1$. Hence $A_3 \cdot Z = A_3 \cdot (A_1 + A_2 + Z_{B_1}) = 2 - 1 = 1$. This is again a contradiction.

In the third case, there are three subcases.

(IIIA) $A_1 \cdot Z = -3$. In this case, $z_1 = 1$, $0 \geqslant A_1 \cdot (-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = -3 + 1 = -2$. Either (i) $A_1 \cdot A_1 = -3$, $A_1 \cap B_2 \neq \varnothing$ and $A_1 \cap B_3 = \varnothing$, or (ii) $A_1 \cdot A_1 = -4$ and $A_1 \cap B_2 = \varnothing$, or (iii) $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 \neq \varnothing \neq A_1 \cap B_3$. If (i) holds, then $A_1 \cdot Z \geqslant -2$, since $z_1 = 1$. This is a contradiction. If (iii) holds, then $A_1 \cdot Z \geqslant -1$. This is also impossible. Suppose $A_1 \cdot A_1 = -4$ and $A_1 \cap B_2 = \varnothing$. Let $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$. Since $A_1 \cdot Z = -3$, we have $z_2 = 1$ and $A = A_1 \cup B_1$. Moreover, $Z = A_1 + Z_{B_1}$ and hence $A_2 \cdot Z_{B_1} = 1$. So we are in (1).
(IIIB) $A_1 \cdot Z = -2$. In this case, we have $z_1 = 1$. Otherwise $z_1 \geqslant 2$ would imply that $Z \cdot Z \leqslant -4$, which is absurd. $0 \geqslant A_1 \cdot (-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = -2 + 1 = -1$. Either (i) $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 \neq \varnothing$ or (ii) $A_1 \cdot A_1 = -3$ and $A_1 \cap B_2 = \varnothing$. If (i) holds, then $A_1 \cdot Z \geqslant -1$, since $z_1 = 1$. This is a contradiction. Suppose $A_1 \cdot A_1 = -3$ and $A_1 \cap B_2 = \varnothing$. Since $A_1 \cdot Z = -2$ and $z_1 = 1$, there is no $A_i \nsubseteq B_1$ such that $A_i \cdot A_1 = 1$. It follows that $A = A_1 \cup B_1$ and $Z = A_1 + Z_{B_1}$. But then $Z \cdot Z = (A_1 + Z_{B_1}) \cdot Z = A_1 \cdot Z = -2$. This is a contradiction.
(IIIC) $A_1 \cdot Z = -1$. Then $0 \geqslant A_1 \cdot (-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = 0$. So $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 = \varnothing$. z_1 cannot equal 1: otherwise $A = A_1 \cup B_1$ and $Z = A_1 + Z_{B_1}$, which implies that $Z \cdot Z = (A_1 + Z_{B_1}) \cdot Z = A_1 \cdot Z = -1$. This is a contradiction. Therefore either $z_1 = 2$ or $z_1 = 3$. Let $A_2 \subseteq B_1$ such that $A_2 \cdot A_1 = 1$, $A_2 \nsubseteq B_2$.
(IIIC α) $z_1 = 2$. Let $A_3 \nsubseteq B_1$, $A_3 \neq A_1$ such that $A_3 \cdot Z < 0$. Then $A_3 \cdot Z = -1$ and $z_3 = 1$. Since $2 + A_3 \cdot A_3 = A_3 \cdot (-K') = A_3 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) = A_3 \cdot Z = -1$, $A_3 \cdot A_3 = -3$. For any $A_i \nsubseteq B_1$, $A_3 \neq A_i \neq A_1$, we have $A_i \cdot Z = 0$ and $A_i \cap B_1 = \varnothing$. Hence, $A_i \cdot (-K') = A_i (\Sigma_{i=0}^{l} Z_{B_i} + E) = A_i \cdot Z = 0$ and $A_i \cdot A_i = -2$. There are four subcases.
(IIIC α i) $z_2 = 1$. In this case $Z/B_1 = Z_{B_1}$. Therefore $A_2 \cdot Z = 2A_2 \cdot A_1 + A_2 \cdot Z_{B_1} = 2 - 1 = 1$. This is impossible.

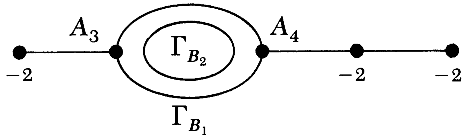
(IIIC α ii) $z_2=2$. Since $z_1=2$ and $A_1\cdot A_1=-2$, there exists $A_3 \not\subseteq B_1$, $A_3\cdot A_1=1$ and $z_3=1$. $A_3\cdot A_3$ is either -2 or -3. If $A_3\cdot A_3=-2$, then $A=B_1\cup A_1\cup A_3$ and $Z=A_3+2A_1+D$, where D is a positive cycle with $|D|=B_1$. Then $Z\cdot Z=(A_3+2A_1+D)\cdot Z=2A_1\cdot Z=-2$. This is a contradiction. So $A_3\cdot A_3=-3$ and we are in (2).
(IIIC α iii) $z_2=3$. Then $A=A_1\cup B_1$ and $Z=2A_1+D$, where D is a positive cycle with $|D|=B_1$. It follows that $Z\cdot Z=2A_1\cdot Z=-2$. This is a contradiction.
(IIIC α iv) $z_2\geqslant 4$. Then $A_1\cdot Z\geqslant 0$. This is impossible by our hypothesis.
(IIIC β) $z_1=3$. Since $Z\cdot Z=-3$ and $A_1\cdot Z=-1$, $A_i\cdot Z=0$ for any $A_i\not\subseteq B_1$, $A_i\neq A_1$. Moreover $0\geqslant A_i\cdot(-K')=A_i\cdot(\Sigma_{j=0}^{l}Z_{B_j}+E)=A_i\cdot Z=0$. Hence $A_i\cdot A_i=-2$.
(IIIC β i) $z_2=1$. Then $Z/B_1=Z_{B_1}$. Since $z_1=3$, we have $A_2\cdot Z\geqslant 3A_1\cdot A_2+A_2\cdot Z_{B_1}=3-1=2$. This is a contradiction.
(IIIC β ii) $z_2=2$. Let Γ_1 be the subgraph of Γ consisting of those $A_i\not\subseteq B_1$ such that $A_i\neq A_1$. Since $z_1=3$, $z_2=2$, $A_1\cdot A_1=-2$ and $A_1\cdot Z=-1$, we have $\deg A_1=2$. As $A_i\cdot A_i=-2$ for all A_i in Γ_1, Γ_1 is a graph of a rational double point. There exists a unique $A_3\subseteq\Gamma_1$ such that $z_3=3$. Because $A_3\cdot Z=0$ and $z_1=3$, $\deg A_3=2$. There exists a unique $A_4\subseteq\Gamma_1$ such that $z_4=3$ and $A_4\cdot A_3=1$. By induction Γ_1 is of the following form:



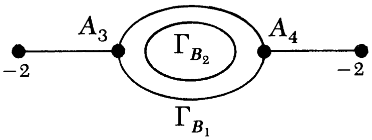
$Z=3A_n+3A_{n-1}+\cdots+3A_4+3A_3+3A_1+D$, where D is a positive cycle with $|D|=B_1$. Then $A_n\cdot Z=-3$ and $Z\cdot Z<-3$. This is a contradiction.
(IIIC β iii) $z_2=3$. It is easy to see that $\deg A_1=2$. Hence we are in (3).
(IIIC β iv) $z_2=4$. Since $z_1=3$, $A_1\cdot A_1=-2$ and $A_1\cdot Z=-1$, there exists a unique $A_3\not\subseteq B_1$ such that $A_3\cdot A_1=1$ and $z_3=1$. Then $A_3\cdot Z\geqslant 1$. This is a contradiction.
(IIIC β v) $z_2=5$. We are in (5).
(IIIC β vi) $z_2\geqslant 6$. In this case, $A_1\cdot Z\geqslant 0$. This is a contradiction. Q.E.D.

PROPOSITION 3.2. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists and $Z=Z_{B_0},\ldots,Z_{B_l},Z_E$ be the elliptic sequence. If $Z\cdot Z=-3$ and $Z_{B_1}\cdot Z_{B_1}=-2$, then Γ must be one of the following forms:*

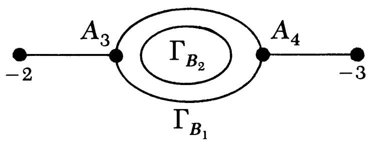
(1) $Z=1D21$, $z_3=1$, $z_4=2$, D is a positive cycle, $|D|=B_1$, $A_3\cdot Z_{B_1}<0$, $A_4\cdot Z_{B_1}=0$, $A_3,A_4\not\subseteq B_2$, $A_3\neq A_4$:



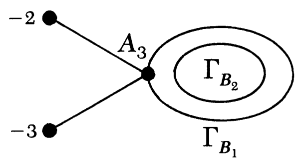
(2) $Z=1D2$, $z_3=1$, $z_4=3$, D is a positive cycle, $|D|=B_1$, $A_3\cdot Z_{B_1}<0$, $A_4\cdot Z_{B_1}=0$, $A_3,A_4\subseteq B_1$, $A_3,A_4\not\subseteq B_2$, $A_3\neq A_4$:



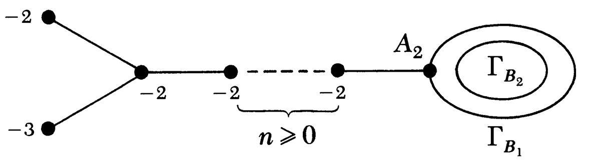
(3) $Z=1Z_{B_1}1$, $z_3=z_4=1$, $A_3\cdot Z_{B_1}=-1=A_4\cdot Z_{B_1}$, $A_3,A_4\subseteq B_1$, $A_3,A_4\not\subseteq B_2$, $A_3\neq A_4$:



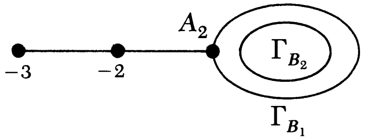
(4) $Z=\begin{smallmatrix}1\\1\end{smallmatrix}Z_{B_1}$, $z_3=1$, $A_3\cdot Z_{B_1}=-2$, $A_3\subseteq B_1$, $A_3\not\subseteq B_2$:



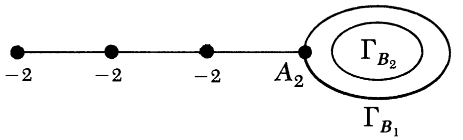
(5) $Z=\begin{smallmatrix}1\\1\end{smallmatrix}22\ldots2Z_{B_1}$, $z_2=1$, $A_2\cdot Z_{B_1}=-2$, $A_2\subseteq B_1$, $A_2\not\subseteq B_2$:



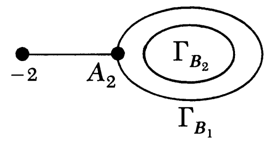
(6) $Z=12D$, D is a positive cycle, $|D|=B_1$, $z_2=2$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$, $A_2 \cdot Z_{B_1}=0$:



(7) $Z=123D$, D is a positive cycle, $|D|=B_1$, $z_2=3$, $A_2 \cdot Z_{B_1}=0$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$:



(8) $Z=3D$, D is a positive cycle, $|D|=B_1$, $z_2=5$, $A_2 \cdot Z_{B_1}=0$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$:



where Γ_{B_i} is the graph of B_i.

Proof. Since $Z \cdot Z=-3$, by the definition of elliptic sequence, we have the following three cases:

(I) There are only three distinct $A_1, A_2, A_3 \not\subseteq B_1$ such that $A_i \cap B_1 \neq \varnothing$, $1 \leqslant i \leqslant 3$.

(II) There are only two distinct $A_1, A_2 \not\subseteq B_1$ such that $A_i \cap B_1 \neq \varnothing$, $1 \leqslant i \leqslant 2$.

(III) There exists a unique $A_1 \not\subseteq B_1$ such that $A_1 \cap B_1 \neq \varnothing$.

In the first case, we have $A_1 \cdot Z=-1=A_2 \cdot Z=A_3 \cdot Z$ and $z_1=z_2=z_3=1$, $0 \geqslant A_i \cdot(-K')=A_i(\Sigma_{i=0}^{l} Z_{B_i}+E) \geqslant A_i(Z+Z_{B_1})=0$ for $i=1,2,3$. So $A_i \cdot A_i=-2$ and $A_i \cap B_2=\varnothing$, $1 \leqslant i \leqslant 3$. Let $A_4, A_5, A_6 \subseteq B_1$ such that $A_1 \cdot A_4=1=A_2 \cdot A_5=A_3 \cdot A_6$. Then $z_4=z_5=z_6=1$ and $Z=A_1+A_2+A_3+Z_{B_1}$. Hence $A_4 \cdot Z_{B_1}<0$, $A_5 \cdot Z_{B_1}<0$, and $A_6 \cdot Z_{B_1}<0$. If A_4, A_5, A_6 are distinct, then $Z_{B_1} \cdot Z_{B_1} \leqslant -3$. This is a contradiction. If $A_4=A_5 \neq A_6$, then $A_4 \cdot Z_{B_1}=-2$ because $A_1 \cdot A_4=A_2 \cdot A_4=1$ and $A_4 \cdot Z=0$. Again we get $Z_{B_1} \cdot Z_{B_1} \leqslant -3$, which is absurd. If $A_4=A_5=A_6$, then $A_4 \cdot Z_{B_1} \leqslant -3$. In particular, $Z_{B_1} \cdot Z_{B_1} \leqslant -3$. This is absurd.

In the second case, there are two subcases.

(IIA) $A_1 \cdot Z = -1 = A_2 \cdot Z$. We claim that there are no $A_3 \not\subseteq B_1$ such that $A_1 \neq A_3 \neq A_2$ and $A_3 \cdot Z < 0$. Otherwise $A_1 \cdot Z = A_2 \cdot Z = A_3 \cdot Z = -1$ and $z_1 = z_2 = z_3 = 1$. By our hypothesis, for any $A_i \not\subseteq B_1$, $A_2 \neq A_i \neq A_1$, we have $A_i \cap B_1 = \varnothing$. Since A is connected, $\exists A_j \not\subseteq B_1$, $A_1 \neq A_j \neq A_2$ such that $A_j \cdot A_1 = 1$ or $A_j \cdot A_2 = 1$. As $0 \geqslant A_i \cdot (-K') = A_i(\sum_{i=0}^{l} Z_{B_i} + E) \geqslant A_i \cdot (Z + Z_{B_1}) = 0$ for $i = 1, 2$, we have $A_1 \cdot A_1 = -2 = A_2 \cdot A_2$ and $A_1 \cap B_2 = \varnothing = A_2 \cap B_2$. It follows that either $A_1 \cdot Z \geqslant 0$ or $A_2 \cdot Z \geqslant 0$. This is a contradiction. Our claim is proved. Without loss of generality, we may assume that $z_1 = 1$, $z_2 = 2$. There is no $A_i \not\subseteq B_1$ such that $A_i \cdot A_1 = 1$. For any $A_j \not\subseteq B_1$, $A_2 \neq A_j \neq A_1$, we have $A_j \cdot Z = 0 = A_j \cdot Z_{B_1}$. So $A_j \cdot (-K') = A_j(\sum_{i=0}^{l} Z_{B_i} + E) = 0$ and $A_j \cdot A_j = -2$. Let $A_3, A_4 \subseteq B_1$ such that $A_4, A_3 \not\subseteq B_2$ and $A_1 \cdot A_3 = 1 = A_2 \cdot A_4$. Then $z_3 = 1$ and $A_3 \cdot Z_{B_1} < 0$. If $A_3 = A_4$, then $Z/B_1 = Z_{B_1}$. So $A_3 \cdot Z = 2A_1 \cdot A_3 + A_2 \cdot A_3 + A_3 \cdot Z_{B_1} \geqslant 1$. This is a contradiction. We conclude that $A_3 \neq A_4$. z_4 cannot equal 1. Otherwise $Z/B_1 = Z_{B_1}$ and $A_4 \cdot Z_{B_1} = -2$. This would imply that $Z_{B_1} \cdot Z_{B_1} \leqslant -3$, which is absurd. Suppose $z_4 = 2$. Then there exists unique $A_5 \not\subseteq B_1$ such that $A_2 \neq A_5 \neq A_1$, $A_5 \cdot A_2 = 1$ and $z_5 = 1$. It follows that $A = A_1 \cup A_2 \cup A_5 \cup B_1$. If $A_4 \cdot Z_{B_1} < 0$, then $Z = A_1 + A_2 + A_5 + Z_{B_1}$. This is a contradiction. So $A_4 \cdot Z_{B_1} = 0$ and we are in (1). Suppose $z_4 = 3$ then $A = A_1 \cup A_2 \cup B_1$. Similar argument to the above will show that $A_4 \cdot Z_{B_1} = 0$ and we are in (2).

(IIB) $A_1 \cdot Z = -1$ *and* $A_2 \cdot Z = -2$. In this case, $z_1 = 1 = z_2$. $0 \geqslant A_1 \cdot (-K') = A_1 \cdot (\sum_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = 0$. Hence $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 = \varnothing$. Since $0 \geqslant A_2 \cdot (-K') = A_2 \cdot (\sum_{i=0}^{l} Z_{B_i} + E) \geqslant A_2 \cdot (Z + Z_{B_1}) = -1$, either $A_2 \cdot A_2 = -2$ and $A_2 \cap B_2 \neq \varnothing$ or $A_2 \cdot A_2 = -3$ and $A_2 \cap B_2 = \varnothing$. $A_2 \cdot A_2$ cannot equal -2: otherwise $A_2 \cdot Z \geqslant -1$, which is a contradiction. Therefore $A_2 \cdot A_2 = -3$ and $A_2 \cap B_2 = \varnothing$. Let $A_3, A_4 \subseteq B_1$, $A_3, A_4 \not\subseteq B_2$ such that $A_1 \cdot A_3 = 1 = A_2 \cdot A_4$. Then $A = A_1 \cup A_2 \cup B_1$ and $z_3 = z_4 = 1$. Moreover, $Z = A_1 + A_2 + Z_{B_1}$ and $A_3 \cdot Z_{B_1} < 0$, $A_4 \cdot Z_{B_1} < 0$. If $A_3 \neq A_4$, then $A_3 \cdot Z_{B_1} = -1 = A_4 \cdot Z_{B_1}$ and we are in (3). If $A_3 = A_4$, then $A_3 \cdot Z_{B_1} = -2$ and we are in (4).

In the third case, there are three subcases.

(IIIA) $A_1 \cdot Z = -3$. In this case $z_1 = 1$. Since $0 \geqslant A_1 \cdot (-K') = A_1(\sum_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = -3 + 1 = -2$, either (i) $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 \neq \varnothing \neq A_1 \cap B_3$, or (ii) $A_1 \cdot A_1 = -3$, $A_1 \cap B_2 \neq \varnothing$ and $A_1 \cap B_3 = \varnothing$, or (iii) $A_1 \cdot A_1 = -4$ and $A_1 \cap B_2 = \varnothing$. In case (i), $A_1 \cdot Z \geqslant -1$, which is absurd. In case (ii), $A_1 \cdot Z \geqslant -2$, which is also absurd. In case (iii), it is easy to see that $A = A_1 \cup B_1$. Let $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$ such that $A_1 \cdot A_2 = 1$. Then $z_2 = 1$ and hence $Z = A_1 + Z_{B_1}$. Since $A_2 \cdot Z = 0$, this implies that $A_2 \cdot Z_{B_1} = -1$. However, $Z_{B_1} \cdot Z_{B_1} = -2$, so there exists $A_3 \subseteq B_1$, $A_3 \neq A_2$ such that $A_3 \cdot Z_{B_1} = -1$. It follows that $A_3 \cdot Z = (A_1 + Z_{B_1}) \cdot A_3 = A_3 \cdot Z_{B_1} = -1 < 0$. This is a contradiction.

(IIIB) $A_1 \cdot Z = -2$. In this case, $z_1 = 1$. Otherwise $z_1 \geqslant 2$ and $Z \cdot Z \leqslant -4$. Since $0 \geqslant A_1 \cdot (-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = -2 + 1 = -1$, either (i) $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 \neq \varnothing$ or (ii) $A_1 \cdot A_1 = -3$ and $A_1 \cap B_2 = \varnothing$. If (i) holds, then $A_1 \cdot Z \geqslant 0$, which is absurd. Suppose (ii) holds. It follows easily that $A = A_1 \cup B_1$ and $Z = A_1 + Z_{B_1}$. Then $Z \cdot Z = (A_1 + Z_{B_1}) \cdot Z = A_1 \cdot Z = -2$. This is a contradiction.

(IIIC) $A_1 \cdot Z = -1$. Since $0 \geqslant A_1 \cdot (-K') = A_1 (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_1 \cdot (Z + Z_{B_1}) = 0$, we have $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 = \varnothing$. There are three subcases.

(IIIC α) $z_1 = 1$. In this case, $A = A_1 \cup B_1$ and $Z = A_1 + Z_{B_1}$. Hence $Z \cdot Z = (A_1 + Z_{B_1}) \cdot Z = A_1 \cdot Z = -1$. This is a contradiction.

(IIIC β) $z_1 = 2$. Let $A_3 \not\subseteq B_1$, $A_3 \neq A_1$ such that $A_3 \cdot Z < 0$. Then $z_3 = 1$, $A_3 \cdot Z = -1$ and $A_3 \cap B_1 = \varnothing$. Since $A_3 \cdot (-K') = A_3 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) = A_3 \cdot Z = -1$, we have $A_3 \cdot A_3 = -3$. For $A_i \not\subseteq B_1$, $A_3 \neq A_i \neq A_1$, we have $A_i \cdot Z = 0$ and $A_i \cap B_1 = \varnothing$. Because $A_i \cdot (-K') = A_i \cdot (\Sigma_{i=0}^{l} Z_{B_0} + E) = A_i \cdot Z = 0$, we have $A_i \cdot A_i = -2$. Let $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$ such that $A_1 \cdot A_2 = 1$.

(IIIC β i) $z_2 = 1$. It is easy to see that we are in (5).

(IIIC β ii) $z_2 = 2$. It is easy to see that we are in (6).

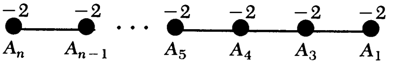
(IIIC β iii) $z_2 = 3$. In this case, $A = A_1 \cup B_1$ and $Z = 2A_1 + D$, where D is a positive cycle with $|D| = B_1$. It follows that $Z \cdot Z = (2A_1 + D) \cdot Z = 2A_1 \cdot Z = -2$. This is a contradiction.

(IIIC β iv) $z_2 \geqslant 4$. In this case, $A_1 \cdot Z \geqslant 0$, which is absurd.

(IIIC γ) $z_1 = 3$. Since $Z \cdot Z = -3$ and $A_1 \cdot Z = -1$, $A_i \cdot Z = 0$ for any $A_i \not\subseteq B_1$, $A_i \neq A_1$. Moreover, $0 \geqslant A_i \cdot (-K') = A_i \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_i \cdot (Z + Z_{B_1}) = 0$. Hence $A_i \cdot A_i = -2$ for all $A_i \neq A_1$ and $A_i \not\subseteq B_1$. Let $A_2 \subseteq B_1$ such that $A_2 \not\subseteq B_2$ and $A_1 \cdot A_2 = 1$.

(IIIC γ i) $z_2 = 1$. In this case $Z / B_1 = Z_{B_1}$. So $A_2 \cdot Z = 3A_1 \cdot A_2 + A_2 \cdot Z_{B_1} \geqslant 3 - 2 = 1$. This is a contradiction.

(IIIC γ ii) $z_2 = 2$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \not\subseteq B_1$. Since $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of a rational double point. Since $z_1 = 3$, $z_2 = 2$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, it is easy to see that $\deg A_1 = 2$. Hence there exists a unique $A_3 \subseteq \Gamma_1$ such that $z_3 = 3$. Since $A_3 \cdot Z = 0$ and $z_1 = 3$, $\deg A_3 = 2$. There exists a unique $A_4 \subseteq \Gamma_1$ such that $z_4 = 3$, $A_4 \neq A_1$ and $A_4 \cdot A_3 = 1$. By induction Γ_1 is of the following form:

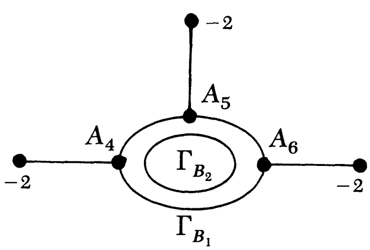


$Z = 3A_4 + \cdots + 3A_3 + 3A_1 + D$, where D is a positive cycle with $|D| = B_1$. Then $A_n \cdot Z = -3$ and $Z \cdot Z < -3$. This is a contradiction.

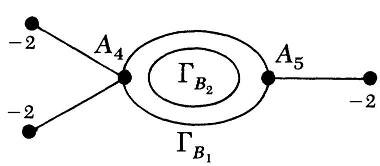
(IIIC γ iii) $z_2=3$. Since $z_1=3$, $A_1\cdot A_1=-2$ and $A_1\cdot Z=-1$, we have $2\leqslant \deg A_1 \leqslant 3$. It is not hard to see that $\deg A_1=3$ cannot occur. Therefore $\deg A_1=2$. It follows that we are in (7).
(IIIC γ iv) $z_2=4$. There exists a unique $A_3 \not\subseteq B_1$ such that $A_1\cdot A_3=1$ and $z_3=1$. Therefore $A_3\cdot Z\geqslant 3-2=1$. This is a contradiction.
(IIIC γ v) $z_2=5$. Since $z_1=3$, $z_2=5$, $A_1\cdot A_1=-2$ and $A_1\cdot Z=-1$, we have $A=A_1\cup B_1$. So we are in (8).
(IIIC γ vi) $z_2\geqslant 6$. Then $A_1\cdot Z\geqslant 0$, which is a contradiction.

PROPOSITION 3.3. *Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z=Z_{B_0},\ldots, Z_{B_l}, Z_E$ be the elliptic sequence. If $Z\cdot Z\leqslant -3$ and $Z_{B_1}\cdot Z_{B_1}=-3$, then Γ must be one of the following forms:*

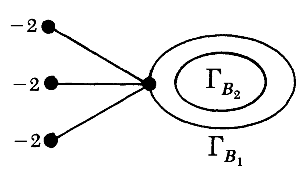
(1) $Z=1\,\overset{1}{Z_{B_1}}\,1$, $z_4=z_5=z_6=1$, $A_4\cdot Z_{B_1}=-1=A_5\cdot Z_{B_1}=A_6\cdot Z_{B_1}$: $A_4,A_5,A_6 \subseteq B_1$, $A_4,A_5,A_6 \not\subseteq B_2$, $A_4\neq A_5\neq A_6\neq A_4$:



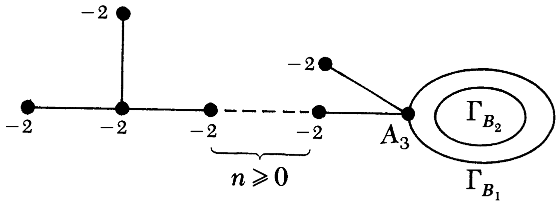
(2) $Z=\begin{smallmatrix}1\\1\end{smallmatrix} Z_{B_1}\,1$, $z_4=z_5=1$, $A_4\cdot Z_{B_1}=-2$, $A_5\cdot Z_{B_1}=-1$, $A_4,A_5\subseteq B_1$, $A_4,A_5 \not\subseteq B_2$, $A_4\neq A_5$:



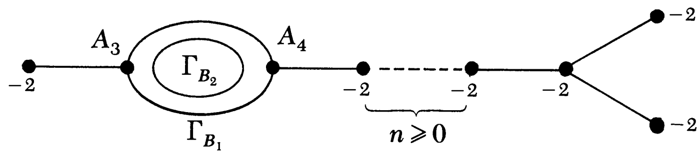
(3) $Z=\begin{matrix}1\\1\\1\end{matrix} Z_{B_1}$, $z_4=1$, $A_4\cdot Z_{B_1}=-3$, $A_4\subseteq B_1$, $A_4\not\subseteq B_2$:



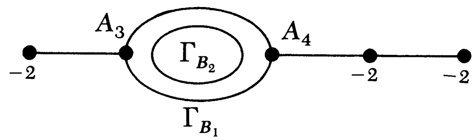
(4) $Z=1\overset{1}{2}2\ldots2^{1}Z_{B_1}$, $z_3=1$, $A_3\cdot Z_{B_1}=-3$, $A_3\subseteq B_1$, $A_3\not\subseteq B_2$:



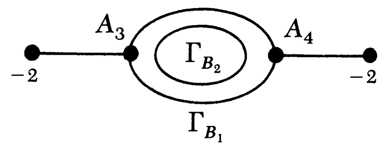
(5) $Z=1Z_{B_1}2\ldots22^{1}_{1}$, $z_3=z_4=1$, $A_3\cdot Z_{B_1}=-1$, $A_4\cdot Z_{B_1}=-2$, $A_3,A_4\subseteq B_1$, $A_3,A_4\not\subseteq B_2$, $A_3\neq A_4$:



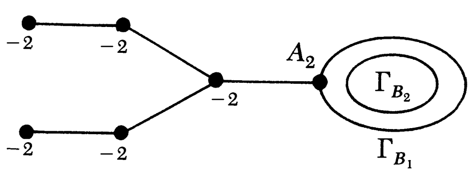
(6) $Z=1D21D$ is a positive cycle, $|D|=B_1$, $z_3=1$, $z_4=2$, $A_3\cdot Z_{B_1}<0$, $A_4\cdot Z_{B_1}=0$, $A_3,A_4\subseteq B_1$, $A_3,A_4\not\subseteq B_2$, $A_3\neq A_4$:



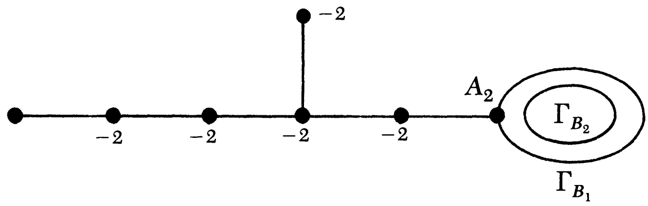
(7) $Z=1D2$, D is a positive cycle, $|D|=B_1$, $z_3=1$, $z_4=3$, $A_3\cdot Z_{B_1}<0$, $A_4\cdot Z_{B_1}=0$, $A_3,A_4\subseteq B_1$, $A_3,A_4\not\subseteq B_2$, $A_3\neq A_4$:



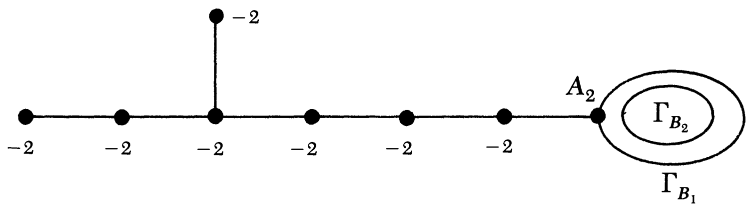
(8) $Z=\begin{smallmatrix}1&2\\1&2\end{smallmatrix}3Z_{B_1}$, $z_3=1$, $A_2\cdot Z_{B_1}=-3$, $A_2\subseteq B_1$, $A_2\not\subseteq B_2$:



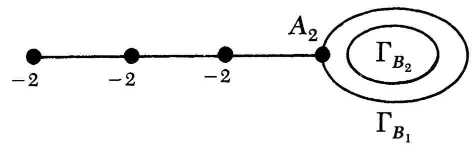
(9) $Z = 123\overset{2}{4}3 Z_{B_1}$, $z_2 = 1$, $A_2 \cdot Z_{B_1} = -3$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$:



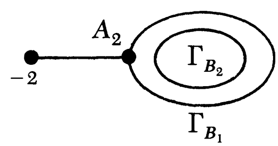
(10) $Z = 24\overset{3}{6}543 Z_{B_1}$, $z_2 = 1$, $A_2 \cdot Z_{B_1} = -3$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$:



(11) $Z = 123D$, D is a positive cycle, $|D| = B_1$, $z_2 = 3$, $A_2 \cdot Z_{B_1} = 0$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$:



(12) $Z = 3D$, D is a positive cycle, $|D| = B_1$, $z_2 = 5$, $A_2 \cdot Z_{B_1} = 0$, $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$:



where Γ_{Bi} is the graph of B_i.

Proof. Since $Z \cdot Z = Z_{B_1} \cdot Z_{B_1}$, $(Z - Z_{B_1}) \cdot (Z + Z_{B_1}) = 0$. For all $A_i \not\subseteq B_1$, we have $0 \geqslant A_i \cdot (-K') = A_i \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) \geqslant A_i \cdot (Z + Z_{B_1}) = 0$. Therefore, $A_i \cdot A_i = -2$ and $A_i \cap B_2 = \varnothing$ for all $A_i \not\subseteq B_1$. As $Z \cdot Z = -3$, by the definition of elliptic sequence, we have the following three cases.

(I) There exist $A_1, A_2, A_3 \not\subseteq B_1$ such that $A_i \cap B_1 \neq \varnothing$, $1 \leqslant i \leqslant 3$, and A_1, A_2, A_3 are distinct.

(II) There are only $A_1, A_2 \not\subseteq B_1$, $A_1 \neq A_2$, such that $A_1 \cap B_1 \neq \emptyset \neq A_2 \cap B_1$.
(III) There exists unique $A_1 \not\subseteq B_1$ such that $A_1 \cap B_1 \neq \emptyset$.

In the first case, we have $A_1 \cdot Z = -1 = A_2 \cdot Z = A_3 \cdot Z$ and $z_1 = z_2 = z_3 = 1$. It follows that there is no $A_i \not\subseteq B_1$, $A_i \neq A_j$, $1 \leqslant j \leqslant 3$, such that $A_i \cdot A_1 > 0$ or $A_i \cdot A_2 > 0$ or $A_i \cdot A_3 > 0$, i.e., $A = A_1 \cup A_2 \cup A_3 \cup B_1$. Let $A_4, A_5, A_6 \subseteq B_1$ such that $A_4, A_5, A_6 \not\subseteq B_2$ and $A_1 \cdot A_4 = 1 = A_2 \cdot A_5 = A_3 \cdot A_6$. Then $z_4 = z_5 = z_6$ and $A_4 \cdot Z < 0$, $A_5 \cdot Z < 0$, $A_6 \cdot Z < 0$. If A_4, A_5, A_6 are distinct, then $Z = A_1 + A_2 + A_3 + Z_{B_1}$ and we are in (1). If $A_4 = A_5 \neq A_6$, then $A_4 \cdot Z_{B_1} = -2$ because $A_4 \cdot Z = 0$ and $A_1 \cdot A_4 = A_2 \cdot A_4 = 1$. Hence $A_6 \cdot Z_{B_1} = -1$. We are in (2). Suppose $A_4 = A_5 = A_6$. Since $A_4 \cdot Z = 0$, we have $A_4 \cdot Z_{B_1} = -3$. We are in (3).

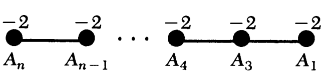
In the second case, we claim that there is no $A_3 \not\subseteq B_1$ such that $A_1 \neq A_3 \neq A_2$ and $A_3 \cdot Z < 0$. Otherwise $A_1 \cdot Z = -1 = A_2 \cdot Z = A_3 \cdot Z$ and $z_1 = z_2 = z_3 = 1$. By our hypothesis, $A_i \cap B_1 = \emptyset$ for any $A_i \not\subseteq B_1$, $A_1 \neq A_i \neq A_2$. Since A is connected, there exists $A_j \not\subseteq B_1$, $A_2 \neq A_j \neq A_1$, such that $A_j \cdot A_1 = 1$ or $A_j \cdot A_2 = 1$. Consequently, either $A_1 \cdot Z \geqslant 0$, or $A_2 \cdot Z > 0$. This is a contradiction. Our claim is proved. There are two subcases:

(IIA) $A_1 \cdot Z = -1$, $A_2 \cdot Z = -2$. Since $Z \cdot Z = -3$, we conclude that $z_1 = 1$, $z_2 = 1$. However, $A_2 \cdot A_2 = -2$, so $A_2 \cdot Z \geqslant -1$. This is a contradiction.
(IIB) $A_1 \cdot Z = -1 = A_2 \cdot Z$. Without loss of generality, we may assume that $z_1 = 1$, $z_2 = 2$. Let $A_3, A_4 \subseteq B_1$, $A_3, A_4 \not\subseteq B_2$ such that $A_1 \cdot A_3 = 1 = A_2 \cdot A_4$. Since $z_1 = 1$ and $A_1 \cdot A_1 = -2$, we have $z_3 = 1$ and $A_3 \cdot Z_{B_1} < 0$. If $A_3 = A_4$, then $A_3 \cdot Z_{B_1} = -3$ because $A_3 \cdot Z = A_3(2A_2 + A_1 + Z_{B_1}) = 0$. As $A_2 \cdot A_2 = -2$ and $A_2 \cdot Z = -1$, we have $2 \leqslant \deg A_2 \leqslant 3$. We are in (4). Suppose $A_3 \neq A_4$. Because $A_2 \cdot Z = -1$, the proof breaks up into four subcases.
(IIB i) *There exist* $A_5, A_6 \not\subseteq B_1$, $A_1 \neq A_5 \neq A_6 \neq A_1$, *such that* $A_5 \cdot A_2 = A_6 \cdot A_2 = 1$. In this case, we have $z_4 = z_5 = z_6 = 1$. Hence $Z = A_1 + A_5 + A_6 + 2A_2 + Z_{B_1}$. $A_4 \cdot Z = 0$, $A_3 \cdot Z = 0$ imply that $A_4 \cdot Z_{B_1} = -2$, $A_3 \cdot Z_{B_1} = -1$. Then we are in (5).
(IIB ii) *There exists a unique* $A_5 \not\subseteq B_1$, $A_1 \neq A_5 \neq A_2$, *such that* $A_5 \cdot A_2 = 1$, $z_4 = 1$ *and* $z_5 = 2$. In this case, $Z/B_1 = Z_{B_1}$. So $A_4 \cdot Z_{B_1} = -2$ and $A_3 \cdot Z_{B_1} = -1$. Since $A_5 \cdot A_5 = -2$, $A_5 \cdot Z = 0$ and $z_5 = z_2 = 2$, we have $2 \leqslant \deg A_5 \leqslant 3$. It follows easily that we are in (5).
(IIB iii) *There exists a unique* $A_5 \not\subseteq B_1$, $A_1 \neq A_5 \neq A_2$ *such that* $A_5 \cdot A_2 = 1$, $z_5 = 1$ *and* $z_4 = 2$. In this case $A = A_1 \cup A_2 \cup A_5 \cup B_1$. Since $z_4 > 1$, it is easy to see that $A_4 \cdot Z_{B_1} = 0$. We are in (6).
(IIB iv) $z_4 = 3$. In this case, $A = A_1 \cup A_2 \cup B_1$. Hence $A_4 \cdot Z_{B_1} = 0$. We are in (7).

In the third case, there are three subcases.

(IIIA) *There exist* $A_2, A_3 \not\subseteq B_1$ *such that* A_1, A_2, A_3 *are distinct and* $A_2 \cdot Z < 0$,

$A_3 \cdot Z < 0$. Because $Z \cdot Z = -3$, we have $A_1 \cdot Z = A_2 \cdot Z = A_3 \cdot Z = -1$ and $z_1 = z_2 = z_3 = 1$. There exists $A \not\subseteq B_1$ such that $A_i \cdot A_1 = 1$. Since $z_1 = 1$ and $A_1 \cdot A_1 = -2$, we have $A_1 \cdot Z \geqslant 0$. This is a contradiction.
(IIIB) *There exists a unique* $A_2 \not\subseteq B_1$ *such that* $A_2 \neq A_1$ *and* $A_2 \cdot Z < 0$. Since $Z \cdot Z = -3$, we have three subcases.
(IIIB α) $z_1 = 2$, $z_2 = 1$. In this case, we have $A_1 \cdot Z = -1 = A_2 \cdot Z$. Let $A_3 \subseteq B_1$, $A_3 \not\subseteq B_2$ such that $A_1 \cdot A_3 = 1$. If $z_3 = 2$, then there exists a unique $A_4 \not\subseteq B_1$, such that $A_4 \cdot A_1 = 1$ and $z_4 = 1$. It follows easily that $A = A_1 \cup A_4 \cup B_1$. Since $z_4 = 1$, $z_1 = 2$ and $A_4 \cdot A_4 = -2$, we have $A_4 \cdot Z = 0$. This implies that $A_4 \neq A_2$, which is absurd. If $z_3 = 1$, then $Z/B_1 = Z_{B_1}$. Since $0 = A_3 \cdot Z = A_3 \cdot (2A_1 + Z_{B_1}) = 2 + A_3 \cdot Z_{B_1}$, we have $A_3 \cdot Z_{B_1} = -2$. As $Z_{B_1} \cdot Z_{B_1} = -3$, there exists $A_i \not\subseteq B_1$ such that $A_i \cdot Z_{B_1} = -1$. It follows that $A_i \cdot Z = A_i \cdot Z_{B_1} = -1 < 0$. This is a contradiction.
(IIIB β) $z_1 = 1$, $z_2 = 2$. In this case, there exists $A_i \not\subseteq B_1$, $A_i \cdot A_1 = 1$. Since $z_1 = 1$, $A_1 \cdot A_1 = -2$, we have $A_1 \cdot Z \geqslant 0$. This is a contradiction.
(IIIB γ) $z_1 = 1 = z_2$. The same argument as (IIIB β) shows that this case cannot occur.
(IIIC) *There is no* $A_i \not\subseteq B_1$, $A_i \neq A_1$, *such that* $A_i \cdot Z < 0$. In this case, $A_1 \cdot Z = -1$ and $z_1 = 3$. Otherwise, $A_1 \cdot Z \leqslant -2$ and $A_1 \cdot (-K') = A_1 \cdot (\Sigma_{i=0}^{l} Z_{B_i} + E) = A_1 \cdot (Z + Z_{B_1}) \leqslant -1$. This would imply that $A_1 \cdot A_1 \leqslant -3$, which is a contradiction. Let $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$ and $A_2 \not\subseteq B_2$. There are five subcases.
(IIC i) $z_2 = 1$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \not\subseteq B_1$. Since $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of a rational double point. Since $z_1 = 3$, $z_2 = 1$ and $A_1 \cdot Z = -1$, it is not hard to check that we are in (8), (9) or (10).
(IIC ii) $z_2 = 2$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \not\subseteq B_1$. Since $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of a rational double point. As $z_1 = 3$, $z_2 = 2$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, we have $\deg A_1 = 2$. There exists a unique $A_3 \subseteq \Gamma_1$ such that $z_3 = 3$. Since $A_3 \cdot Z = 0$ and $z_1 = 3$, we have $\deg A_3 = 2$. There exists a unique $A_4 \subseteq \Gamma_1$ such that $z_4 = 3$, $A_4 \neq A_1$ and $A_4 \cdot A_3 = 1$. By induction, Γ_1 is of the following form:



$Z = 3A_n + \cdots + 3A_3 + 3A_1 + D$, where D is a positive cycle with $|D| = B_1$. Then $A_n \cdot Z = -3$ and $Z \cdot Z < -3$. This is a contradiction.
(IIIC iii) $z_2 = 3$. Then we are in (11).
(IIIC iv) $z_2 = 4$. Since $z_1 = 3$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, there exists a unique $A_3 \not\subseteq B_1$ such that $A_3 \cdot A_1 = 1$ and $z_3 = 1$. Then $A_3 \cdot Z \geqslant 1 > 0$, which is absurd.
(IIIC v) $z_2 = 5$. Then we are in (12). Q.E.D.

THEOREM 3.4. *Let $\pi: M \to V$ be a resolution of normal two dimensional Stein space with p as its only singular point. If* $\dim H^1(M, \mathcal{O}) \leqslant 2$ *and p is a hypersurface singularity, then the multiplicity ${}_V\mathcal{O}_p$ is less than or equal to* 3.

Proof. This is a trivial consequence of Theorem 1.3.

COROLLARY 3.5. *Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only singular point. Suppose $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$. If p is a hypersurface singularity, then the elliptic sequence is one of the following forms:*

(I) *elliptic sequence is* $\{Z, Z_E\}$,
- (a) $Z \cdot Z = -3$, $Z_E \cdot Z_E = -1$,
- (b) $Z \cdot Z = -3$, $Z_E \cdot Z_E = -2$,
- (c) $Z \cdot Z = -3 = Z_E \cdot Z_E$,
- (d) $Z \cdot Z = -1$, $Z_E \cdot Z_E = -1$,
- (e) $Z \cdot Z = -2$, $Z_E \cdot Z_E = -1$,
- (f) $Z \cdot Z = -2$, $Z_E \cdot Z_E = -2$;

(II) *elliptic sequence is* $\{Z, Z_{B_1}, Z_E\}$
- (g) $Z \cdot Z = -2$, $Z_{B_1} \cdot Z_{B_1} = -1 = Z_E \cdot Z_E$,
- (h) $Z \cdot Z = -1 = Z_{B_1} \cdot Z_{B_1} = Z_E \cdot Z_E$;

(III) *elliptic sequence is* $\{Z, Z_{B_1}, Z_{B_2}, Z_E\}$,
- (i) $Z \cdot Z = -1 = Z_{B_1} \cdot Z_{B_1} = Z_{B_2} \cdot Z_{B_2} = Z_E \cdot Z_E$.

Proof. This is an easy consequence of Proposition 2.1, Theorem 3.4 and Theorem 0.15.

THEOREM 3.6. *Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only singular point. Suppose $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$ and p is a hypersurface singularity. Then the associated weighted dual graph is one of the forms in Table 4.*

Table 4. The Weighted Dual Graphs for Hypersurface Singularities with Geometric Genus $h = 2$.

	Dual Graph	$A_* \cdot A_*$
	I(a) $Z \cdot Z = -3$, $Z_E \cdot Z_E = -1$	
(1)	$\underline{\mathrm{El}} + F_{-4}$	$\underline{-1}$
(2)	$\underline{N_0} + F_{-4}$	$\left\{ \begin{matrix} \underline{-1}, r=0 \\ \underline{-3}, r \geqslant 1 \end{matrix} \right\}$
(3)	$\underline{\mathrm{Cu}} + F_{-4}$	$\underline{-1}$
(4)	$\underline{\mathrm{Ta}} + F_{-4}$	$-2, \underline{-3}$
(5)	$\underline{\mathrm{Tr}} + F_{-4}$	$-2, -2, \underline{-3}$
(6)	$\underline{A_{1,****}} + F_{-4}$	$-2, -2, -2, \underline{-3}$

Table 4. (cont.)

	Dual Graph	$A_* \cdot A_*$
(7)	$\underline{A_{n,****}}+F_{-4}$	$-2,-2,-2,\underline{-3}$
(8)	$\underline{D_{4,***}}+F_{-4}$	$-2,-2,\underline{-3}$
(9)	$\underline{E_{6,**}}+F_{-4}$	$-2,\underline{-3}$
(10)	$\underline{E_{8,*}}+F_{-4}$	$\underline{-3}$
(11)	$\underline{\mathrm{Tr}}+A_1$	$\underline{-2},-2,-3$
(12)	$\underline{N_0}+A_1\ (r=s=t=0)$	$\underline{-2},-2,-3$
	I(b) $Z\cdot Z=-3,\ Z_E\cdot Z_E=-2$	
(13)	$N_0+A_1+F_{-3}\ (r\geqslant 0,\ s\geqslant 0)$	$\underline{-3},-3$
(14)	$\underline{\mathrm{Ta}}+A_1+F_{-3}$	$\underline{-3},-3$
(15)	$\underline{\mathrm{Tr}}+A_1+F_{-3}$	$-2,\underline{-3},\underline{-3}$
(16)	$\underline{A_{1,****}}+A_1+F_{-3}$	$-2,-2,\underline{-3},\underline{-3}$
(17)	$\underline{A_{n,****}}+A_1+F_{-3}$	$-2,-2,\underline{-3},\underline{-3}$
(18)	$\underline{A_{n,****}}+A_1+F_{-3}$	$-2,\underline{-3},\underline{-3},-2$
(19)	$\underline{D_{4,***}}+A_1+F_{-3}$	$-2,\underline{-3},\underline{-3}$
(20)	$\underline{E_{6,**}}+A_1+F_{-3}$	$\underline{-3},\underline{-3}$
(21)	$\underline{\mathrm{El}}+A_1+F_{-3}$	$\underline{\underline{-2}}$
(22)	$\underline{N_0}+A_1+F_{-3}$	$\begin{cases}\underline{\underline{-2}},\ r=0\\ \underline{\underline{-4}},\ r\geqslant 1\end{cases}$
(23)	$\underline{\mathrm{Cu}}+A_1+F_{-3}$	$\underline{\underline{-2}}$
(24)	$\underline{\mathrm{Ta}}+A_1+F_{-3}$	$-2,\underline{\underline{-4}}$
(25)	$\underline{\mathrm{Tr}}+A_1+F_{-3}$	$-2,-2,\underline{\underline{-4}}$
(26)	$\underline{A_{1,****}}+A_1+F_{-3}$	$-2,-2,-2,\underline{\underline{-4}}$
(27)	$\underline{A_{n,****}}+A_1+F_{-3}$	$-2,-2,-2,\underline{\underline{-4}}$
(28)	$\underline{D_{4,***}}+A_1+F_{-3}$	$-2,-2,\underline{\underline{-4}}$
(29)	$\underline{E_{6,**}}+A_1+F_{-3}$	$-2,\underline{\underline{-4}}$
(30)	$\underline{E_{8,*}}+A_1+F_{-3}$	$\underline{\underline{-4}}$
(31)	$\underline{\mathrm{El}}+A'(m+2,-3)\ (m\geqslant 0)$	$\underline{-2}$
(32)	$\underline{N_0}+A'(m+2,-3)\ (m\geqslant 0)$	$\begin{cases}\underline{-2},\ r=0\\ \underline{-4},\ r\geqslant 1\end{cases}$
(33)	$\underline{\mathrm{Cu}}+A'(m+2,-3)\ (m\geqslant 0)$	$\underline{-2}$

Table 4. (cont.)

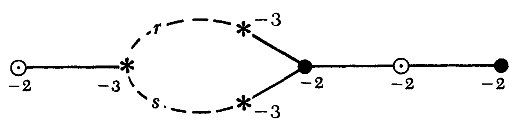
	Dual Graph	$A_* \cdot A_*$
(34)	$\underline{\mathrm{Ta}} + A'(m+2, -3)\ (m \geqslant 0)$	$-2,\ \underline{-4}$
(35)	$\underline{\mathrm{Tr}} + A'(m+2, -3)\ (m \geqslant 0)$	$-2, -2,\ \underline{-4}$
(36)	$\underline{A_{1,****}} + A'(m+2, -3)\ (m \geqslant 0)$	$-2, -2, -2,\ \underline{-4}$
(37)	$\underline{A_{n,****}} + A'(m+2, -3)\ (m \geqslant 0)$	$-2, -2, -2,\ \underline{-4}$
(38)	$\underline{D_{4,***}} + A'(m+2, -3)\ (m \geqslant 0)$	$-2, -2,\ \underline{-4}$
(39)	$\underline{E_{6,**}} + A'(m+2, -3)\ (m \geqslant 0)$	$-2,\ \underline{-4}$
(40)	$\underline{E_{8,*}} + A'(m+2, -3)\ (m \geqslant 0)$	$\underline{-4}$
(41)	$\underline{N_0} + AAA(0,0,-3,1)\ (r=0, s=0)$	$\underline{-2}, -4$
(42)	$\underline{\mathrm{Ta}} + AAA(0,0,-3,1)$	$\underline{-2}, -4$
(43)	$\underline{N_0} + AAA(0,0,-3,1)\ (r=0, s=0, t \geqslant 0)$	$-3, \underline{-2}, -3$
(44)	$\underline{\mathrm{Tr}} + AAA(0,0,-3,1)$	$\underline{-2}, -3, -3$
(45)	$\underline{A_{*,} + A_{*,0} + A_{*,} + A_{*,0} + A_{*,0}} + AAA(0,0,-3,1)$	$\underline{-2}, -2, -2, -2, -2$
(46)	$\underline{A_{*,} + A_{*,0} + A_{*,0} + A_{n,**,0}} + AAA(0,0,-3,1)$	$\underline{-2}, -2, -2, -2, -2$
(47)	$\underline{A_{*,} + A_{*,0} + A'_{3,**,0}} + AAA(0,0,-3,1)$	$\underline{-2}, -2, -2, -2$
(48)	$\underline{A_{*,} + A_{*,0} + D_{5,*,0}} + AAA(0,0,-3,1)$	$\underline{-2}, -2, -2$
(49)	$\underline{A_{*,} + A_{*,0} + E_{7,0}} + A_{*,} + A_{*,0} +$	$\underline{-2}, -2$
(50)	$\underline{A_{*,0} + A_{n,**,0} + A_{m,**,0}} + AAA(0,0,-3,1)$	$\underline{-2}, -2, -2, -2, -2$
(51)	$\underline{A_{*,0} + A'_{5,**,0}} + AAA(0,0,-3,1)$	$\underline{-2}, -2, -2$
(52)	$\underline{A_{*,0} + D_{7,*,0}} + AAA(0,0,-3,1)$	$\underline{-2}, -2$
	I(c) $Z \cdot Z = -3 = Z_E \cdot Z_E$	
(53)	$\underline{N_0} + A_1 + A_1 + A_1\ (r \geqslant 0, s \geqslant 0, t \geqslant 0)$	$\underline{-3}, \underline{-3}, \underline{-3}$
(54)	$\underline{\mathrm{Tr}} + A_1 + A_1 + A_1$	$\underline{-3}, \underline{-3}, \underline{-3}$
(55)	$\underline{A_{1,****}} + A_1 + A_1 + A_1$	$-2, \underline{-3}, \underline{-3}, \underline{-3}$
(56)	$\underline{A_{n,****}} + A_1 + A_1 + A_1$	$-2, \underline{-3}, \underline{-3}, \underline{-3}$
(57)	$\underline{D_{4,***}} + A_1 + A_1 + A_1$	$\underline{-3}, \underline{-3}, \underline{-3}$
(58)	$\underline{N_0} + A_1 + A_1 + A_1\ (r \geqslant 0, s \geqslant 0)$	$\underline{-3}, \underline{\underline{-4}}$
(59)	$\underline{\mathrm{Ta}} + A_1 + A_1 + A_1$	$\underline{-3}, \underline{\underline{-4}}$
(60)	$\underline{\mathrm{Tr}} + A_1 + A_1 + A_1$	$-2, \underline{-3}, \underline{\underline{-4}}$
(61)	$\underline{A_{1,****}} + A_1 + A_1 + A_1$	$-2, -2, \underline{-3}, \underline{\underline{-4}}$
(62)	$\underline{A_{n,****}} + A_1 + A_1 + A_1$	$-2, -2, \underline{-3}, \underline{\underline{-4}}$

Table 4. (cont.)

	Dual Graph	$A_* \cdot A_*$
(63)	$\underline{A_{n,****}} + A_1 + A_1 + A_1$	$-2, \underline{-3}, \underline{\underline{-4}}, -2$
(64)	$\underline{D_{4,****}} + A_1 + A_1 + A_1$	$-2, \underline{-3}, \underline{\underline{-4}}$
(65)	$\underline{E_{6,**}} + A_1 + A_1 + A_1$	$\underline{-3}, \underline{\underline{-4}}$
(66)	$\underline{\mathrm{El}} + A_1 + A_1 + A_1$	$\underline{\underline{\underline{-3}}}$
(67)	$\underline{N_0} + A_1 + A_1 + A_1$	$\begin{Bmatrix} \underline{\underline{-3}}, & r=0 \\ \underline{\underline{-5}}, & r \geqslant 1 \end{Bmatrix}$
(68)	$\underline{\mathrm{Cu}} + A_1 + A_1 + A_1$	$\underline{\underline{\underline{-3}}}$
(69)	$\underline{\mathrm{Ta}} + A_1 + A_1 + A_1$	$-2, \underline{\underline{\underline{-5}}}$
(70)	$\underline{\mathrm{Ta}} + A_1 + A_1 + A_1$	$-2, -2, \underline{\underline{\underline{-5}}}$
(71)	$\underline{A_{1,****}} + A_1 + A_1 + A_1$	$-2, -2, -2, \underline{\underline{\underline{-5}}}$
(72)	$\underline{A_{n,****}} + A_1 + A_1 + A_1$	$-2, -2, -2, \underline{\underline{\underline{-5}}}$
(73)	$\underline{D_{4,***}} + A_1 + A_1 + A_1$	$-2, -2, \underline{\underline{\underline{-5}}}$
(74)	$\underline{E_{6,**}} + A_1 + A_1 + A_1$	$-2, \underline{\underline{\underline{-5}}}$
(75)	$\underline{E_{8,*}} + A_1 + A_1 + A_1$	$\underline{\underline{\underline{-5}}}$
(76)	$\underline{\mathrm{El}} + D(l+4) + A_1\ (l \geqslant -1)$	$\underline{\underline{-3}}$
(77)	$\underline{N_0} + D(l+4) + A_1\ (l \geqslant -1)$	$\begin{Bmatrix} \underline{\underline{-3}}, & r=0 \\ \underline{\underline{-5}}, & r \geqslant 1 \end{Bmatrix}$
(78)	$\underline{\mathrm{Cu}} + D(l+4) + A_1\ (l \geqslant -1)$	$\underline{\underline{-3}}$
(79)	$\underline{\mathrm{Ta}} + D(l+4) + A_1\ (l \geqslant -1)$	$-2, \underline{\underline{-5}}$
(80)	$\underline{\mathrm{Tr}} + D(l+4) + A_1\ (l \geqslant -1)$	$-2, -2, \underline{\underline{-5}}$
(81)	$\underline{A_{1,****}} + D(l+4) + A_1\ (l \geqslant -1)$	$-2, -2, -2, \underline{\underline{-5}}$
(82)	$\underline{A_{n,****}} + D(l+4) + A_1\ (l \geqslant -1)$	$-2, -2, -2, \underline{\underline{-5}}$
(83)	$\underline{D_{4,***}} + D(l+4) + A_1\ (l \geqslant -1)$	$-2, -2, \underline{\underline{-5}}$
(84)	$\underline{E_{6,**}} + D(l+4) + A_1\ (l \geqslant -1)$	$-2, \underline{\underline{-5}}$
(85)	$\underline{E_{8,*}} + D(l+4) + A_1\ (l \geqslant -1)$	$\underline{\underline{-5}}$

Table 4. (cont.)

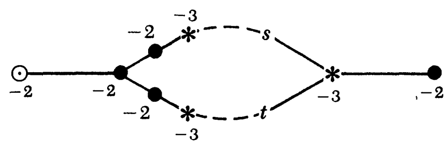
	Dual Graph	$A_* \cdot A_*$
(86)	$\underline{N_0} + A_1 + D(l+4)\ (l \geqslant -1)$	$\underline{-3}$, $\underline{-4}$
(87)	$\underline{\mathrm{Ta}} + A_1 + D(l+4)\ (l \geqslant -1)$	$\underline{-3}$, $\underline{-4}$
(88)	$\underline{\mathrm{Tr}} + A_1 + D(l+4)\ (l \geqslant -1)$	-2, $\underline{-3}$, $\underline{-4}$
(89)	$\underline{A_{1,****}} + A_1 + D(l+4)\ (l \geqslant -1)$	-2, -2, $\underline{-3}$, $\underline{-4}$
(90)	$\underline{A_{n,****}} + A_1 + D(l+4)\ (l \geqslant -1)$	-2, -2, $\underline{-3}$, $\underline{-4}$
(91)	$\underline{A_{n,****}} + A_1 + D(l+4)\ (l \geqslant -1)$	-2, $\underline{-3}$, $\underline{-4}$, -2
(92)	$\underline{D_{4,***}} + A_1 + D(l+4)\ (l \geqslant -1)$	-2, $\underline{-3}$, $\underline{-4}$
(93)	$\underline{E_{6,**}} + A_1 + D(l+4)\ (l \geqslant -1)$	$\underline{-3}$, $\underline{-4}$
(94)	$\underline{N_0} + A_1 + A_2\ (r=0,\ s=0)$	$\underline{-5}$, $\underline{-2}$
(95)	$\underline{\mathrm{Ta}} + A_1 + A_2$	$\underline{-5}$, $\underline{-2}$
(96)	$\underline{N_0} + A_2 + A_1\ (r=0,\ t=0,\ s \geqslant 0)$	$\underline{-2}$, -3, $\underline{-4}$
(97)	$\underline{\mathrm{Tr}} + A_2 + A_1$	$\underline{-2}$, -3, $\underline{-4}$
(98)	$\underline{N_0} + A_1 + A_2\ (s=1,\ r \geqslant 0,\ t \geqslant 0)$	$\underline{-3}$, -3, -3



	Dual Graph	$A_* \cdot A_*$
(99)	$\underline{A_{*,0}+A_{*,0}+A_{*,0}+A_{*,0}+A_{*,0}} + A_1 + A_2$	$\underline{-3}$, -2, -2, -2, $\underline{-2}$
(100)	$\underline{A_{*,0}+A_{*,0}+A_{*,0}+A_{n,**,0}} + A_1 + A_2$	$\underline{-3}$, $\underline{-2}$, -2, -2, -2
(101)	$\underline{A_{*,0}+A_{*,0}+A_{*,0}+A_{n,**,0}} + A_1 + A_2$	$\underline{-2}$, -2, -2, -2, $\underline{-3}$
(102)	$\underline{A_{*,0}+A_{*,0}+A'_{3,**,0}} + A_2 + A_1$	$\underline{-2}$, -2, -2, $\underline{-3}$
(103)	$\underline{A_{*,0}+A_{*,0}+A'_{3,**,0}} + A_1 + A_2$	$\underline{-3}$, $\underline{-2}$, -2, -2
(104)	$\underline{A_{*,0}+A_{*,0}+D_{5,*,0}} + A_1 + A_2$	$\underline{-3}$, $\underline{-2}$, -2
(105)	$\underline{A_{*,0}+A_{*,0}+D_{5,*,0}} + A_2 + A_1$	$\underline{-2}$, -2, $\underline{-3}$
(106)	$\underline{A_{*,0}+A_{*,0}+E_{7,0}} + A_1 + A_2$	$\underline{-3}$, $\underline{-2}$
(107)	$\underline{A_{*,0}+A_{n,**,0}+A_{m,**,0}} + A_2 + A_1$	$\underline{-2}$, -2, -2, -2, $\underline{-3}$
(108)	$\underline{A_{*,0}+A'_{5,**,0}} + A_2 + A_1$	$\underline{-2}$, -2, $\underline{-3}$
(109)	$\underline{A_{*,0}+D_{7,*,0}} + A_2 + A_1$	$\underline{-2}$, $\underline{-3}$
(110)	$\underline{N_0} + A_1 + A_1\ (r=1,\ s=1)$	$\underline{-5}$, $\underline{-2}$
(111)	$\underline{A_{1,****}} + A_1 + A_1$	-2, $\underline{-2}$, -2, $\underline{-5}$

Table 4. (cont.)

	Dual Graph	$A_* \cdot A_*$
(112)	$\underline{A_{n,****}} + A_1 + A_1$	$-2, -2, \underline{-2}, \underline{-5}$
(113)	$\underline{N_0} + A_1 + A_1$ $(r=1, t=1, s \geqslant 0)$	$\underline{-2}, -3, \underline{-4}$
(114)	$\underline{A_{2,****}} + A_1 + A_1$	$-2, \underline{-2}, -3, \underline{-4}$
(115)	$\underline{D_{4,***}} + A_1 + A_1$	$\underline{-2}, -3, \underline{-4}$
(116)	$\underline{A_{*,0} + A_{*,0} + A_{*,0} + A_{2,**,0}} + A_1 + A_1$	$\underline{-3}, -2, -2, -2, \underline{-2}$
(117)	$\underline{N_0} + A_1 + A_1$ $(r=3, s \geqslant 0, t \geqslant 0)$	$\underline{-3}, -3, -3$



(118)	$\underline{A_{*,0} + A_{*,0} + D_{5,*,0}} + A_1 + A_1$	$\underline{-3}, -2, \underline{-2}$
(119)	$\underline{A_{*,0} + A_{n,**,0} + A_{2,**,0}} + A_1 + A_1$	$\underline{-3}, -2, -2, -2, -2$
(120)	$\underline{A_{*,0} + A_{n,**,0} + A_{2,**,0}} + A_1 + A_1$	$-2, \underline{-3}, -2, -2, \underline{-2}$
(121)	$\underline{A_{*,0} + D_{7,*,0}} + A_1 + A_1$	$\underline{-3}, \underline{-2}$
(122)	$\underline{A_{2,**,0} + A'_{3,**,0}} + A_1 + A_1$	$\underline{-2}, -2, -2, \underline{-3}$
(123)	$\underline{A_{2,**,0} + D_{5,*,0}} + A_1 + A_1$	$\underline{-2}, -2, \underline{-3}$
(124)	$\underline{\mathrm{El}} + A''_5$	$\underline{-3}$
(125)	$\underline{N_0} + A''_5$	$\left\{\begin{matrix} \underline{-3}, r=0 \\ \underline{-5}, r \geqslant 1 \end{matrix}\right\}$
(126)	$\underline{\mathrm{Cu}} + A''_5$	$\underline{-3}$
(127)	$\underline{\mathrm{Ta}} + A''_5$	$-2, \underline{-5}$
(128)	$\underline{\mathrm{Tr}} + A''_5$	$-2, -2, \underline{-5}$
(129)	$\underline{A_{1,***}} + A''_5$	$-2, -2, -2, \underline{-5}$
(130)	$\underline{A_{n,***}} + A''_5$	$-2, -2, -2, \underline{-5}$
(131)	$\underline{D_{4,***}} + A''_5$	$-2, -2, \underline{-5}$
(132)	$\underline{E_{6,**}} + A''_5$	$\underline{-5}, -2$
(133)	$\underline{E_{8,*}} + A''_5$	$\underline{-5}$
(134)	$\underline{\mathrm{El}} + D_6$	$\underline{-3}$
(135)	$\underline{N_0} + D_6$	$\left\{\begin{matrix} \underline{-3}, r=0 \\ \underline{-5}, r \geqslant 1 \end{matrix}\right\}$

Table 4. (cont.)

	Dual Graph	$A_*\cdot A_*$
(136)	$\underline{\mathrm{Cu}} + D_6$	$\underline{-3}$
(137)	$\underline{\mathrm{Ta}} + D_6$	$-2, \underline{-5}$
(138)	$\underline{\mathrm{Tr}} + D_6$	$-2, -2, \underline{-5}$
(139)	$\underline{A_{1,****}} + D_6$	$-2, -2, -2, \underline{-5}$
(140)	$\underline{A_{n,****}} + D_6$	$-2, -2, -2, \underline{-5}$
(141)	$\underline{D_{4,***}} + D_6$	$-2, -2, \underline{-5}$
(142)	$\underline{E_{6,**}} + D_6$	$-2, \underline{-5}$
(143)	$\underline{E_{8,*}} + D_6$	$\underline{-5}$
(144)	$\underline{\mathrm{El}} + E_7$	$\underline{-3}$
(145)	$\underline{N_0} + E_7$	$\left\{\begin{array}{l}\underline{-3}, r=0\\ \underline{-5}, r\geqslant 1\end{array}\right\}$
(146)	$\underline{\mathrm{Cu}} + E_7$	$\underline{-3}$
(147)	$\underline{\mathrm{Ta}} + E_7$	$-2, \underline{-5}$
(148)	$\underline{\mathrm{Tr}} + E_7$	$-2, -2, \underline{-5}$
(149)	$\underline{A_{1,****}} + E_7$	$-2, -2, -2, \underline{-5}$
(150)	$\underline{A_{n,****}} + E_7$	$-2, -2, -2, \underline{-5}$
(151)	$\underline{D_{4,***}} + E_7$	$-2, -2, \underline{-5}$
(152)	$\underline{E_{6,***}} + E_7$	$-2, \underline{-5}$
(153)	$\underline{E_{8,*}} + \mathrm{E}_7$	$\underline{-5}$
(154)	$\underline{N_0} + A_3$ $(r=0, s=1)$	$\underline{-2}, -5$
(155)	$\underline{\mathrm{Tr}} + A_3$	$\underline{-2}, -2, -5$
(156)	$\underline{A_{1,****}} + A_3$	$\underline{-2}, -3, -3, -3$
(157)	$\underline{A_{*,0} + A_{*,0} + A_{*,0} + A_{1,**,0}} + A_3$	$-2, -2, -2, \underline{-2}, -3$
(158)	$\underline{A_{*,0} + A_{1,**,0} + A_{n,**,0}} + A_3$	$-2, -3, \underline{-2}, -2, -2$
(159)	$\underline{A_{1,**,0} + A'_{3,**,0}} + A_3$	$\underline{-2}, -3, -2, -2$
(160)	$\underline{A_{1,**,0} + D_{5,*,0}} + A_3$	$\underline{-2}, -3, -2$
(161)	$\underline{A_{1,**,0} + E_{7,0}} + A_3$	$\underline{-2}, -3$
(162)	$\underline{A_{1,*,0} + A_{1,*,0} + A_{1,*,0} + A_{1,*,0}} + A_3$	$\underline{-2}, -2, -2, -2$
(163)	$\underline{A_{1,*,0} + A_{1,*,0} + A_{4,*,0}} + A_3$	$\underline{-2}, -2, -2$

Table 4. (cont.)

	Dual Graph	$A_* \cdot A_*$
(164)	$\underline{A_{1,*,0}+A_{1,*,0}+E_{6,0}}+A_3$	$\underline{-2}, -2$
(165)	$\underline{A_{1,*,0}+A_{7,*,0}}+A_3$	$\underline{-2}, -2$
(166)	$\underline{N_0}+A_1\ (r=1, s=3)$	$\underline{-2}, -5$
(167)	$\underline{D_{4,***}}+A_1$	$\underline{-2}, -2, -5$
(168)	$\underline{N_0}+A_1\ (r\geqslant 0, s=3, t=1)$	$-4, -3, \underline{-2}$
	I(d) $Z\cdot Z=-1,\ Z_E\cdot Z_E=-1$	
(169)	$\underline{\mathrm{El}}+A_1$	$\underline{-1}$
(170)	$\underline{N_0}+A_1$	$\left\{\begin{matrix}\underline{-1}, r=0\\ \underline{-3}, r\geqslant 1\end{matrix}\right\}$
(171)	$\underline{\mathrm{Cu}}+A_1$	$\underline{-1}$
(172)	$\underline{\mathrm{Ta}}+A_1$	$-2, \underline{-3}$
(173)	$\underline{\mathrm{Tr}}+A_1$	$-2, -2, \underline{-3}$
(174)	$\underline{A_{1,****}}+A_1$	$-2, -2, -2, \underline{-3}$
(175)	$\underline{A_{n,****}}+A_1$	$-2, -2, -2, \underline{-3}$
(176)	$\underline{D_{4,***}}+A_1$	$-2, -2, \underline{-3}$
(177)	$\underline{E_{6,**}}+A_1$	$-2, \underline{-3}$
(178)	$\underline{E_{8,*}}+A_1$	$\underline{-3}$
	I(e) $Z\cdot Z=-2,\ Z_E\cdot Z_E=-1$	
(179)	$\mathrm{El}+F_{-3}$	-1
(180)	$\underline{N_0}+F_{-3}$	$\left\{\begin{matrix}\underline{-1}, r=0\\ \underline{-3}, r\geqslant 1\end{matrix}\right\}$
(181)	$\underline{\mathrm{Cu}}+F_{-3}$	$\underline{-1}$
(182)	$\underline{\mathrm{Ta}}+F_{-3}$	$-2, \underline{-3}$
(183)	$\underline{\mathrm{Tr}}+F_{-3}$	$-2, -2, \underline{-3}$
(184)	$\underline{A_{1,****}}+F_{-3}$	$-2, -2, -2, \underline{-3}$
(185)	$\underline{A_{n,****}}+F_{-3}$	$-2, -2, -2, \underline{-3}$
(186)	$\underline{D_{4,***}}+F_{-3}$	$-2, -2, \underline{-3}$
(187)	$\underline{E_{6,**}}+F_{-3}$	$-2, \underline{-3}$
(188)	$\underline{E_{8,*}}+F_{-3}$	$\underline{-3}$
(189)	$\underline{\mathrm{Ta}}+A_1$	$\underline{-2}, -3$

Table 4. (cont.)

	Dual Graph	$A_* \cdot A_*$
	I(f) $Z \cdot Z = -2$, $Z_E \cdot Z_E = -2$	
(190)	$\underline{N_0} + A_1 + A_1$ $(r \geqslant 0, s \geqslant 0)$	$\underline{-3}, \underline{-3}$
(191)	$\underline{\mathrm{Ta}} + A_1 + A_1$	$\underline{-3}, \underline{-3}$
(192)	$\underline{\mathrm{Tr}} + A_1 + A_1$	$-2, \underline{-3}, \underline{-3}$
(193)	$\underline{A_{1,****}} + A_1 + A_1$	$\underline{-3}, -2, -2, \underline{-3}$
(194)	$\underline{A_{n,****}} + A_1 + A_1$	$-2, -2, \underline{-3}, \underline{-3}$
(195)	$\underline{A_{n,****}} + A_1 + A_1$	$\underline{-3}, -2, -2, \underline{-3}$
(196)	$\underline{D_{4,***}} + A_1 + A_1$	$-2, \underline{-3}, \underline{-3}$
(197)	$\underline{E_{6,**}} + A_1 + A_1$	$\underline{-3}, \underline{-3}$
(198)	$\underline{\mathrm{El}} + A_1 + A_1$	$\underline{\underline{-2}}$
(199)	$\underline{N_0} + A_1 + A_1$	$\begin{Bmatrix} \underline{\underline{-2}}, r=0 \\ \underline{\underline{-4}}, r \geqslant 1 \end{Bmatrix}$
(200)	$\underline{\mathrm{Cu}} + A_1 + A_1$	$\underline{\underline{-2}}$
(201)	$\underline{\mathrm{Ta}} + A_1 + A_1$	$-2, \underline{\underline{-4}}$
(202)	$\underline{\mathrm{Tr}} + A_1 + A_1$	$-2, -2, \underline{\underline{-4}}$
(203)	$\underline{A_{1,****}} + A_1 + A_1$	$-2, -2, -2, \underline{\underline{-4}}$
(204)	$\underline{A_{n,****}} + A_1 + A_1$	$-2, -2, -2, \underline{\underline{-4}}$
(205)	$\underline{D_{4,***}} + A_1 + A_1$	$-2, -2, \underline{\underline{-4}}$
(206)	$\underline{E_{6,**}} + A_1 + A_1$	$-2, \underline{\underline{-4}}$
(207)	$\underline{E_{8,*}} + A_1 + A_1$	$\underline{\underline{-4}}$
(208)	$\underline{\mathrm{El}} + D(l+4)$ $l \geqslant -1$	$\underline{-2}$
(209)	$\underline{N_0} + D(l+4)$ $l \geqslant -1$	$\begin{Bmatrix} \underline{-2}, r=0 \\ \underline{-4}, r \geqslant 1 \end{Bmatrix}$
(210)	$\underline{\mathrm{Cu}} + D(l+4)$ $l \geqslant -1$	$\underline{-2}$
(211)	$\underline{\mathrm{Ta}} + D(l+4)$ $l \geqslant -1$	$-2, \underline{-4}$
(212)	$\underline{\mathrm{Tr}} + D(l+4)$ $l \geqslant -1$	$-2, -2, \underline{-4}$
(213)	$\underline{A_{1,****}} + D(l+4)$ $l \geqslant -1$	$-2, -2, -2, \underline{-4}$
(214)	$\underline{A_{n,****}} + D(l+4)$ $l \geqslant -1$	$-2, -2, -2, \underline{-4}$
(215)	$\underline{D_{4,***}} + D(l+4)$ $l \geqslant -1$	$-2, -2, \underline{-4}$
(216)	$\underline{E_{6,**}} + D(l+4)$ $l \geqslant -1$	$-2, \underline{-4}$
(217)	$\underline{E_{8,*}} + D(l+4)$ $l \geqslant -1$	$\underline{-4}$

Table 4. (cont.)

	Dual Graph	$A_* \cdot A_*$
(218)	$\underline{\mathrm{Ta}} + A_2$	$\underline{-2}, -4$
(219)	$\underline{N_0} + A_2\ (r=0, s=0)$	$\underline{-2}, -4$
(220)	$\underline{\mathrm{Tr}} + A_2$	$\underline{-2}, -3, -3$
(221)	$\underline{N_0} + A_2\ (r=0, t=0, s \geqslant 0)$	$\underline{-2}, -3, -3$
(222)	$\underline{A_{*,0} + A_{*,0} + A_{*,0} + A_{*,0} + A_{*,0}} + A_2$	$\underline{-2}, -2, -2, -2, -2$
(223)	$\underline{A_{*,0} + A_{*,0} + A_{*,0} + A_{n,**,0}} + A_2$	$\underline{-2}, -2, -2, -2, -2$
(224)	$\underline{A_{*,0} + A_{*,0} + A'_{3,**,0}} + A_2$	$\underline{-2}, -2, -2, -2$
(225)	$\underline{A_{*,0} + A_{*,0} + D_{5,*,0}} + A_2$	$\underline{-2}, -2, -2$
(226)	$\underline{A_{*,0} + A_{*,0} + E_{7,0}} + A_2$	$\underline{-2}, -2$
(227)	$\underline{A_{*,0} + A_{n,**,0} + A_{m,**,0}} + A_2$	$\underline{-2}, -2, -2, -2, -2$
(228)	$\underline{A_{*,0} + A'_{5,**,0}} + A_2$	$\underline{-2}, -2, -2$
(229)	$\underline{A_{*,0} + D_{7,*,0}} + A_2$	$\underline{-2}, -2$
(230)	$\underline{N_0} + A_1\ (r=1, s=1)$	$\underline{-2}, -4$
(231)	$\underline{A_{1,****}} + A_1$	$\underline{-2}, -2, -2, -4$
(232)	$\underline{A_{n,****}} + A_1$	$\underline{-2}, -4, -2, -2$
(233)	$\underline{N_0} + A_1\ (r=1, s=1, t \geqslant 0)$	$\underline{-2}, -3, -3$
(234)	$\underline{A_{2,****}} + A_1$	$\underline{-2}, -2, -3, -3$
(235)	$\underline{D_{4,***}} + A_1$	$\underline{-2}, -2, -3$
(236)	$\underline{A_{*,0} + A_{*,0} + A_{*,0} + A_{2,**,0}} + A_1$	$-2, -2, -2, -2, \underline{-2}$
(237)	$\underline{A_{*,0} + A_{*,0} + D_{5,*,0}} + A_1$	$-2, -2, \underline{-2}$
(238)	$\underline{A_{*,0} + A_{2,**,0} + A_{m,**,0}} + A_1$	$-2, \underline{-2}, -2, -2, -2$
(239)	$\underline{A_{*,0} + D_{7,*,0}} + A_1$	$-2, \underline{-2}$
(240)	$\underline{D_{9,*,0}} + A_1$	$\underline{-2}$
(241)	$\underline{A_{2,**,0} + A'_{3,**,0}} + A_1$	$\underline{-2}, -2, -2, -2$
(242)	$\underline{A_{2,**,0} + D_{5,*,0}} + A_1$	$\underline{-2}, -2, -2$
(243)	$\underline{A_{n,**,0} + D_{5,*,0}} + A_1$	$-2, -2, \underline{-2}$
(244)	$\underline{A_{2,**,0} + E_{7,0}} + A_1$	$\underline{-2}, -2$
	II(g) $Z \cdot Z = -2,\ Z_{B_1} \cdot Z_{B_1} = Z_E \cdot Z_E = -1$	
(245)	$\underline{\mathrm{El}} + AF$	$\underline{-1}$

Table 4. (cont.)

		Dual Graph	$A_*\cdot A_*$
(246)	$\underline{N_0}+AF$		$\left\{\begin{matrix}\underline{-1}, r=0\\ \underline{-3}, r\geqslant 1\end{matrix}\right\}$
		II(h) $Z\cdot Z=-1=Z_{B_1}\cdot Z_{B_1}=Z_E\cdot Z_E$	
(247)	$\underline{\mathrm{El}}+A_2$		$\underline{-1}$
(248)	$\underline{N_0}+A_2$		$\left\{\begin{matrix}\underline{-1}, r=0\\ \underline{-3}, r\geqslant 1\end{matrix}\right\}$
		III(i) $Z\cdot Z=-1=Z_{B_1}\cdot Z_{B_1}=Z_{B_2}\cdot Z_{B_2}=Z_E\cdot Z_E$	
(249)	$\underline{\mathrm{El}}+A_3$		$\underline{-1}$
(250)	$\underline{N_0}+A_3$		$\left\{\begin{matrix}\underline{-1}, r=0\\ \underline{-3}, r\geqslant 1\end{matrix}\right\}$

Proof of Theorem 3.6. This is a consequence of Proposition 3.1, Proposition 3.2, Proposition 3.3 and Corollary 3.5.

By the virtue of Theorem 3.6 we have the following theorem.

THEOREM 3.7. *Let $\pi: M\to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only singular point. Suppose $H^1(M,\mathcal{O})\cong\mathbb{C}^2$ and p is a hypersurface singularity. Let A be the exceptional set. If $H^1(A,\mathbb{Z})=0$, then p is an almost minimally elliptic singularity.*

Proof. The condition $H^1(A,\mathbb{Z})=0$ rules out cases (245), (246), (247), (248), (249) and (250) in Theorem 3.6. All the remaining cases in Theorem 3.6 are almost minimally elliptic by Theorem 0.16. Q.E.D.

Remark 3.8. Using Propositions 3.1, 3.2 and 3.3, we can list all possible weighted dual graphs of weakly elliptic singularities such that K' exists and $Z\cdot Z\leqslant -3$. By Corollary 1.2, we know that all hypersurface maximally elliptic singularities must be one of these forms. However, the list is too long to be included here. We remark only that the condition on the elliptic sequence of Theorem 0.16 is automatically satisfied if $Z\cdot Z\leqslant -3$ and K' exists.

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