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Mathematics

Index theory for the boundaries of complex analytic varieties

(boundary p-index/strongly pseudoconvex boundary/partially complex structure/maximally complex condition/complex Plateau problem)

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ABSTRACT A result of boundary p-index is presented for a vector bundle on a complex analytic variety with boundary. A characterization of which odd-dimensional, real submanifolds of $\mathbb{C}^N$ are boundaries of complex subvarieties (respectively submanifolds) in $\mathbb{C}^N$ is given.

Let Y be Stein analytic variety of pure dimension n with only finite number of isolated singularities. Let $V \subset\subset Y$ be an open set with strongly pseudoconvex boundary X, a C^∞ manifold. Suppose that X does not contain any singular point of Y. Assume X defined by $r = 0$, where $dr \neq 0$ on Y, and $V = \{y \in Y: r(y) < 0\}$. For E a holomorphic vector bundle on a neighborhood of $\bar{V}$, define vector spaces as follows:

$$\mathcal{A}^{p,q}(E) = \{\phi \in C^\infty(V_{\text{reg}}, E \otimes^{p,q}): \phi \text{ extends smoothly across } X\}$$

$$\mathcal{C}^{p,q}(E) = \{\phi \in \mathcal{A}^{p,q}(E): \bar{\partial} r \wedge \phi|_X = 0\}.$$

$\bar{\partial}$ on $\mathcal{A}^{p,*}(E)$ preserves $\mathcal{C}^{p,*}(E)$, and so passes to $\mathfrak{B}^{p,*}(E)$: $= \mathcal{A}^{p,*}(E)/\mathcal{C}^{p,*}(E)$. $\mathfrak{B}^{p,*}(E)$ is the $\bar{\partial}_b$-complex associated with E. Denote its cohomology groups by $H^{p,q}(\mathfrak{B}(E))$. It is well known that $\dim H^{p,q}(\mathfrak{B}(E)) < \infty$ for $1 \leq q \leq n-2$.

Definition 1: The boundary p-index of the vector bundle E is $\chi_b^p(E)$: $= \Sigma_{q=1}^{n-2} (-1)^q H^{p,q}(\mathfrak{B}(E))$.

THEOREM 1. *Suppose that* V *has* $y_1, \ldots, y_s$ *as its only isolated singularities. Assume all of them are hypersurface singular points. Let* E *be a vector bundle of rank* m. *Then*

$$\dim H^{p,q}(\mathfrak{B}(E)) = \begin{cases} 0 & p+q \leq n-2 \quad 1 \leq q \leq n-2 \\ m(\tau_1 + \ldots + \tau_s) & p+q = n-1 \quad 1 \leq q \leq n-2 \\ m(\tau_1 + \ldots + \tau_s) & p+q = n \quad 1 \leq q \leq n-2 \\ 0 & p+q = n-1 \quad 1 \leq q \leq n-2, \end{cases}$$

in which τ_i *is the number of local moduli of* V *at* y_i.
In particular, for $n \geq 4$,

$$\chi_b^p(E) = \begin{cases} 0 & \textit{if } p = 0 \\ (-1)^{n-2} m(\tau_1 + \ldots + \tau_2) & \textit{if } p = 1 \\ 0 & \textit{if } 2 \leq p \leq n-2 \\ -m(\tau_1 + \ldots + \tau_s) & \textit{if } p = n-1 \\ 0 & \textit{if } p = n. \end{cases}$$

Remark 1: Let f be a holomorphic function in $\mathbb{C}^{n+1}$. Suppose $V = \{f = 0\}$ has isolated singularity at the origin. Then the number of local moduli τ of V at 0 is given by the following:

$$\tau = \dim \mathbb{C}\{z_0, z_1, \ldots, z_n\} / \left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right).$$

It seems that one of the natural fundamental questions of complex geometry is the classical complex Plateau problem. Specifically, the problem asks which odd-dimensional, real submanifolds of $\mathbb{C}^N$ are boundaries of complex submanifolds in $\mathbb{C}^N$.

With regard to this problem, Harvey and Lawson (1) have recently developed a very interesting theory. Their theorems are important and very general. The paper gives a fundamental contribution to complex geometry. They first observed the following necessary condition for the problem to be solvable. Let X be a real C^∞ submanifold of a complex manifold W that is a C^∞-boundary of a complex submanifold. If $\dim_{\mathbb{R}} X = 2n - 1$, then at each point $z \in X$ we must have

$$\dim_{\mathbb{R}}(T_z X \cap J(T_z X)) = 2n - 2, \qquad [1]$$

in which J is the almost complex structure (i.e., scalar multiplication by i) in $T_z(W)$. The condition 1 asserts that the complex linear space of $T_z X$ is as large as possible (i.e., of real codimension one). Therefore, a submanifold X of dimensional $2n - 1$ that satisfies Eq. 1 at all points will be called maximally complex.

Of course, maximal complexity only imposes a condition on X if the real dimension of X is greater than one. However, there is a natural replacement for Eq. 1 that is necessary for real compact oriented curve γ in W to be the boundary of a complex curve V. Suppose $\gamma = dV$ and let ω be a holomorphic 1-form on W. Then by Stoke's theorem

$$\int_\gamma \omega = \int_{dV} \omega = \int_V d\omega = \int_V \partial\omega$$

because $d = \partial + \bar{\partial}$ and $\bar{\partial}\omega = 0$. From simple type considerations the restriction of a (2,0)-form to an holomorphic curve is identically zero. Therefore the following moment condition is necessary.

$$\int_\gamma \omega = 0 \text{ for all holomorphic 1-forms } \omega. \qquad [2]$$

Suppose now that X is a compact oriented submanifold in a Stein manifold W. Let $[X]$ denote that $(2n - 1)$-dimensional current in W given by integration over X. Similarly, if V is a piece of n-dimensional complex subvariety of W, we let $[V]$ denote the $2p$-dimensional current given by integration over the manifold points of V with the canonical orientation. The current $[X]$ is the boundary of $[V]$ in the sense of currents (written $d[V] = [X]$) if $[X](\alpha) = [V](d\alpha)$ for all C^∞ $(2p - 1)$-forms α on W. The main result of ref. 1 is that, if X is in $\mathbb{C}^N$ as above, then there exists a unique complex variety V with compact support so that $d[V] = \pm[X]$. [For $n = 1$, this can be

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deduced from the work of Wermer (2) and others on the polynomical hull of a curve in $\mathbb{C}^N$.] After 4 yr of laborious work attempting to understand the deep work of Harvey and Lawson, finally we have come up with a somewhat simpler proof for the case when the Levi form of X is not identically zero at every point of X.

THEOREM 2. *Let* X *be a compact, orientable, real manifold of dimension* $2n - 1$, $n \geq 2$ *with partially complex structure in Stein manifold* W. *Suppose the Levi form of* X *is not identically zero at every point of* X. *Then there exists a complex analytic subvariety* V *of dimension* n *of* W − X *such that the boundary of* V *is* X *in the sense of point set topology.*

We should emphasize that when the Levi form is zero at some point the method we use breaks down completely. Indeed, there are examples of this kind such that one cannot find V as in *Theorem 2*. This explains why the result of Harvey and Lawson is interesting and important.

The idea of the proof of *Theorem 2* for $W = \mathbb{C}^N$ goes as follows. We first extend X to a "strip" of variety in $\mathbb{C}^N$ by the Lewy theorem (3). Then we apply the deep theorem of Rothstein and Sperling. Their result (Theorem I, p. 547 of ref. 4) provides us a normal variety V' over $\mathbb{C}^N$ such that *Theorem 2* is true. When we project this back to $\mathbb{C}^N$, we may get an extra component of variety coming from the interior of V'. This extra variety intersects the original strip of variety in complex codimension one subvariety, hence real codimension one in X which is of $(2n - 1)$-measure zero. Therefore we do not have boundary regularity at every point but instead boundary regularity outside a set of $(2n - 1)$-measure zero.

The problem of nonexistence of singularities inside V has not been solved. In order to solve the problem, obviously we have to put some more conditions on the boundary X. However, it seems very difficult to get a right condition because of the following example. Let $V = \{(z_0, \ldots, z_n): f(z) = 0\}$ be a hypersurface with the origin as its only singularity in $\mathbb{C}^{n+1}$. Let $S(0;\delta)$ and $B(0;\delta)$ be the sphere and ball with center 0 and radius δ respectively in $\mathbb{C}^{n+1}$. Let $X_t = S(0;\delta) \cap V_t$ where $V_t = \{(z_0, \ldots, z_n): f(z) = t\}$. Then X_0 bounds the variety $B(0;\delta) \cap V$ with singularity at the origin and X_t bounds complex submanifold $B(0;\delta) \cap V_t$, $t \neq 0$ and t small. However, X_0 is diffeomorphic to X_t. In ref. 5, Donnelly has found a necessary condition depending on η invariants of Atiyah and Singer. In *Theorem 3* below, we will find a necessary and sufficient condition that depends only on the C–R structure of the boundary.

THEOREM 3. *Let* X *be a compact, connected, orientable, real manifold of dimension* $2n - 1$, $n \geq 3$ *with partially complex structure in a Stein manifold* W *of dimension* $n + 1$. *Suppose that* X *is strongly pseudoconvex. Then* X *is a boundary of complex submanifold* $V \subset W - X$ *if and only if the* $\bar{\partial}_b$*-cohomology groups* $H^{p,q}(\mathfrak{B})$ *of Kohn and Rossi* (*cf. refs.* 6 *and* 7) *are zero for* $1 \leq q \leq n - 2$.

Proof: This is an easy consequence of *Theorem 1* and *Theorem 2* and the fact that the number of local moduli for isolated hypersurface singularity is never zero.

Remark 2: There are theorems similar to *Theorem 1* and *Theorem 3* if one replaces strongly pseudoconvexity of the boundary by some other Levi-convexity condition.

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