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Kohn-Rossi cohomology and its application to the complex Plateau problem, I*

By STEPHEN S.-T. YAU

1. Introduction

It seems that one of the natural fundamental questions of complex geometry is the classical complex Plateau problem. Specifically the problem asks which odd-dimensional, real submanifolds of $\mathbf{C}^N$ are boundaries of complex submanifolds in $\mathbf{C}^N$.

With regard to this problem, Harvey and Lawson [12] have recently developed a very interesting theory. Their theorems are important and very general, and the paper is a fundamental contribution to complex geometry. In order to state their main theorem precisely, we need some preliminary remarks. In [12], they first observed the following necessary condition for the problem to be solvable: Let X be a real, C^1 submanifold of a complex manifold W which is a C^1 boundary of a complex submanifold. If $\dim_{\mathbf{R}} X = 2n - 1$, then at each point $z \in X$ we must have

$$\dim_{\mathbf{R}}\big(T_z X \cap J(T_z X)\big) = 2n - 2 \tag{1.1}$$

where J is the almost complex structure (i.e., scalar multiplication by i) in $T_z(W)$. The condition (1.1) asserts that the complex linear subspace of $T_z X$ is as large as possible (i.e., of real codimension one). Therefore, a submanifold M of dimension $2n - 1$ which satisfies (1.1) at all points will be called maximally complex.

Of course, maximal complexity only imposes a condition on X if the real dimension of X is greater than one. However, there is a natural replacement for (1.1) which is necessary for the real compact oriented curve γ in W to be the boundary of a complex curve V. Suppose $\gamma = dV$ and let ω be a holomorphic 1-form on W. Then by Stoke's theorem,

$$\int_\gamma \omega = \int_{dV} \omega = \int_V d\omega = \int_V \partial\omega$$

since $d = \partial + \bar\partial$ and $\bar\partial\omega = 0$. From simple considerations, the restriction of

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a (2, 0)-form to a holomorphic curve is identically zero. Therefore the following moment condition is necessary:

$$\int_{\gamma} \omega = 0 \quad \text{for all holomorphic 1-forms } \omega . \tag{1.2}$$

Suppose now that X is a compact oriented submanifold in a Stein manifold W. Let $[X]$ denote that $(2n-1)$-dimensional current in W given by integration over X. Similarly, if V is a piece of n-dimensional complex subvariety of W, we let $[V]$ denote the $2p$-dimensional current given by integration over the manifold points of V with the canonical orientation. The current X is the boundary of $[V]$ in the sense of currents (written $d[V] = X$) if $[X](\alpha) = [V](d\alpha)$ for all C^∞ $(2p-1)$-forms α on W. By a holomorphic p-chain on a complex manifold W we mean a locally finite sum $T = \sum n_i[V_i]$, where $n_i \in \mathbf{Z} - \{0\}$ and V_i is an irreducible, complex n-dimensional subvariety supp $T = \bigcup_i V_i$.

THEOREM (Harvey-Lawson). *Let X be a compact, oriented submanifold of real dimension $2n-1$ and of class C^1 in a Stein manifold W. Or, more generally, allow X to have a small scar set S. (That is, suppose that S is a compact set of Hausdorff $(2n-1)$-measure zero, which is contained in X, and that X is a compact subset of W such that $X - S$ is an oriented submanifold of $W - S$ of class C^1 with finite volume and $d[X] = 0$. Actually it suffices to assume that $X - S$ is an oriented immersed submanifold of $W - S$ instead of an embedded submanifold.)*

Suppose that X is maximally complex, or if $n = 1$, suppose X satisfies the moment condition. Then there exists a unique holomorpic p-chain T in $W - X$ with supp $T \Subset W$ *and with finite mass, such that*

$$dT = [X] \quad in \quad W . \tag{1.3}$$

Furthermore, there is a compact nowhere dense subset $A \subset X$ such that each point of $X - A$, near which X is of class C^k, $1 \leqq k \leqq \infty$, has a neighborhood in which (supp T) $\cup$ *X is a regular C^k submanifold with boundary (if $k \geqq 2$ then A can be chosen to have Hausdorff $(2n-1)$-measure zero).*

In particular, if X is connected, then there exists a unique precompact irreducible complex n-dimensional subvariety of $W - X$ such that $d[V] = \pm[X]$ with boundary regularity as above.

For $p = 1$, the theorem can be deduced from the work of Wermer [33], Bishop, Alexander and others on the polynomial hull of a curve in $\mathbf{C}^n$ (cf. Gamelin [8]). This function algebraic approach encounters some difficulties in generalization, whereas Harvey-Lawson's proof works uniformly in all dimensions. After four years of laborious work attempting to understand the deep work of Harvey-Lawson, we have come up with a somewhat simpler proof for the case when the Levi form of X is not identically zero at every point of X. We produce a variety V such that the boundary of V is exactly X. For the definition of Levi form, partially complex structure etc., we refer to Section 2.

THEOREM A. *Let X be a compact, orientable, real manifold of dimension $2n-1$, $n \geqq 2$, with partially complex structure in a Stein manifold W. Suppose the Levi form of X is not identically zero at every point of X. Then there exists a complex analytic subvariety V of dimension n of $W - X$ such that the boundary of V is X in the sense of point-set topology. Moreover outside a set of $(2n-1)$-measure zero in X, V has boundary regularity.*

The idea of the proof of Theorem A for $W = \mathbf{C}^N$ goes as follows. We first extend X to a "strip" of a variety in $\mathbf{C}^N$ by H. Lewy's theorem. Then we apply the deep theorem of Rothstein and Sperling (cf. [23], [24], [25], [26] and [31]). Their results [Theorem 1, p. 547 of 20] provide us a normal variety V' over $\mathbf{C}^N$ such that Theorem A is true. When we project this back to $\mathbf{C}^N$, we may get an extra component of a variety coming from the interior of V'. This extra component of a variety intersects the original strip of a variety in a complex codimension one subvariety, hence real codimension one in X which is of $(2n-1)$-measure zero. Therefore Theorem A is true only in the sense of point-set topology and hence also in the sense of distribution. We should emphasize that when the Levi form is zero at some point, the method we use breaks down completely. Indeed there are examples of this kind such that one cannot find V as in Theorem A. This explains why Harvey-Lawson's result is interesting and important.

The problem of nonexistence of singularities inside V has not been solved. In [6], Donnelly has found a necessary condition depending on eta-invariants of Atiyah and Singer. In this paper we will find a necessary and sufficient condition which depends only on the C-R structure of the boundary. However, it seems very difficult to get the right condition as shown in the following example: Let $V = \{(z_0, \cdots, z_n): f(z) = 0\}$ be a hypersurface with the origin as its only singularity in $\mathbf{C}^{n+1}$. Let $S(0; \delta)$ and $B(0; \delta)$ be the sphere and ball respectively in $\mathbf{C}^{n+1}$. Let $X_t = S(0; \delta) \cap V_t$ where $V_t = \{(z_0, \cdots, z_n): f(z) = t\}$. Then X_0 bounds the variety $B(0; \delta) \cap V$ with singularity at the origin and X_t bounds the complex submanifold $B(0; \delta) \cap V_t$, $t \neq 0$ and t small. However X_0 is diffeomorphic to X_t.

It seems to us that the first fundamental invariant of this kind was first introduced by Kohn and Rossi ([18], [17] and [7]), the so-called Kohn-Rossi $\bar{\partial}_b$-cohomology groups $H^{p,q}(\mathfrak{B})$ (cf. § 2). They proved the finite dimensionality of their cohomology groups under certain natural conditions. (Cf. Proposition 2.14.) Of course it would be of interest to compute the dimensions of these $\bar{\partial}_b$-cohomology groups. In general, a strongly pseudoconvex manifold M is a modification of a Stein space V with isolated singularities. In [18], Kohn-Rossi made the following conjecture: In general, either there is no boundary cohomology of the boundary of M (or V) in degree (p, q) $p \neq 0$, $n-1$, or it must result from the interior singularities of V. The following theorem answers the above questions affirmatively.

THEOREM B. *Let M be a strongly pseudoconvex manifold M of dimension n $(n \geqq 3)$ which is a modification of a Stein space V at the isolated singularities $x_1, \cdots, x_m$. Then*

$$\dim H^{p,q}(\mathfrak{B}) = \sum_{i=1}^{n} b_{x_i}^{p,q+1}\,, \qquad 1 \leqq q \leqq n-2$$

where $H^{p,q}(\mathfrak{B})$ is the Kohn-Rossi $\bar{\partial}_b$-cohomology group of type (p, q) and $b_{x_i}^{p,q+1}$ is the Brieskorn invariant of type $(p, q+1)$ at x_i which is a local invariant of the singularity x_i (cf. § 3).

Suppose $x_1, \cdots, x_m$ are hypersurface singularities. Then

$$\dim H^{p,q}(\mathfrak{B}) = \begin{cases} 0\,, & p+q \leqq n-2\,, \quad 1 \leqq q \leqq n-2\,, \\ \tau_1 + \cdots + \tau_m\,, & p+q = n-1\,, \quad 1 \leqq q \leqq n-2\,, \\ \tau_1 + \cdots + \tau_m\,, & p+q = n\,, \quad 1 \leqq q \leqq n-2\,, \\ 0\,, & p+q \geqq n+1\,, \quad 1 \leqq q \leqq n-2\,, \end{cases}$$

where τ_i is the number of moduli of V at x_i (cf. Remark 3.3).

As a result of Theorem A and Theorem B, we can answer the classical complex Plateau problem in the affirmative sense.

THEOREM C. *Let X be a compact, orientable, real manifold of dimension $2n-1$, $n \geqq 3$ with partially complex structure in a Stein manifold W of dimension $n+1$. Suppose that X is strongly pseudoconvex. Then X is a boundary of the complex submanifold $V \subset W - X$ if and only if Kohn-Rossi*'s *$\bar{\partial}_b$-cohomology groups $H^{p,q}(\mathfrak{B})$ are zero for* $1 \leqq q \leqq n-2$.

Proof. This is an easy consequence of the proofs of Theorem A and Theorem B and the fact that the local moduli for isolated hypersurface singularity are never zero.

We remark that the last part of Theorem B remains true if x_i is a local complete intersection singularity for all i. Actually Theorem B and Theorem C remain true if one replaces strong pseudoconvexity of the boundary by some other Levi-convexity condition. In Section 2, following Folland and Kohn [7], we collect all the definitions and theorems we need later on. From this section, the reader can figure out what Levi condition we need in order to solve the complex Plateau problem affirmatively. In Section 4, we prove the duality theorem (cf. Theorem 4.1) for certain 1-convex manifolds. This sharpens the result we had in [34] although the idea was already there.

H. Lewy [21] first studied $\bar{\partial}_b$, and his work was extended by Kohn and Rossi [18], who first formalized the notion of boundary complex. It is a general method to reduce questions about boundary value problems on M to the study of operators on the boundary of M, which is a compact manifold without boundary. Much progress in this area and others has recently been made by M. Kuranishi.

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2. Kohn-Rossi's $\bar{\partial}_b$-complex

In this section we recall Kohn-Rossi's theory for the $\bar{\partial}_b$-complex and fix our notations. The reference for this section is [7]. Let M be a Hermitian complex manifold M of complex dimension n with smooth boundary bM such that $\bar{M} = M \cup bM$ is compact. We shall assume, without loss of generality, that M is imbedded in a slightly large open manifold M' and that bM is defined by the equation $r = 0$ where r is a real C^∞ function with $r < 0$ inside M, $r > 0$ outside $\bar{M}$, and $|dr| = 1$ on bM. Let $\mathcal{A}^{p,q}(M)$ be the space of C^∞ (p, q)-forms on M. $\mathcal{A}^{(p,q)}(\bar{M})$ is the subspace of $\mathcal{A}^{p,q}(M)$ whose elements can be extended smoothly to M'. $\mathcal{A}_c^{p,q}(M)$ is the subspace of $\mathcal{A}_c^{p,q}(\bar{M})$ whose elements have compact support disjoint from bM. Recall that a Hermitian metric on an almost-complex manifold M is a Hermitian inner product $\langle\ ,\ \rangle_x$ on each $\pi_{1,0}(CT_xM)$ varying smoothly in x, where $\pi_{1,0}\colon CT_xM \to T_{1,0}M$ is the natural projection from the complexified tangent bundle to the subbundle consisting of the (1, 0) vectors. For $\xi, \eta \in CT_xM$, we set

$$\langle \xi, \eta \rangle_x = \langle \pi_{1,0}\xi, \pi_{1,0}\eta \rangle_x + \overline{\langle \pi_{1,0}\bar{\xi}, \pi_{1,0}\bar{\eta} \rangle_x}\,.$$

The inner product $\langle\ ,\ \rangle_x$ then extends naturally to all the spaces $\Lambda^{p,q}\mathbf{C}T_x^*M$. If $\omega_1, \cdots, \omega_n$ is an orthonormal basis for $\Lambda^{1,0}\mathbf{C}T_x^*M$, then $\omega_1 \wedge \bar{\omega}_1 \wedge \cdots \wedge \omega_n \wedge \bar{\omega}_n = \gamma$ is the volume element on M at x. We define global scalar products for forms by

$$(\phi, \psi) = \int_M \langle\phi, \psi\rangle\gamma \qquad \text{for} \quad \phi, \psi \in \mathcal{A}^{p,q}(M) .$$

The formal adjoint ϑ of $\bar\partial$ is the differential operator from $\mathcal{A}^{p,q}(M)$ to $\mathcal{A}^{p,q-1}(M)$ defined by $(\vartheta\phi, \psi) = (\phi, \bar\partial\psi)$ for all $\psi \in \mathcal{A}^{p,q-1}(M)$ with compact support. The operator $\square = \bar\partial\vartheta + \vartheta\bar\partial$ is called the complex Laplacian. Let $H_0^{p,q}$ be the space of square integrable (p, q)-forms on M. We shall henceforth use the symbol $\bar\partial$ to mean the closure of $\bar\partial/\mathcal{A}^{p,q}(\bar M)$ with respect to $H_0^{p,q}$; in other words, the operator whose graph is the closure of the graph of $\bar\partial/\mathcal{A}^{p,q}(\bar M)$ in $H_0^{p,q} \times H_0^{p,q+1}$. The following proposition is obtained by integration by parts.

PROPOSITION 2.1. *For all* $\phi \in \mathcal{A}^{p,q}(\bar M)$, $\theta \in \mathcal{A}^{p,q+1}(\bar M)$, $\psi \in \mathcal{A}^{p,q-1}(\bar M)$,

$$(\bar\partial\phi, \theta) = (\phi, \vartheta\theta) + \int_{bM} \langle\sigma(\bar\partial, dr)\phi, \theta\rangle ,$$

$$(\vartheta\phi, \psi) = (\phi, \bar\partial\psi) + \int_{bM} \langle\sigma(\vartheta, dr)\phi, \psi\rangle ,$$

where $\sigma(\bar\partial, dr)$ and $\sigma(\vartheta, dr)$ are the symbols of the differential operators $\bar\partial$ and ϑ at dr respectively. The relation between the Hilbert space adjoint $\bar\partial^*$ of $\bar\partial$ and its formal adjoint ϑ is given by the following proposition. Recall that the Hilbert space adjoint $\bar\partial^*$ of $\bar\partial$ is defined on the domain $\mathrm{Dom}(\bar\partial^*)$ consisting of all $\phi \in H_0^{p,q}$ such that for some constant $c > 0$, $|(\phi, \bar\partial\psi)| \leqq c\|\psi\|$ for all $\psi \in H^{p,q-1}(\bar M)$. For such a ϕ, $\psi \to (\phi, \bar\partial\psi)$ extends to a bounded functional on $H_0^{p,q}$ and $\bar\partial^*\phi$ its dual vector.

PROPOSITION 2.2. *Let* $\mathcal{D}^{p,q} = \mathrm{Dom}(\bar\partial^*) \cap \mathcal{A}^{p,q}(\bar M)$. *Then*

$$\mathcal{D}^{p,q} = \{\phi \in \mathcal{A}^{p,q}(\bar M) : \sigma(\vartheta, dr)\phi = 0 \quad on \quad bM\}$$

and

$$\bar\partial^* = \vartheta \quad on \quad \mathcal{D}^{p,q} .$$

For each $p \in bM$, the *Levi form* at p is the Hermitian form on the $(n-1)$-dimensional space $(\pi_{1,0}\mathbf{C}T_pM) \cap \mathbf{C}T_pbM$ given by

$$(L_1, L_2) \longrightarrow 2\langle\partial\bar\partial r, L_1 \wedge \bar L_2\rangle .$$

(It is Hermitian because $\partial\bar\partial = -\bar\partial\partial = -\overline{\partial\bar\partial}$.) We shall be working in special boundary charts U, with the special basis, $\{\omega_i\}$, $1 \leqq i \leqq n$, $\omega_n = \sqrt{2}\,\partial r$ for $\mathcal{A}^{1,0}(U)$. Let $L_1, \cdots, L_n$ be the dual vector fields. Then $\{(L_i)_p\}$, $1 \leqq i \leqq n-1$ is an orthonormal basis of the space $(\pi_{1,0}\mathbf{C}T_pM) \cap \mathbf{C}T_pbM$ and the Levi form, which is defined with respect to this basis, is given by the matrix coefficients of the Levi form, $c_{ij} = 2\langle \partial\bar{\partial} r, L_i \wedge \bar{L}_j \rangle$. The following proposition shows that this form depends only on the outward normal dr to bM and is therefore intrinsically defined.

PROPOSITION 2.3.

$$c_{ij} = \frac{1}{\sqrt{2}} \langle \omega_n, [L_i, \bar{L}_j] \rangle \ .$$

In other words, c_{ij} is the coefficient of L_n in the basis expansion of $1/\sqrt{2}\,[L_i, \bar{L}_j]$.

Definition 2.4. (a) M is said to be *pseudoconvex* (pseudoconcave) if the Levi form is positive (negative) semi-definite at each point of bM and *strongly pseudoconvex* (pseudoconcave) if it is positive (negative) definite at each point of bM.

(b) We say that M satisfies condition $Z(q)$ if the Levi form has at least $n - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues at each point of bM. (Thus a strongly pseudoconvex manifold satisfies condition $Z(q)$ for all $q > 0$.)

Suppose H is a Hilbert space and Q is a Hermitian form defined on a dense subspace D of H satisfying $Q(\phi, \phi) \geqq ||\phi||^2$ for $\phi \in D$. Suppose further that D is a Hilbert space under the inner product Q. Then there is a canonical self-adjoint operator F on H associated with Q as follows. For each $\alpha \in H$, $\psi \to (\alpha, \psi)$ is a Q-bounded functional on D. Thus there is a unique $\phi \in D$ such that $Q(\phi, \psi) = (\alpha, \psi)$ for all $\psi \in D$. Define $T: H \to D \subset H$ by $T\alpha = \phi$. Then T is bounded, self-adjoint and injective. Set $F = T^{-1}$. We have the following famous Friedrichs Extension Theorem.

PROPOSITION 2.5. *F is the unique self-adjoint operator with* $\mathrm{Dom}(F) \subseteqq D$ *satisfying $Q(\phi, \psi) = (F\phi, \psi)$ for all $\phi \in \mathrm{Dom}(F)$ and $\psi \in D$.*

In our case, we define the form Q on $\mathfrak{D}^{p,q}$ by

$$Q(\phi, \psi) = (\bar{\partial}\phi, \bar{\partial}\psi) + (\vartheta\phi, \vartheta\psi) + (\phi, \psi)$$

and let $\tilde{\mathfrak{D}}^{p,q}$ be the completion of $\mathfrak{D}^{p,q}$ under Q. The inclusion $\mathfrak{D}^{p,q} \to H_0^{p,q}$ extends uniquely to a norm-decreasing map $\tilde{\mathfrak{D}}^{p,q} \to H_0^{p,q}$. This map is injective. Hence we can identify $\tilde{\mathfrak{D}}^{p,q}$ with a subspace of $H_0^{p,q}$ and apply the Friedrichs construction. We denote the Friedrichs operator associated to Q by F. Since for $\phi, \psi \in \mathfrak{A}_c^{p,q}(M)$, $Q(\phi, \psi) = ((\square + I)\phi, \psi)$, we see that F is a self-adjoint extension of the Hermitian operator $(\square + I) | \mathfrak{A}_c^{p,q}(M)$. The smooth elements of $\tilde{\mathfrak{D}}^{p,q}$ are described by the boundary condition $\sigma(\vartheta, dr)\phi = 0$ on

bM; the smooth elements of $\mathrm{Dom}(F)$ are characterized by a further first-order boundary condition (the so-called "free boundary condition").

PROPOSITION 2.6. *If* $\phi \in \mathcal{D}^{p,q}$, *then* $\phi \in \mathrm{Dom}(F)$ *if and only if* $\bar\partial\phi \in \mathcal{D}^{p,q+1}$, *in which case* $F\phi = (\Box + I)\phi$.

Let $\Box_F = F - I$ and let the harmonic space $\mathcal{H}^{p,q} = \eta(\Box_F)$ be the nullspace of the operator $\Box_F$. In [7, p. 51], Kohn proved that the harmonic space $\mathcal{H}^{p,q}$ is a finite-dimensional subspace of $\mathcal{A}^{p,q}(\bar M)$ provided M satisfies condition $Z(q)$. As a consequence of his beautiful solution of the $\bar\partial$-Neumann problem, Kohn proved the following:

THEOREM 2.7. *If* M *satisfies condition* $Z(q)$, *then* $H^{(p,q)}(\bar M) \cong \tilde H^{p,q}(M) \cong \mathcal{H}^{p,q}$ *where*

$$H^{p,q}(\bar M) = \frac{\{\phi \in \mathcal{A}^{p,q}(\bar M): \bar\partial\phi = 0\}}{\bar\partial \mathcal{A}^{p,q-1}(\bar M)}\,,$$

$$H^{p,q}(M) = \frac{\{\phi \in \mathcal{A}^{p,q}(M): \bar\partial\phi = 0\}}{\bar\partial \mathcal{A}^{p,q-1}(M)}\,,$$

$$\tilde H^{p,q}(M) = \frac{\{\phi \in H_0^{p,q} \cap \mathrm{Don}(\bar\partial): \bar\partial\phi = 0\}}{\bar\partial\left(H_0^{p,q+1} \cap \mathrm{Dom}(\bar\partial)\right)}\,.$$

On the other hand, the Dolbeault theorem asserts that $H^{p,q}(M) \cong H^q(M, \Omega^p)$ where Ω^p is the sheaf of germs of holomorphic $(p, 0)$-forms. The relationship between these important groups and the preceding one is due to Hormander [14].

THEOREM 2.8. *If* M *satisfies condition* $Z(q)$ *and* $Z(q+1)$ *then* $H^q(M, \Omega^p) \cong \mathcal{H}^{p,q}$.

Let $\mathcal{C}^{p,q} = \{\phi \in \mathcal{A}^{p,q}(\bar M): \bar\partial r \wedge \phi = 0 \text{ on } bM\}$. Since $\sigma(\bar\partial, dr) = \bar\partial r \wedge (\,\cdot\,)$, we may also write

$$\mathcal{C}^{p,q} = \{\phi \in \mathcal{A}^{p,q}(\bar M): \sigma(\bar\partial, dr)\phi = 0 \text{ on } bM\}\,.$$

Recall that the Hodge star operator $*: \mathcal{A}^{p,q}(\bar M) \to \mathcal{A}^{n-p,n-q}(\bar M)$ is defined by the equation $\psi \wedge *\phi = \langle \psi, \bar\phi\rangle \gamma$ where γ is the volume form on M. It is not hard to prove that $** = (-1)^{p+q}$, $*\bar\phi = \overline{*\phi}$ and $\vartheta = -*\partial*$. There is a duality of the space $\mathcal{C}^{p,q}$ and $\mathcal{D}^{p,q}$ and the spaces $\mathcal{C}^{p,q}$ form a complex under $\bar\partial$.

PROPOSITION 2.9.

$$\mathcal{C}^{p,q} = *\,\overline{\mathcal{D}^{n-p,n-q}}\,,$$

$$\bar\partial \mathcal{C}^{p,q} \subset \mathcal{C}^{p,q+1}\,.$$

We may therefore form the cohomology

$$H^{p,q}(\mathcal{C}) = \{\phi \in \mathcal{C}^{p,q}: \bar\partial\phi = 0\}/\bar\partial\mathcal{C}^{p,q-1}\,.$$

In [18], Kohn-Rossi introduced the zero-boundary-value cohomology,

$$H^{p,q}(0) = \frac{\{\phi \in \mathcal{A}^{p,q}(\bar{M})\colon \bar{\partial}\phi = 0,\ \phi|_{bM} = 0\}}{\bar{\partial}\{\phi \in \mathcal{A}^{p,q-1}(\bar{M})\colon \phi|_{bM} = 0,\ \bar{\partial}\phi|_{bM} = 0\}} .$$

PROPOSITION 2.10.

$$H^{p,q}(\mathcal{C}) \cong H^{p,q}(0) .$$

They also proved the following important Kohn-Rossi duality on pseudoconvex manifolds.

PROPOSITION 2.11. *If M satisfies condition $Z(q)$, $H^{p,q}(\bar{M})$ is naturally dual to $H^{n-p,n-q}(\mathcal{C})$; in particular, $H^{n-p,n-q}(0) \cong (H^{p,q}(\bar{M}))^*$.*

Following Folland and Kohn [7], we now introduce space $\mathcal{B}^{p,q}$ of forms on bM according to the following equivalent definitions:

(1) $\mathcal{B}^{p,q}$ is the space of (smooth) sections of the vector bundle $\Lambda^{p,q}CT^*M \cap \Lambda^{p+q}CT^*bM$ on bM.

(2) $\mathcal{B}^{p,q}$ is the space of (p, q)-forms restricted to bM which are pointwise orthogonal to the ideal generated by $\bar{\partial}r$ (i.e., to all forms of the type $\bar{\partial}r \wedge \theta$).

(3) $\mathcal{B}^{p,q}$ is the space of restrictions of elements of $\mathcal{D}^{p,q}$ to bM.

(1) says that $\mathcal{B}^{p,q}$ is the space of tangential (p, q)-forms on bM. Using the language of sheaves, there is another way to express $\mathcal{B}^{p,q}$, which is clearly equivalent to (2).

(4) Let $\tilde{\mathcal{A}}^{p,q}$ and $\tilde{\mathcal{C}}^{p,q}$ denote the sheaves of germs of $\mathcal{A}^{p,q}$ and $\mathcal{C}^{p,q}$ on $\bar{M}$, respectively. Then there is a natural injection $0 \to \tilde{\mathcal{C}}^{p,q} \to \tilde{\mathcal{A}}^{p,q}$. The quotient sheaf $\tilde{\mathcal{B}}^{p,q} = \tilde{\mathcal{A}}^{p,q}/\tilde{\mathcal{C}}^{p,q}$ is a locally free sheaf supported on bM, and $\mathcal{B}^{p,q}$ is its space of sections.

In view of Proposition 2.9, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathcal{C}}^{p,q+1} & \longrightarrow & \tilde{\mathcal{A}}^{p,q+1} & \longrightarrow & \tilde{\mathcal{B}}^{p,q+1} & \longrightarrow & 0 \\
 & & \bar{\partial}\Big| & & \bar{\partial}\Big| & & \bar{\partial}_b\Big| & & \\
0 & \longrightarrow & \tilde{\mathcal{C}}^{p,q} & \longrightarrow & \tilde{\mathcal{A}}^{p,q} & \longrightarrow & \tilde{\mathcal{B}}^{p,q} & \longrightarrow & 0
\end{array}$$

where $\bar{\partial}_b$ is the quotient map which is induced by $\bar{\partial}$. $\bar{\partial}_b$ may be explicitly described on sections as follows: if $\phi \in \mathcal{B}^{p,q}$, choose $\phi' \in \mathcal{A}^{p,q}$ such that $\phi'|_{bM} = \phi$. Then $\bar{\partial}_b\phi$ is the projection of $\bar{\partial}\phi'|_{bM}$ onto $\mathcal{B}^{p,q}$. It is easy to check that this is independent of the choice of ϕ'.

Since $\bar{\partial}^2 = 0$, it follows that $\bar{\partial}_b^2 = 0$, so we have the boundary complex

$$0 \longrightarrow \mathcal{B}^{p,0} \xrightarrow{\bar{\partial}_b} \mathcal{B}^{p,1} \xrightarrow{\bar{\partial}_b} \cdots \xrightarrow{\bar{\partial}_b} \mathcal{B}^{p,n-1} \longrightarrow 0 .$$

(Note that $\mathcal{B}^{p,q} = 0$.)

Definition 2.12. The cohomology of the above boundary complex is called Kohn-Rossi cohomology and is denoted by $H^{p,q}(\mathcal{B})$.

We recall the following two propositions.

PROPOSITION 2.13. *If* $\phi \in \mathcal{B}^{p,q}$, *then* $\bar{\partial}_b \phi = 0$ *if and only if* $\vartheta * \bar{\phi}' \in \mathcal{D}^{n-p,n-q-1}$ *for any smooth extension* ϕ' *of* ϕ.

PROPOSITION 2.14. *If M satisfies conditions* $Z(q)$ *and* $Z(n-q-1)$ *then* $H^{(p,q)}(\mathcal{B})$ *is finite-dimensional, and the range of* $\bar{\partial}_b$: $\mathcal{B}^{p,q-1} \to \mathcal{B}^{p,q}$ *is closed in the* C^∞ *topology.* (Actually the range of the Hilbert space operator $\bar{\partial}_b$ is also closed.)

We remark that conditions $Z(q)$ and $Z(n-q-1)$ mean together: $\max(q+1, n-q)$ eigenvalues of the Levi form have the same sign, or there are $\min(q+1, n-q)$ pairs of eigenvalues with opposite signs.

3. Computation of Kohn-Rossi's $\bar{\partial}_b$-cohomology

In this section, we will compute Kohn-Rossi's $\bar{\partial}_b$-cohomology explicitly. Let us first adopt the following convention.

Definition 3.1. Let X be a complex analytic space of dimension n with x as its isolated singularity. Then $b^{p,q} = \dim H^q_{\{x\}}(X, \Omega^p)$, $p \leqq n$ and $1 \leqq q \leqq n$, will be called the Brieskorn numbers of type (p, q) at the singular point x.

It is well known that a strongly pseudoconvex manifold M is a modification of a Stein space V with isolated singularities. According to Kohn-Rossi, it would be of very much interest to compute $\bar{\partial}_b$ cohomology in this general case.

THEOREM 3.2. *Let M be a strongly pseudoconvex manifold M of dimension n which is a modification of a Stein space V at the isolated hypersurface singularities* $x_1, \cdots, x_m$. *Then*

(3.1) $\dim H^{p,q}(\mathcal{B})$

$$= \begin{cases} 0 & p+q \leqq n-1 \quad 1 \leqq q \leqq n-2 \\ \tau_1 + \cdots \tau_m & p+q = n-1 \quad 1 \leqq q \leqq n-2 \\ \tau_1 + \cdots + \tau_m & p+q = n \quad\quad\; 1 \leqq q \leqq n-2 \\ 0 & p+q \geqq n+1 \quad 1 \leqq q \leqq n-2, \end{cases}$$

where τ_i *is the number of moduli of* V *at* x_i.

Remark 3.3. Let f be a holomorphic function in $\mathbf{C}^{n+1}$. Suppose $V = \{f = 0\}$ has an isolated singularity at the origin. Then the number of moduli τ of V at 0 is given by the following:

$$\tau = \dim \mathbf{C}[z_0, z_1, \cdots, z_n] \Big/ \left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right).$$

Proof of Theorem 3.2. We first calculate $b_{x_i}^{p,q}$. Since $b_{x_i}^{p,q}$ is a local invariant, we may assume that x_i is the origin of $\mathbf{C}^{n+1}$. Let f be a defining function of V in a neighborhood of 0 in $\mathbf{C}^{n+1}$. Recall that on p. 91, (2.4) of [34], we gave an explicit resolution of $\Omega^p_{v\,0}$ as follows:

(3.2)

$$0 \longrightarrow \Omega^0_{\mathbf{C}^{n+1}} \xrightarrow{\partial_p} \Omega^1_{\mathbf{C}^{n+1}} \oplus \Omega^0_{\mathbf{C}^{n+1}} \longrightarrow \cdots \longrightarrow \Omega^{p-i}_{\mathbf{C}^{n+1}} \oplus \Omega^{p-i-1}_{\mathbf{C}^{n+1}} \xrightarrow{\partial_i} \Omega^{p-i+1}_{\mathbf{C}^{n+1}} \oplus \Omega^{p-i}_{\mathbf{C}^{n+1}}$$

$$\longrightarrow \cdots \longrightarrow \Omega^{p-1}_{\mathbf{C}^{n+1}} \oplus \Omega^{p-2}_{\mathbf{C}^{n+1}} \xrightarrow{\partial_1} \Omega^{p}_{\mathbf{C}^{n+1}} \oplus \Omega^{p-1}_{\mathbf{C}^{n+1}} \xrightarrow{\partial_0} \Omega^{p}_{\mathbf{C}^{n+1}} \xrightarrow{\varepsilon} \Omega^p_V \longrightarrow 0$$

is exact at 0 in $\mathbf{C}^{n+1}$ where

$$\begin{aligned} \partial_0(\alpha, \beta) &= f\alpha + df \wedge \beta\,, && (\alpha, \beta) \in \Omega^p_{\mathbf{C}^{n+1}} \oplus \Omega^{p-1}_{\mathbf{C}^{n+1}} \\ \partial_i(\alpha, \beta) &= \big(df \wedge \alpha, df \wedge \beta + (-1)^i f\alpha\big)\,, && (\alpha, \beta) \in \Omega^{p-i}_{\mathbf{C}^{n+1}} \oplus \Omega^{p-i-1}_{\mathbf{C}^{n+1}} \\ & && 1 \leqq i \leqq p-1 \\ \partial_p(\alpha) &= \big(df \wedge \alpha, (-1)^p f\alpha\big)\,, && \alpha \in \Omega^0_{\mathbf{C}^{n+1}} \\ \varepsilon &= \text{the natural quotient map} \end{aligned}$$

are $\mathcal{O}$-linear. By local duality

(3.3)
$$\begin{aligned} b^{p,q} &= \dim H^q_{\{0\}}(V, \Omega^p) \\ &= \dim \operatorname{Ext}^{n+1-q}_{\mathcal{O}_{\mathbf{C}^{n+1},0}}(\Omega^p_{V,0}, \mathcal{O}_{\mathbf{C}^{n+1},0})\,. \end{aligned}$$

Dualizing (3.3) and abbreviating $\Omega^p_{\mathbf{C}^{n+1}}$ as Ω^p, we get

(3.4)

$$\operatorname{Hom}(\Omega^p, \Omega^0) \xrightarrow{\partial_0^t} \operatorname{Hom}(\Omega^p \oplus \Omega^{p-1}, \Omega^0) \xrightarrow{\partial_1^t} \operatorname{Hom}(\Omega^{p-1} \oplus \Omega^{p-2}, \Omega^0)$$

$$\longrightarrow \cdots \longrightarrow \operatorname{Hom}(\Omega^{p-i+1} \oplus \Omega^{p-i}, \Omega^0) \xrightarrow{\partial_i^t} \operatorname{Hom}(\Omega^{p-i} \oplus \Omega^{p-i-1}, \Omega_0) \longrightarrow$$

$$\longrightarrow \cdots \longrightarrow \operatorname{Hom}(\Omega^2 \oplus \Omega^1, \Omega^0) \xrightarrow{\partial_{p-1}^t} \operatorname{Hom}(\Omega^1 \oplus \Omega^0, \Omega_0) \xrightarrow{\partial_p^t} \operatorname{Hom}(\Omega_0, \Omega_0) \longrightarrow 0$$

where ∂_i^t is the transpose of ∂_i, $0 \leqq i \leqq p$. Before we can continue the proof of Theorem 3.2, we need some facts about the Koszul complex as well as Serre's beautiful theory of "algébriques locales multipicités". For the sake of convenience to the reader, we recall briefly what they are.

Let A be a commutative ring and M an A-module. Let $x_1, \cdots, x_r$ be elements in A. We denote $K(x_1, \cdots, x_r; M)$ to be the Koszul complex for elements $x_1, \cdots, x_r$. If $M = A$, we simply denote $K(x_1, \cdots, x_r)$ for $K(x_1, \cdots, x_r; A)$.

Definition 3.4. Let A be a ring and M an A-module. A sequence $a_1, a_2, \cdots, a_r$ of elements of A is said to be *M-regular* if, for each $1 \leqq i \leqq r$, a_i is not a zero divisor on $M/(a_1M + \cdots + a_{i-1}M)$. When all a_i belong to an ideal I we say $a_1, \cdots, a_r$ is an *M-regular sequence in I*. If, moreover, there is no $b \in I$ such that $a_1, \cdots, a_r, b$ is M-regular, then $a_1, \cdots, a_r$ is said to be a maximal M-regular sequence in I.

Remark 3.5. In general, the definition of an M-regular sequence depends on the order of elements. However, if A is a noetherian ring, M a finite A-module and $I \subseteq \mathrm{rad}(A)$, then the definition of an M-regular sequence in I is independent of the permutation of $a_1, \cdots, a_r$. Also, since A is noetherian, every M-sequence in I can be extended to a maximal M-sequence in I.

Let us now recall a theorem due to Auslander and Buchsbaum (cf. Theorem 1.7 of [3]).

THEOREM 3.6. *Let I be an ideal in A, a noetherian ring, and M a finite A-module such that $IM \neq M$. Let $x_1, \cdots, x_s$ be a maximal M-sequence in I, and let $J = (y_1, \cdots, y_n)$ be an ideal contained in I, such that $J + \mathrm{Ann}(M)$ contains I^k for some integer $k > 0$ where $\mathrm{Ann}(M)$ is the annihilator of M. Then $s + q = n$ where q is the largest integer, such that $H_q(y_1, \cdots, y_n; M) \neq 0$.*

In particular, let $L = K(a_1, \cdots, a_n)$ and let q be the largest integer such that $H_q(L) \neq 0$. Let s be the number of elements in a maximal A-sequence in $(a_1, \cdots, a_n)$. Then $s + q = n$.

Let $A = \mathcal{O}_{\mathbf{C}^{n+1},0}$. Consider the Koszul complex of the elements $(\partial f/\partial z_0, \cdots, \partial f/\partial z_n, (-1)^p f)$ in A:

$$(3.5)\quad 0 \longrightarrow A^{\binom{n+2}{n+2}} \xrightarrow{\partial_{n+2}} A^{\binom{n+2}{n+1}} \xrightarrow{\partial_{n+1}} \cdots \longrightarrow A^{\binom{n+2}{2}} \xrightarrow{\partial_2} A^{\binom{n+2}{1}} \xrightarrow{\partial_1} A \longrightarrow 0 .$$

We claim that (3.4) is a complex which is isomorphic to a part of the Koszul complex in (3.5).

For this, it suffices to observe the following more general, well-known statement. A Koszul complex is self-dual in the sense that, if all terms are replaced by their duals and all arrows by their adjoints, then the new complex is isomorphic to the original one.

The Koszul complex for elements $g_1, \cdots, g_k$ in a ring A can be viewed as follows: Identify an element $(a_{i_1 \ldots i_p})_{1 \leq i_1 < \ldots < i_p \leq k}$ of $A^{\binom{k}{p}}$ with $a = \sum a_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and with dx^i, $1 \leq i \leq k$, as an indeterminate. Let $g = \sum_{i=1}^{p} g_i dx^i$. The map $d_p\colon A^{\binom{k}{p}} \to A^{\binom{k}{p+1}}$ is given by $a \mapsto g \wedge a$.

Using the identification $(\alpha, \beta) \in \Omega^q \oplus \Omega^{q-1}$ with $\alpha + \beta \wedge dz_{n+1}$ in $A^{\binom{n+2}{q}}$, one can see that the complex (3.2), after the arrow ε is removed, is isomorphic to a part of the Koszul complex for $\partial f/\partial z_0, \cdots, \partial f/\partial z_{n+1}, (-1)^p f$.

One can also see that a Koszul complex is self-dual in the following way: Define $(*)\colon A^{\binom{k}{p}} \to A^{\binom{k}{k-p}}$ by the equation $b \wedge *a = \langle b, a\rangle dx^1 \wedge \cdots \wedge dx^k$ where

$$\langle b, a\rangle = \sum_{i_1<\cdots<i_p} b_{i_1\dots i_p} a_{i_1\dots i_p} .$$

The transpose ${}^t d_{k-p-1}$ of d_{k-p-1} is simply $(-1)^p * d_p *^{-1}$, because

$$\begin{aligned}\langle b, {}^t d_{k-p-1}(*a)\rangle &= \langle d_{k-p-1}b, *a\rangle \\ &= d_{k-p-1}b \wedge **a \\ &= (-1)^{p(k-p)} d_{k-p-1}b \wedge a \\ &= (-1)^{p(k-p)} g \wedge b \wedge a \\ &= (-1)^{p(k-p)+(k-p-1)} b \wedge g \wedge a \\ &= (-1)^{p(k-p)+(k-p-1)+(p+1)(k-p-1)} b \wedge **(g \wedge a) \\ &= (-1)^p \langle b, *(g \wedge a)\rangle \\ &= (-1)^p \langle b, *d_p a\rangle .\end{aligned}$$

By (3.2), (3.3) and (3.4), it follows that

(3.6) $$b^{p,q} = 0 \quad \text{if} \quad n+1-q \geqq p+2 , \quad \text{i.e.,}\ p+q \leqq n-1 ,$$

and

$$\begin{aligned} b^{p,q} &= \dim \operatorname{Ext}^{n+1-q}_{\mathcal{O}_{\mathbf{C}^{n+1},0}}(\Omega^p_{V,0}, \mathcal{O}_{\mathbf{C}^{n+1},0}) \\ &= \dim \ker \delta^t_{n+1-q} / \operatorname{Im} \delta^t_{n-q} . \end{aligned}$$

Hence

(3.7) $$\begin{aligned} b^{p,q} &= \dim \ker \partial_{p+q-n} / \operatorname{Im} \partial_{p+q-n+1} \\ &= H_{p+q-n}\left(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}, (-1)^p f; A\right) \\ &\qquad \text{for} \quad n-q \geqq 1 , \quad \text{i.e.,} \quad q \leqq n-1 \end{aligned}$$

where H_i denotes the i^{th} homology of the Koszul complex of the elements $(\partial f/\partial z_0, \cdots, \partial f/\partial z_n, (-1)^p f)$. Since the number of elements in a maximal A-sequence in $(\partial f/\partial z_0, \cdots, \partial f/\partial z_n, (-1)^p f)$ is $n+1$, then

(3.8) $$H_q\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}, (-1)^p f; A\right) = 0 \qquad \text{for} \quad q > 1$$

by Theorem 3.6. Hence

(3.9) $$b^{p,q} = 0 \qquad \text{for} \quad p+q \geqq n+2 \quad \text{and} \quad q \leqq n-1 .$$

It remains to compute $b^{p,n-p}$ and $b^{p,n-p+1}$.

(3.10) $$\begin{aligned} b^{p,n-p} &= \dim \operatorname{Ext}^{p+1}_{\mathcal{O}_{\mathbf{C}^{n+1}}}(\Omega^p_{V,0}, \mathcal{O}_{\mathbf{C}^{n+1},0}) \\ &= \dim \operatorname{Hom}(\Omega^0, \Omega^0)/\operatorname{Im} \delta^t_p , & p \geqq 1 , \\ &= \dim A/\operatorname{Im} \partial_1 , & p \geqq 1 , \\ &= \dim H_0\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}, (-1)^p f; A\right) \\ &= \dim \mathbf{C}[z_0, z_1, \cdots, z_n]\Big/\left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right) , & p \geqq 1 ; \end{aligned}$$

$$
\begin{aligned}
(3.11)\qquad b^{p,n-p+1} &= \dim \operatorname{Ext}^{p}_{\mathcal{O}_{\mathbf{C}^{n+1},0}}(\Omega^{p}_{V,0}, \mathcal{O}_{\mathbf{C}^{n+1},0}) \\
&= \dim \ker \delta^{t}_{p}/\operatorname{Im} \delta^{t}_{p-1} \\
&= \dim \ker \partial_1/\operatorname{Im} \partial_2 , \qquad \text{for} \quad p \geqq 2 , \\
&= \dim H_1\Big(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}, (-1)^p f; A\Big) , \quad \text{for} \quad p \geqq 2 .
\end{aligned}
$$

Now let us recall the following interesting theorem of Serre. Suppose E is a module of finite type over a Noetherian local ring A. Let $\underline{q}=(x_1, \cdots, x_r)$ be an ideal of definition; i.e., $\underline{q}$ contains some power of the maximal ideal m. Then the multiplicity $e_{\underline{q}}(E, r)$ is by definition equal to the coefficient of $n^r/r!$ in the characteristic polynomial $l_A(E/\underline{q}^n E)$. (We denote $l_A(F)$ as the length of an A-module F.) Serre [28] proved that

$$
(3.12)\qquad e_{\underline{q}}(E, r) = \textstyle\sum_{i=0}^{r} (-1)^i l_A\big(H_i(x_1, \cdots, x_r : E)\big)
$$

where $H_i(x_1, \cdots, x_r; E)$ denotes the i^{th} homology of the Koszul complex of the elements $(x_1, \cdots, x_r)$.

Apply Serre's result in our case where $E = A$ and $\underline{q}=\big(\partial f/\partial z_0, \cdots, \partial f/\partial z_n, (-1)^p f\big)$. Now $e_{\underline{q}}(A, n + 2) = 0$ since $\dim A = n + 1 < n + 2$. By (3.8) and (3.12), we have

$$
\begin{aligned}
(3.13)\qquad &\dim H_1\Big(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}, (-1)^p f; A\Big) \\
&\quad = \dim H_0\Big(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}, (-1)^p f; A\Big) .
\end{aligned}
$$

It follows from (3.10), (3.11) and (3.12) that

$$
(3.14)\qquad b^{(p,n-p+1)} = \dim \mathbf{C}[z_0, z_1, \cdots, z_n]\Big/\Big(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}, f\Big) \qquad \text{for } p \geqq 2 .
$$

Consider the following local cohomology exact sequence:

$$
\begin{aligned}
&H^1_Z(V, \Omega^p) \longrightarrow H^1(V, \Omega^p) \longrightarrow H^1(V - Z, \Omega^p) \longrightarrow \cdots \longrightarrow H^{n-2}_Z(V, \Omega^p) \\
&\longrightarrow H^{n-2}(V, \Omega^p) \longrightarrow H^{n-2}(V - Z, \Omega^p) \longrightarrow H^{n-1}_Z(V, \Omega^p) \longrightarrow H^{n-1}(V, \Omega^p)
\end{aligned}
$$

where $Z = \{x_1, \cdots, x_m\}$. By Cartan's Theorem B, we have

$$
(3.15)\qquad H^q(V - Z, \Omega^p) \cong H^{q+1}_Z(V, \Omega^p) \qquad \text{for} \quad q \geqq 1 .
$$

It follows that from (3.6), (3.9), (3.10), (3.14) and (3.15) that

$$
(3.16)\quad H^q(V - Z, \Omega^p) = \begin{cases}
0 & p + q \leqq n - 2 \quad 1 \leqq q \leqq n - 2 \\
\tau_1 + \cdots + \tau_m & p + q = n - 1 \quad 1 \leqq q \leqq n - 2 \\
\tau_1 + \cdots + \tau_m & p + q = n \qquad 1 \leqq q \leqq n - 2 \\
0 & p + q \geqq n + 1 \quad 1 \leqq q \leqq n - 2 .
\end{cases}
$$

Let $\pi: M \to V$ be the modification of V at the points $x_1, \cdots, x_m$. Let $A = \bigcup A_i$,

$1 \leqq i \leqq m$, where $A_i = \pi^{-1}(x_i)$. Then (3.16) is equivalent to the following:

$$(3.17)\quad H^q(M - A, \Omega^p) = \begin{cases} 0 & p+q \leqq n-2 \quad 1 \leqq q \leqq n-2 \\ \tau_1 + \cdots + \tau_m & p+q = n-1 \quad 1 \leqq q \leqq n-2 \\ \tau_1 + \cdots + \tau_m & p+q = n \quad 1 \leqq q \leqq n-2 \\ 0 & p+q \geqq n+1 \quad 1 \leqq q \leqq n-2 . \end{cases}$$

Following Laufer [20], we consider the sheaf cohomology with support at infinity. Let us recall briefly the definition. Let $\mathcal{F} = \mathcal{O}(E)$ be the sheaf of germs of sections of a holomorphic vector bundle E over M; then $H^*_\infty(M, \mathcal{O}(E))$ is by definition the cohomology of the quotient complex $C^\infty(M, E \otimes \Lambda^{0,\infty})/C^\infty_c(M, E \otimes \Lambda^{0,*})$. Here $C^\infty(M, E \otimes \Lambda^{0,*})$ is the C^∞-Dolbeault complex, and $C^\infty_c(M, E \otimes \Lambda^{0,*})$ the subcomplex of smooth compactly supported E-valued $(0, q)$ forms. Take a 1-convex exhaustion function φ on M such that $\varphi \geqq 0$ on M and $\varphi(y) = 0$ if and only if $y \in A$. Put $M_r = \{y \in M: \varphi(y) \leqq r\}$. Then by Laufer [20], $\varinjlim_r H^i(M - M_r, \Omega^p) \cong H^i_\infty(M, \Omega^p)$. On the other hand, by Andreotti and Grauert (Théorème 15 of [1]), $H^i(M - A, \Omega^p)$ is isomorphic to $H^i(M - M_r, \Omega^p)$ for $i \leqq n - 2$ and $H^{n-1}(M - A, \Omega^p) \to H^{n-1}(M - M_r, \Omega^p)$ is injective. By (3.17), we have

$$(3.18)\quad H^q_\infty(M, \Omega^p) = \begin{cases} 0 & p+q \leqq n-2 \quad 1 \leqq q \leqq n-2 \\ \tau_1 + \cdots + \tau_m & p+q = n-1 \quad 1 \leqq q \leqq n-2 \\ \tau_1 + \cdots + \tau_m & p+q = n \quad 1 \leqq q \leqq n-2 \\ 0 & p+q \geqq n+1 \quad 1 \leqq q \leqq n-2 . \end{cases}$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{A}^{p,*}_c & \longrightarrow & \mathfrak{A}^{p,*}(\bar{M}) & \longrightarrow & \bar{\mathfrak{A}}^{p,*}_\infty & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{A}^{p,*}_c & \longrightarrow & \mathfrak{A}^{p,*}(M) & \longrightarrow & \mathfrak{A}^{p,*}_\infty & \longrightarrow & 0 . \end{array}$$

It follows from Theorem 2.7, Theorem 2.8 and the five lemma that

$$(3.19)\quad H^q(\bar{\mathfrak{A}}^{p,*}_\infty) = H^q(\mathfrak{A}^{p,*}_\infty) , \qquad q \geqq 1 .$$

We claim that the natural inclusion map i from $\mathfrak{A}^{p,*}_c$ to $\mathcal{C}^{p,*}$ induces isomorphisms from $H^q(\mathfrak{A}^{p,*}_c)$ to $H^q(\mathcal{C}^{p,*})$ for $1 \leqq q \leqq n - 1$. To see this, recall that $H^q(\mathfrak{A}^{p,*}_c)$ is Serre dual to $H^{n-q}(M, \Omega^{n-p})$ by integration pairing. On the other hand, $H^q(\mathcal{C}^{p,*})$ is Kohn-Rossi dual to $H^{n-q}(\bar{M}, \Omega^{n-p})$ and hence to $H^{n-q}(M, \Omega^{n-p})$ for $q \leqq n - 1$, again by integration pairing (cf. Proposition 2.10 and 2.11). Since i is compatible with these integration pairings, our claim follows easily. Now the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}_c^{p,*} & \longrightarrow & \mathcal{A}^{p,*}(\bar{M}) & \longrightarrow & \mathcal{A}_\infty^{p,*} & \longrightarrow & 0 \\ & & | & & | & & | & & \\ 0 & \longrightarrow & \mathcal{C}^{p,*} & \longrightarrow & \mathcal{A}^{p,*}(\bar{M}) & \longrightarrow & \mathcal{B}^{p,*} & \longrightarrow & 0 \end{array}$$

gives

(3.20) $$H^q(\mathcal{A}_\infty^{p,*}) = H^q(\mathcal{B}^{p,*}) \qquad \text{for} \quad 1 \leqq q \leqq n-2$$

by the five lemma. Finally (3.1) follows from (3.20), (3.19) and (3.18).

As a corollary to the proof of Theorem 3.2, we have the following:

COROLLARY 3.7. *Let M be a strongly pseudoconvex manifold M of dimension n ($n \geqq 3$) which is a modification of a Stein space V at the isolated singularities $x_1, \cdots, x_m$. Then*

$$\dim H^{p,q}(\mathcal{B}) = \sum_{i=1}^m b_{x_i}^{p,q+1} .$$

Example 3.8. Let $V = \{z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0\} \subseteqq \mathbf{C}^{n+1}$ where a_i are positive integers. Let $X = V \cap S^{2n+1}$ where S^{2n+1} is a sphere centered at the origin. Then the dimension of Kohn-Rossi's $\bar{\partial}_b$-cohomology group is given by the following formula:

$$\dim H^{p,q}(\mathcal{B}) = \begin{cases} 0 & p+q \leqq n+2 \quad 1 \leqq q \leqq n-2 \\ (a_0-1)(a_1-1)\cdots(a_n-1) & p+q = n-1 \quad 1 \leqq q \leqq n-2 \\ (a_0-1)(a_1-1)\cdots(a_n-1) & p+q = n \quad\quad\; 1 \leqq q \leqq n-2 \\ 0 & p+q \geqq n+1 \quad 1 \leqq q \leqq n-2 . \end{cases}$$

4. Brieskorn numbers and Serre duality for strongly pseudoconvex manifolds

Duality theorems for compact complex manifolds (such as Serre duality) are well-known. Serre duality is still true for open manifolds but one has to use cohomology with compact support. It is natural to ask for a duality theorem for 1-convex manifolds without using cohomology with compact support. A partial result was obtained in our previous paper [34]. In Section 3 we introduced Brieskorn invariants for the singularities which are obtained by blowing down the exceptional set in the strongly pseudoconvex manifold. These numerical invariants for the isolated singularities turn out to be exactly the obstructions for the Serre duality to be true in 1-convex manifolds.

THEOREM 4.1. *Let M be a 1-convex manifold M of dimension n which is a modification of a Stein space V at the isolated hypersurface singularities $x_1, \cdots, x_m$. Then*

(1)

(a) $b^{p,q}_{x_i} = 0$ *for* $p + q \leqq n - 1$, $q \geqq 1$, $1 \leqq i \leqq m$;

(b) $b^{p,n-p}_{x_i} = \tau_i$ *for* $p \geqq 1$, $1 \leqq i \leqq m$;

(c) $b^{p,n-p+1}_{x_i} = \tau_i$ *for* $p \geqq 2$, $1 \leqq i \leqq m$;

(d) $b^{p,q}_{x_i} = 0$ *for* $p + q \geqq n + 2$, $q < n$, $1 \leqq i \leqq m$.

(2)

(a) $h^{p,q} = h^{n-p,n-q}$ *for* $p + q \leqq n - 2$, $q \geqq 1$, *and* $n \geqq 3$;
or $p + q \geqq n + 2$, $q \leqq n - 1$, *and* $n \geqq 3$;

(b) (i) $h^{p,n-1-p} - h^{p,n-p} + h^{p,n-p+1} = -(h^{n-p,p+1} - h^{n-p,p} + h^{n-p,p-1})$
for $2 \leqq p \leqq n - 2$ *and* $n \geqq 4$,

(ii) $h^{1,n-2} - h^{1,n-1} + h^{n-1,1} - h^{n-1,2} = \tau_1 + \cdots + \tau_m - s^{(n-1)}$
for $n \geqq 4$

where τ_i *is the number of moduli of* V *at* x_i, *where*

$h^{p,q} = \dim H^q(M, \Omega^p)$, *and* $s^{(n-1)} = \dim H^0(M - A, \Omega^{n-1})/H^0(M, \Omega^{n-1})$.

Proof. Statement (1) follows from (3.6), (3.9), (3.10) and (3.14). Now by Laufer [20], we have the following exact sequence:

$$0 \longrightarrow H^0_c(M, \Omega^p) \longrightarrow H^0(M, \Omega^p) \longrightarrow H^0_\infty(M, \Omega^p) \longrightarrow H^1_c(M, \Omega^p) \longrightarrow H^1(M, \Omega^p) \longrightarrow H^1_\infty(M, \Omega^p) \longrightarrow \cdots \longrightarrow H^{n-1}_c(M, \Omega^p) \longrightarrow H^{n-1}(M, \Omega^p) \longrightarrow H^{n-1}_\infty(M, \Omega^p).$$

Arguing as in the proof of Theorem 3.2, we know that the following sequence is exact:

$$0 \longrightarrow H^0_c(M, \Omega^p) \longrightarrow H^0(M, \Omega^p) \xrightarrow{\alpha} H^0(M - A, \Omega^p) \longrightarrow H^1_c(M, \Omega^p) \longrightarrow H^1(M, \Omega^p) \longrightarrow H^1(M - A, \Omega^p) \longrightarrow \cdots \longrightarrow H^{n-1}_c(M, \Omega^p) \longrightarrow H^{n-1}(M, \Omega^p) \longrightarrow H^{n-1}(M - A, \Omega^p).$$

Recall that in Theorem 2.7 of [34], we proved $H^0(V, \Omega^p) \to H^0(V - Z, \Omega^p)$ is surjective for $p \leqq n - 2$ where $Z = \{x_1, \cdots, x_m\}$ and hence $\alpha\colon H^0(M, \Omega^p) \to H^0(M - A, \Omega^p)$ is surjective for $p \leqq n - 2$. The map $H^{n-1}_c(M, \Omega^p) \to H^{n-1}(M, \Omega^p)$ is surjective as the Serre dual of the injective map $H^1_c(M, \Omega^{n-p}) \to H^1(M, \Omega^{n-p})$ for $n - p \leqq n - 2$. From these and from (3.17) together with Serre duality, we obtain the statement (2) (a) in the theorem. By (3.17), we also have the exact sequence

$$0 \longrightarrow H^{n-1-p}_c(M, \Omega^p) \longrightarrow H^{n-1-p}(M, \Omega^p) \longrightarrow H^{n-1-p}(M - A, \Omega^p) \longrightarrow H^{n-p}_c(M, \Omega^p) \longrightarrow H^{n-p}(M, \Omega^p) \longrightarrow H^{n-p}(M - A, \Omega^p) \longrightarrow H^{n-p+1}_c(M, \Omega^p) \longrightarrow H^{n-p+1}(M, \Omega^p) \longrightarrow 0 \quad \text{for } 2 \leqq p \leqq n - 2.$$

From this and from (3.17) together with Serre duality, we obtain

$$h^{p,n-1-p} - h^{p,n-p} + h^{p,n-p+1} = -(h^{n-p,p+1} - h^{n-p,p} + h^{n-p,p-1}) \quad \text{for} \quad 2 \leqq p \leqq n-2 .$$

Finally from (3.17), we have the exact sequence

$$0 \longrightarrow H^0(M, \Omega^{n-1}) \longrightarrow H^0(M-A, \Omega^{n-1}) \longrightarrow H^1_c(M, \Omega^{n-1}) \longrightarrow H^1(M, \Omega^{n-1}) \longrightarrow H^1(M-A, \Omega^{n-1}) \longrightarrow H^2_c(M, \Omega^{n-1}) \longrightarrow H^2(M, \Omega^{n-1}) \longrightarrow 0 \quad \text{for} \quad n \geqq 4 .$$

Statement (2) (b) (ii) follows from the above exact sequence and Serre duality.

Remark 4.2. After this paper was completed, A. Fujiki informed us that he had proved independently parts of Theorem 4.1, namely (1) (a), (1) (d), (2) (a) and (2) (b) (i). J. Wahl informed us that they also proved (1) (a), (2) (a) and (2) (b) (i) in an algebraic category. His proof of (2) (b) (i) also depends on our previous result, Theorem A of [34].

5. The complex Plateau problem

In [17], Kohn first considered the $\bar{\partial}_b$ complex intrinsically on a compact manifold of real dimension $2n-1$ which satisfies the maximal complex condition. Unfortunately, his definition of the $\bar{\partial}_b$ complex is different from Kohn-Rossi's $\bar{\partial}_b$ complex which we considered in Section 2. Following Tanaka [32], we reformulate the $\bar{\partial}_b$ complex in a way independent of the interior manifold M.

Definition 5.1. Let X be a compact, orientable, real manifold of dimension $2n-1$. A *partially complex structure* on X is an $(n-1)$-dimensional subbundle S of CTX such that

(1) $S \cap \bar{S} = \{0\}$;

(2) If L, L' are local sections of S, then so is $[L, L']$.

In particular, if $X = bM$ where M is a complex manifold, then $S = (\pi_{1,0}CTM) \cap (CTX)$ defines a partially complex structure on X.

Let X be a partially complex manifold with structure S for a complex valued C^∞ function u defined on X. We define $d''u \in \Gamma(\bar{S}^*)$ by

$$(d''u)(\bar{L}) = \bar{L}u , \quad L \in S_x .$$

The differential operator d'' is called the (tangential) Cauchy-Riemann operator, and a solution u of the equation $d''u = 0$ is called a holomorphic function.

Definition 5.2. A complex vector bundle E over X is said to be holomorphic if there is a differential operator

$$\bar{\partial}_E : \Gamma(E) \longrightarrow \Gamma(E \otimes \bar{S}^*)$$

satisfying the following conditions:

(a) $\bar{L}_1(f \cdot u) = \bar{L}_1 f \cdot u + f \cdot \bar{L}_1 u$,

(b) $[\bar{L}_1, \bar{L}_2]u = \bar{L}_1\bar{L}_2 u - \bar{L}_2\bar{L}_1 u$

where $u \in \Gamma(E)$, f is a complex valued function on X, L_1, $L_2 \in \Gamma(S)$ and we put $\bar{Z}u = (\bar{\partial}_E u)(\bar{Z})$, $Z \in \Gamma(S)$. The operator $\bar{\partial}_E$ is called the Cauchy-Riemann operator, and a solution u of the equation $\bar{\partial}_E u = 0$ is called a holomorphic cross section. It is clear that the trivial vector bundle $X \times \mathbf{C}$ is holomorphic with respect to the operator d'' defined above.

Remark 5.3. In the case where X is a complex manifold, our definition of a holomorphic vector bundle is equivalent to the usual one in terms of holomorphic transition functions. We can see this fact, for example, by use of Newlander-Nirenberg's theorem.

We now show that the vector bundle $\hat{T}(X)$: $\mathbf{C}T(X)/\bar{S}$ is a holomorphic vector bundle with respect to the operator $\bar{\partial} = \bar{\partial}_{\hat{T}(X)}$ defined as follows: Let ω be the projection: $\mathbf{C}T(X) \to \hat{T}(X)$. Take any $u \in \Gamma(\hat{T}(X))$ and express it as $u = \omega(Z)$, $Z \in \Gamma(\mathbf{C}T(X))$. For any $L \in \Gamma(S)$, define a cross section $(\bar{\partial}u)(\bar{L})$ of $\hat{T}(X)$ by

$$(\bar{\partial}u)(\bar{L}) = \omega([\bar{L}, Z]) .$$

Then we see easily that $(\bar{\partial}u)(\bar{L})$ does not depend on the choice of Z and that $\bar{\partial}u$ gives a cross section of $\hat{T}(X) \otimes \bar{S}^*$. Furthermore we can show that the operator $u \mapsto \bar{\partial}u$ satisfies (a) and (b) of Definition 5.2, using the Jacobi identity in the Lie algebra $\Gamma(\mathbf{C}T(X))$. The holomorphic vector bundle $\hat{T}(X)$, thus defined, will be called the holomorphic tangent bundle of X.

Remark 5.4. Consider the case where X is a real hypersurface in a complex manifold M. First we note that $\hat{T}(M)$ may be regarded as the holomorphic vector bundle S_M of tangent vectors of type (1, 0) to M. Let E be the restriction of $\hat{T}(M)$ to X. Then the natural map: $\mathbf{C}T(X) \to \mathbf{C}T(M)$ induces a injective homomorphism of $\hat{T}(X)$ to E as homomorphic vector bundles. (Recall that a bundle homomorphism $\varphi: E \to F$ between two holomorphic bundles is called holomorphic if $\bar{L}(\varphi(u)) = \varphi(\bar{L}(u))$, $u \in \Gamma(E)$, $L \in S$.)

Let E be a holomorphic vector bundle over X. We put

$$C^{\infty}(X, E) = E \otimes \Lambda^q \bar{S}^* ,$$
$$\mathcal{C}^{\infty}(X, E) = \Gamma(C^q(X, E))$$

and define differential operators

$$\bar{\partial}^q_E\colon \mathcal{C}^q(X, E) \longrightarrow \mathcal{C}^{q+1}(X, E)$$

by

$$(\bar{\partial}^q_E\varphi)(\bar{L}_1, \cdots, \bar{L}_{p+1}) = \sum_i (-1)^{i+1}\bar{L}_i\varphi(\bar{L}_1, \cdots, \hat{\bar{L}}_i, \cdots, L_{p+1}) + \sum_{i<j}(-1)^{i+j}\varphi([\bar{L}_i, \bar{L}_j], \bar{L}_1, \cdots, \hat{\bar{L}}_i, \cdots, \hat{\bar{L}}_j, \cdots, \bar{L}_{p+1})$$

for all $\varphi \in \mathcal{C}^q(X, E)$ and $L_1, \cdots, L_{p+1} \in \Gamma(S)$. Just as in the case of exterior differentiation d, we can show that $\bar{\partial}^q_E\varphi$ gives an element of $\mathcal{C}^{q+1}(X, E)$ and that $\bar{\partial}^{q+1}_E \circ \bar{\partial}^q_E = 0$. Thus the collection $\{\mathcal{C}^q(X, E), \bar{\partial}^q_E\}$ gives a complex and we denote by $H^q(X, E)$ the cohomology groups of this complex.

Let $\{\mathcal{A}^k(X), d\}$ be the de Rham complex of X with complex coefficients, and $H^k(X)$ the cohomology groups of this complex, the de Rham cohomology groups. If we put $A^k(X) = \Lambda^k(\mathrm{C}T(X))^*$, we have $\mathcal{A}^k(X) = \Gamma(A^k(X))$. For any integers p and k, we denote by $F^p(A^k(X))$ the subbundle of $A^k(X)$ consisting of all $\varphi \in A^k(X)$ which satisfy the equality:

$$\varphi(Y_1, \cdots, Y_{p-1}, \bar{Z}_1, \cdots, \bar{Z}_{k-p+1}) = 0$$

for all $Y_1, \cdots, Y_{p-1} \in \mathrm{C}T(X)_x$ and $Z_1, \cdots, Z_{k-p+1} \in S_x$, x being the origin of φ. Then we have

$$F^p(A^k(X)) \supset F^{p+1}(A^k(X)) ,$$
$$F^0(A^k(X)) = A^k(X) , \qquad F^{p+1}(A^q(X)) = 0 .$$

Furthermore putting $F^p(\mathcal{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we easily find that $dF^p(\mathcal{A}^k(X)) \subset F^p(\mathcal{A}^{k+1}(X))$. Thus the collection $\{F^p(\mathcal{A}^k(X))\}$ gives a filtration of the de Rham complex. Let $\{E_r^{p,q}(X)\}$ denote the spectral sequence associated with this filtration.

The groups $E_1^{p,q}(X)$ are of particular importance; they will be denoted by $H^{p,q}(X)$. We define

$$A^{p,q}(X) := F^p(A^{p+q}(X)) , \qquad \mathcal{A}^{p,q}(X) = \Gamma(A^{p,q}(X)) ,$$
$$C^{p,q}(X) := A^{p,q}(X)/A^{p+1,q-1}(X) , \qquad \mathcal{C}^{p,q}(X) = \Gamma(C^{p,q}(X)) .$$

Then the groups $H^{p,q}(X)$ are the cohomology groups of the complex $\{\mathcal{C}^{p,q}(X), d''\}$, where the operator $d''\colon \mathcal{C}^{p,q}(X) \to \mathcal{C}^{p,q+1}(X)$ is naturally induced from the operator $d\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)$.

Now $E^p = \Lambda^p(\hat{T}(X))^*$ is a holomorphic vector bundle by the rule:

$$(\bar{Y}\varphi)(u_1, \cdots, u_p) = \bar{Y}(\varphi(u_1, \cdots, u_p)) + \sum_i (-1)^i\varphi(\bar{Y}u_i, u_1, \cdots, \hat{u}_i, \cdots, u_p)$$

where

$$\varphi \in \Gamma(E^p),\ u_1, \cdots, u_p \in \Gamma(\hat{T}(X)),\ Y \in S$$

and

$$\bar{Y}\varphi = (\bar{\partial}_{E^p}\varphi)(\bar{Y}),\ \bar{Y}u_i = (\bar{\partial}_{\hat{T}(X)}u_i)(\bar{Y}) .$$

PROPOSITION 5.5 (Tanaka). *$C^{p,q}(X)$ may be identified with $C^q(X, E^p)$ in a natural manner and we have*

$$d''\varphi = (-1)^p \bar{\partial}_{E^p}\varphi,\ \varphi \in \mathcal{C}^{p,q}(X)\ .$$

Proof. Define a map $\iota^p: A^{p,q}(X) \to C^q(X, E^p)$ by

$$(\iota^p\varphi)\big(\omega(Y_1), \cdots, \omega(Y_p); \bar{Z}_1, \cdots, \bar{Z}_q\big) = \varphi(Y_1, \cdots, Y_p, \bar{Z}_1, \cdots, \bar{Z}_q)$$

for all $\varphi \in A^{p,q}(X)_x$, $Y_1, \cdots, Y_p \in CT(X)_x$ and $Z_1, \cdots, Z_q \in S_x$. (It is clear that ι^p is well defined.) Then we have the exact sequence of vector bundles:

$$0 \longrightarrow A^{p+1,q-1}(X) \longrightarrow A^{p,q}(X) \xrightarrow{\iota^p} C^q(X, E^q) \longrightarrow 0$$

whence $C^{p,q}(X) \cong C^q(X, E^p)$. Furthermore we can easily verify the equalities:

$$\bar{\partial}_{E^p}\iota^p\varphi = (-1)^p\iota^p d\varphi = (-1)^p d''\iota^p\varphi,\ \varphi \in \mathcal{A}^{p,q}(X)\ ,$$

proving the proposition.

Remark 5.6. Consider the case when $X = bM$ where M is a complex manifold. Let $\{\underline{C}^{p,q}, d''\}$ be the complex in the sheaf category, associated with the complex $\{\mathcal{C}^{p,q}(X), d''\}$. Then it is easy to see that the complex $\{\underline{C}^{p,q}, d''\}$ coincides with the boundary complex $\{\tilde{\mathcal{B}}^{p,q}, \bar{\partial}_b\}$ introduced by Kohn-Rossi (cf. [18], p. 465). As a consequence, $H^{p,q}(X)$ is isomorphic to $H^{p,q}(\mathcal{B})$.

Definition 5.7. Let $L_1, \cdots, L_{n-1}$ be the local basis for a section of S over $U \subset X$ so that $\bar{L}_1, \cdots, \bar{L}_{n-1}$ is a local basis for sections of $\bar{S}$. Since $S \oplus \bar{S}$ has complex codimension one in CTX, we may choose a local section N of CTX such that $L_1, \cdots, L_{n-1}, \bar{L}_1, \cdots, \bar{L}_{n-1}, N$ span CTX. We may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by

$$[L_i, \bar{L}_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \bar{L}_k + c_{ij}N\ ,$$

is Hermitian, and is called the *Levi form*.

The Levi form is non-invariant; however its essential features are invariant.

PROPOSITION 5.8. *The number of non-zero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \cdots, L_{n-1}, N$.*

Definition 5.9. In view of Proposition 5.8, it makes sense to require that the Levi form have $\max(q + 1, n - q)$ eigenvalues of the same sign or $\min(q + 1, n - q)$ pairs of eigenvalues with opposite signs at each point. If this is true, we say that X satisfies condition $Y(q)$.

Remark 5.10. If $X = bM$, then the new and old Levi forms coincide up to sign and normalization, and X satisfies condition $Y(q)$ if and only if M satisfies conditions $Z(q)$ and $Z(n - q - 1)$.

LEMMA 5.11 (The invariance of the Levi form). *Let $f: \Omega \to \Omega'$ be a holomorphic map between complex manifolds so that $f|_X$ (where X is a partially complex manifold in Ω) is an embedding. Then the number of positive (or negative) eigenvalues of the Levi form of X at x is equal to the number of positive (or negative) eigenvalues of the Levi form of $f(X)$ at $f(x)$.*

Recall that any Stein manifold can be embedded in $\mathbf{C}^N$ for N large. By using the maximum principle, it is easy to see that in order to prove Theorem A in Section 1, it suffices to prove the following.

THEOREM 5.12. *Let $X \subseteqq \mathbf{C}^N$ be a compact, orientable, real manifold of dimension $2n-1$, $n \geqq 2$, with partially complex structure. Suppose the Levi form of X is not identically zero at every point of x. Then there exists a complex analytic subvariety V of dimension n in $\mathbf{C}^N - X$ such that the boundary of V is X.*

Sketch of the Proof. We first extend X to a strip of a variety. For $x \in X$, consider a linear projection from $\mathbf{C}^N$ onto any complex linear space H_x of complex dimension n such that the restriction of the projection to a neighborhood $B(X; x, \varepsilon)$ in X gives an embedding. For instance, we can project along a complex linear space, of complex dimension $N-n$, which is a direct summand of the real tangent space T of X at x.

Let Y_x be the hypersurface which represents the image of the projection restricted to $B(X; x, \varepsilon)$ in H_x. Then $B(X; x, \varepsilon)$ represents the graph of a smooth function $f: Y_x \to \mathbf{C}^{N-n}$, since the graph of f is maximally complex if and only if $\bar{\partial}_b f = 0$ by a theorem of Bochner (cf. [4] or Theorem 5.1 of [12]). (Here $\bar{\partial}_b$ is relative to Y_x in $H_x = \mathbf{C}^n$.) If ε is small enough, we can assume $B(X; x, \varepsilon)$ is connected. Hence Y_x is a smooth connected hypersurface in $B(\mathbf{C}^n; y, \varepsilon)$, an ε-ball in $\mathbf{C}^n$. Let U_y denote $B(\mathbf{C}^n; y, \varepsilon) - Y_x$. Then U_y has two components U_y^+ and U_y^-. Here we denote U_y^+ to be the component such that the usual Levi form of Y_x with respect to it has the same number of positive eigenvalues and the same number of negative eigenvalues as the Levi form induced on Y_x from X by the projection $\pi: \mathbf{C}^N \to \mathbf{C}^n$. Suppose the Levi form has at least one positive (respectively, one negative) eigenvalue. Then there exists $0 < \varepsilon_1 \leqq \varepsilon$ and a unique smooth function F^+ (respectively F^-) on $\bar{U}_y^+ \cap B(\mathbf{C}^n; y, \varepsilon_1)$ (respectively on $\bar{U}_y^- \cap B(\mathbf{C}^n; y, \varepsilon_1)$) such that

$$F^+ \in \mathcal{O}(U_y^+ \cap B(\mathbf{C}^n; y, \varepsilon_1)) \quad \text{and} \quad F^+/Y_x \cap B(\mathbf{C}^n; y, \varepsilon_1) = f$$

and respectively,

$$F^- \in \mathcal{O}(U_y^- \cap B(\mathbf{C}^n; y, \varepsilon_1)) \quad \text{and} \quad F^-/Y_x \cap B(\mathbf{C}^n; y, \varepsilon_1) = f$$

by the Lewy theorem (cf. Theorem 2.6.13, pp. 51–52, [15]).

We denote by $W_x^{\pm}$ the graph of $F^{\pm}$ over $\bar{U}_y^{\pm} \cap B(\mathbf{C}^n; y, \varepsilon_1)$, whenever $W_x^{\pm}$ exists; We claim that these $W_x^{\pm}$, $x \in X$ patch together to give a strip of a variety. Since the restriction of almost all linear projections will give local embedding, by a compactness argument, for $\varepsilon > 0$ small enough, we may assume that for any $x_1, x_2 \in X$, $W_{x_1} \cap W_{x_2} \neq \emptyset$, there exists a linear projection $\pi: \mathbf{C}^N \to \mathbf{C}^N$ such that both W_{x_1} and W_{x_2} represent graphs of holomorphic functions. If the projections of W_{x_1} and W_{x_2} are on the same side of Y in $\mathbf{C}^n$, then one sees easily that they patch together to give a strip of a variety W. If the projections of W_{x_1} and W_{x_2} are on the different sides of Y in $\mathbf{C}^n$, then the following Cauchy theorem tells us that W_{x_1} and W_{x_2} still can patch together to give a strip of a variety.

CAUCHY THEOREM. *Let f be a continuous function on an open set U in $\mathbf{C}^n$. Suppose f is holomorphic outside a smooth real hypersurface. Then f is holomorphic on U.*

Proof. By the Osgood lemma (cf. Theorem I.A. 2 of [9]), it suffices to prove the theorem for $n = 1$. In this case, the standard proof for the Cauchy theorem for one variable works; that is, by using the continuity of f on U, one can prove easily that the line integral of f over any closed loop is zero.
Q.E.D.

In order to finish the proof of Theorem 5.12 we still need to extend the strip of the variety W. At this point, we have to apply the deep theorem of Rothstein and Sperling. Their result (Theorem I, p. 547 of [27]) provides us a normal variety V' over $\mathbf{C}^N$ such that Theorem 5.12 is true. The image V of V' in $\mathbf{C}^N$ is the variety we want. One should be a little bit careful here. When we project V' back to $\mathbf{C}^N$, we may get an extra component of a variety coming from the interior of V'. This extra component may intersect the original strip of the variety in a complex codimension one subvariety, hence real codimension one in X, which is of $(2n - 1)$-measure zero. Therefore we cannot hope to have boundary regularity at every point, but instead we only get boundary regularity outside a set of $(2n - 1)$-measure zero.

Remark 5.13. The last part of the proof of Theorem 5.12 is more or less well-known. It has been discussed in a series of papers by Rothstein [23], [24], [25] and [26]. Rothstein and Sperling [27], and Sperling [31]. For the sake of convenience to the readers, we discuss the theory of extension of a strip of variety in Section 6.

Definition 5.14. Let X be an orientable real manifold of odd dimension with a partially complex structure. Then X is strongly pseudoconvex if the hermitian matrix (c_{ij}) obtained in Definition 5.7 is always nonsingular and its eigenvalues are of the same sign.

THEOREM 5.14. *Let X be a compact, orientable, real manifold of dimension $2n-1$, $n \geqq 3$, with partially complex structure in a Stein manifold W of dimension $n+1$. Suppose that X is strongly pseudoconvex. Then X is a boundary of a complex submanifold $V \subset W - X$ if and only if Kohn-Rossi's $\bar{\partial}_b$-cohomology groups $H^{p,q}(\mathcal{B})$ are zero for $1 \leqq q \leqq n-2$.*

Proof. This is an easy consequence of Theorem 3.2, Theorem 5.12, a result of Rossi [22], and the fact that the number of local moduli for isolated hypersurface singularity is never zero.

6. Extension of an analytic surface-piece

In a series of papers [23], [24], [25], [26], [27] and [31], Rothstein and Sperling have developed a beautiful theory of extension of an analytic surface-piece. The basic theorem which makes this theory work beautifully is the so-called "local extension theorem" of Rothstein [26] (cf. Theorem 6.7). Here we follow Siu's proof of Rothstein's theorem (cf. [29]). The proof uses projections, special analytic polyhedra, analytic covers, elementary symmetric polynomials and the extension of holomorphic functions. The general theorem concerning the continuation of analytic surfaces over $\mathbf{C}^n$ (which are, roughly speaking, spaces whose points are prime germs of analytic sets in $\mathbf{C}^n$) was first proved by Sperling in his Marburg dissertation [31]. The proof uses the "local extension theorem" and Hartogs-type arguments. There are many further consequences due to Rothstein [23]. Lemma 6.17 is a function-theoretic result. Lemmas 6.19, 6.22 and 6.23 are preparations for Hartogs-type arguments. Here we shall follow Rothstein and Sperling's treatment [27]. For further consequences of continuation of analytic spaces, we refer the readers to [23], [24], [25], [26], [27] and [31].

LEMMA 6.1. *Suppose G is an open subset of $\mathbf{C}^N$, K is a compact subset of G, D is an open subset of $\mathbf{C}^n$, and E is a closed subset of $D \times G$. Suppose one of the following two conditions* (i), (ii) *is satisfied:*

(i) *There exist holomorphic functions f_i on $D \times G$ $(i \in I)$ such that $E = \{x \in D \times G : |f_i(x)| \leqq 1$ for all $i \in I\}$.*

(ii) *There exist holomorphic functions g_j on $D \times G$ $(j \in J)$ such that $E = \{x \in D \times G : \operatorname{Re} g_j(x) = 0$ for all $j \in J\}$.*

Then the following conclusions hold:

(a) *If A is a subvariety in $(D \times G) - E$ such that $A \subset D \times K$, then $\dim A \leqq n$.*

(b) *If* A_i *is a subvariety in* $(D \times G) - E$ *whose every branch has dimension* $\geqq n + 1$ $(i = 1, 2)$, *such that*

$$A_1 \cap \big(D \times (G - K)\big) = A_2 \cap \big(D \times (G - K)\big)$$

then $A_1 = A_2$.

Proof. Condition (i) follows from condition (ii) by setting

$$\{f_{i \in I}\} = \{e^{g_j}, e^{-g_j}\}_{j \in J}$$

so we can assume that we have condition (i).

(a) By considering the subvariety $A \cap (\{t\} \times G)$ of $(\{t\} \times G) - E$ for every $t \in D$, we can reduce the general case to the special case $n = 0$.

Suppose $\dim A > n = 0$. We are going to derive a contradiction. We can assume that A is irreducible. Take $x \in A$. Then $|f_i(x)| > 1$ for some $i \in I$. The sup of $|f_i|$ on A is assumed at some point of A; it equals the sup of $|f_i|$ on the compact set,

$$A \cap \{|f_i| \geqq |f_i(x)|\} .$$

By the maximum modulus principle, $f_i = f_i(x)$ on A. It follows that A is compact, contradicting $\dim A > 0$.

(b) Let A'_1 be a branch of A. By (a), $A'_1 \cap \big(D \times (G - K)\big) \neq \varnothing$. Hence a nonempty open subset of A'_1 is contained in A_2. It follows that $A'_1 \subset A_2$. Likewise every branch A'_2 of A_2 is contained in A_1. Consequently $A_1 = A_2$.

Q.E.D.

We introduce the following notations: For $a \in \mathbf{R}^N$, we denote by $a_1, \cdots, a_N$ the coordinates of a. For $a, b \in \mathbf{R}^N$ we say that $a < b$ (respectively $a \leqq b$) if $a_i < b_i$ for $1 \leqq i \leqq N$ (respectively $a_i \leqq b_i$ for $1 \leqq i \leqq N$). For $0 < b$ in $\mathbf{R}^N$, denote by $\Delta^N(b)$ the polydisc

$$\{(z_1, \cdots, z_1) \in \mathbf{C}^N : |z_i| < b_i \quad \text{for} \quad 1 \leqq i \leqq N\} .$$

For $0 \leqq a < b$ denote by $G^N(a, b)$ the set

$$\{(z_1, \cdots, z_N) \in \Delta^N(b) : |z_i| > a_i \quad \text{for some } 1 \leqq i \leqq N\} .$$

When $a_1 = \cdots = a_N = r$ and $b_1 = \cdots = b_N = s$, we write $\Delta^N(s)$, $G^N(r, s)$ instead of $\Delta^N(a)$, $G^N(a, b)$. When $N = 1$, $\Delta^N(s)$ is simply denoted by $\Delta(s)$; also $\Delta^N(1)$ is simply denoted by Δ.

THEOREM 6.2. *Suppose* D *is a connected open subset of* $\mathbf{C}^n$. *Suppose* $0 \leqq a < b$ *in* $\mathbf{R}^N$ *and* V *is a subvariety of* $D \times G^N(a, b)$ *whose every branch has dimension* $\geqq n + 1$. *Suppose* A *is a thick set in* D *such that, for every* $t \in A$, $V \cap (\{t\} \times G^N(a, b))$ *can be extended to a subvariety in* $\{t\} \times \Delta^N(b)$. *Then* V *can be extended uniquely to a subvariety* $\tilde{V}$ *in* $D \times \Delta^N(b)$ *such that every branch of* $\tilde{V}$ *has dimension* $\geqq n + 1$.

(Recall that a subset A of an open subset D of $\mathbf{C}^n$ is called a thin set if $A \subset \bigcup_{i=1}^{\infty} A_i$ and A_i is a subvariety of codimension $\geqq 1$ in some open subset of D. A subset of D is called thick if it is not thin.)

Proof. The uniqueness of $\tilde{V}$ follows from Lemma 6.1 (b).

To prove the existence of $\tilde{V}$, we introduce the following notations: Suppose E is a subset of $\mathbf{C}^n \times \mathbf{C}^N$, H is a subset of $\mathbf{C}^n$, and $0 \leqq c < d$ in $\mathbf{R}^N$. $E(H)$ denotes $E \cap (H \times \mathbf{C}^N)$. When $H = \{t\}$, we write $E(t)$ instead of $E(\{t\})$. $E_{(c,d)}$, $E_{[c,d]}$, and $E_{[c,d)}$ denote respectively $E \cap (\mathbf{C}^n \times G^N(c, d))$, $E \cap (\mathbf{C}^n \times \overline{G^N(c, d)})$, and $E \cap (\mathbf{C}^n \times (G^n(c, d) \cap \Delta^N(d)))$.

Let A' be the set of all $t \in A$ such that, for every open neighborhood U of t in D, $U \cap A$ is thick. It is clear that A' is thick.

(a) We make the following additional assumptions:

$$V \text{ is of pure dimension } d ,$$
$$\dim V(t) \leqq d - n \quad \text{for} \quad t \in A' .$$

We are going to prove that for every $t \in A'$ there exists an open neighborhood W of t in D such that $V(W)$ can be extended to a subvariety in $W \times \Delta^N(b)$. For $t \in A'$, let $V(t)^{\sim}$ be the pure-dimensional subvariety in $\{t\} \times \Delta^N(b)$ which extends $V(t)$. Fix $t^0 \in A'$. Take $a < a' < b' < b$ in $\mathbf{R}^N$. If $V(t^0) = \varnothing$, then $(W \times \mathbf{C}^N) \cap V_{[a',b']} = \varnothing$ for some open neighborhood W of t in D, and, by Lemma 6.1 (a), $W \cap V_{(a,b']} = \varnothing$, which implies that $V(W)$ is a subvariety of $W \times \Delta^N(b)$. Hence we can assume that $V(t^0)$ has pure dimension $d - n$. By the theorem on the existence of special analytic polyhedra (see the appendix of Chapter 2 of [29]) there exist holomorphic functions $f_1, \cdots, f_k$ on $\mathbf{C}^n \times \Delta^N(b)$ (where $k = d - n$) and an open neighborhood U of $\{t^0\} \times \overline{\Delta^N(a')}$ in $D \times \Delta^N(b')$ such that

$$V(t^0)^{\sim}_{[0,a']} \subset U \cap V(t^0)^{\sim} \cap F^{-1}(\Delta^k) \Subset U \cap V(t^0)^{\sim}$$

where $F: \mathbf{C}^n \times \mathbf{C}^N \to \mathbf{C}^k$ is defined by $f_1, \cdots, f_k$. There exists a relatively compact open neighborhood B of $\{t^0\} \times \overline{\Delta^N(a')}$ in U with

$$B \cap V(t^0)^{\sim} = U \cap V(t^0)^{\sim} \cap F^{-1}(\Delta^k) .$$

Choose $0 < \alpha < 1$ such that

$$V(t^0)^{\sim}_{[0,a']} \subset U \cap V(t^0)^{\sim} \cap F^{-1}(\Delta^k(\alpha)) .$$

Take $\alpha < \beta < 1$. There exists an open neighborhood W' of t^0 in D such that

(i) $W' \times \overline{\Delta^N(a')} \subset B$;

(ii) $V \cap (W' \times \partial(\Delta^N(a'))) \subset F^{-1}(\Delta^k(\alpha))$;

(iii) $U \cap V_{[a',b)} \cap F^{-1}(\overline{\Delta^k(\beta)})$ is disjoint from $(\partial B)(W')$.

The map φ from $X := V \cap B(W')_{(a',b')} \cap F^{-1}(G^k(\alpha, \beta))$ to $W' \times G^k(\alpha, \beta)$

defined by $(t_1, \cdots, t_n, f_1, \cdots, f_k)$ (where $t_1, \cdots, t_n$ are the coordinates of $\mathbf{C}^n$) is proper, because, if K is a compact subset of W' and $\alpha < \alpha' < \beta' < \beta$, then the inverse image of $K \times \overline{G^k(\alpha', \beta')}$ under φ is

$$\bar{B} \cap V_{[a',b']}(K) \cap F^{-1}\big(\overline{G^k(\alpha, \beta)}\big)\ ,$$

which is compact. For $t \in W' \cap A'$ the map ψ_t from

$$V(t)^*: = B \cap V(t)^{\sim} \cap F^{-1}\big(\Delta^k(\beta)\big)$$

to $\{t\} \times \Delta^k(\beta)$ is proper, because, if $0 < \beta' < \beta$, then the inverse image of $\{t\} \times \overline{\Delta^k(\beta')}$ under ψ_t is

$$\bar{B} \cap V(t)^{\sim} \cap F^{-1}\big(\overline{\Delta^k(\beta')}\big)\ ,$$

which is compact.

Now let us recall the following lemma which can be found in [29].

LEMMA 6.3. *Suppose Ω is a Stein open subset of $\mathbf{C}^n$ and $\varphi: \Omega \to \mathbf{C}^k$ is a holomorphic map. Suppose U is a connected open subset of $\mathbf{C}^k$, Ω' is an open subset of Ω, and X is a subvariety in Ω' such that φ/X makes X an analytic cover over U. Suppose $\tilde{U}$ is a connected open subset of $\mathbf{C}^k$ containing U.*

(A) *If X_i^* is a subvariety in an open neighborhood Ω_i^* of Ω' in Ω such that $X_i^* \cap \Omega' = X$ and φ/X_i^* makes X_i^* an analytic cover over $\tilde{U}$* ($i = 1, 2$), *then $X_1^* = X_2^*$.*

(B) *Suppose one of the following three conditions* (a), (b) *and* (c) *is satisfied.*

(a) (i) *There exists $K \subset \Omega$ such that $K \cap \varphi^{-1}(\tilde{U}) \to \tilde{U}$, induced by φ, is proper;*

(ii) $X \subset K \cap \varphi^{-1}(U)$;

(iii) *Every holomorphic function on U which is locally bounded on $\tilde{U}$ can be extended to a holomorphic function on $\tilde{U}$.*

(b) *Every holomorphic function on U can be extended to a holomorphic function on $\tilde{U}$.*

(c) (i) *There exist a connected open subset D of $\mathbf{C}^{k-1}$ and $0 < \alpha < \beta$ such that $U = D \times \big(\Delta(\beta) - \overline{\Delta(\alpha)}\big)$ and $\tilde{U} = D \times \Delta(\beta)$.*

(ii) *There exists a thick set A in D satisfying the following property: for $t \in A$, $X_t: = X \cap \varphi^{-1}\big(\{t\} \times \mathbf{C}\big)$ can be extended to a subvariety $\tilde{X}_t$ in some open neighborhood Ω_t of Ω' in Ω such that $\varphi | \tilde{X}_t$ makes $\tilde{X}_t$ an analytic cover over $\{t\} \times \Delta(\beta)$.*

Then X can be extended to a subvariety $\tilde{X}$ in $\Omega \cap \varphi^{-1}(\tilde{U})$ such that $\varphi | \tilde{X}$ makes $\tilde{X}$ an analytic cover over $\tilde{U}$.

By Lemma 6.3 (B) (b) (c) applied to the analytic cover

$$\varphi\colon X \longrightarrow W' \times G^k(\alpha, \beta)\ ,$$

X can be extended to a subvariety $\tilde{X}$ in

$$\bigl(W' \times \Delta^N(b')\bigr) \cap F^{-1}\bigl(\Delta^k(\beta)\bigr)$$

such that the map

$$\tilde{X} \longrightarrow W' \times \Delta^k(\beta)$$

defined by $(t_1, \cdots, t_n, f_1, \cdots, f_k)$ makes $\tilde{X}$ an analytic cover over $W' \times \Delta^k(\beta)$. By Lemma 6.3 (A), $\tilde{X}(t^0) = V(t^0)^\sim$. Hence

$$\tilde{X}(t^0)_{[0,a']} \subset F^{-1}\bigl(\Delta^k(\alpha)\bigr)\ .$$

Take $\alpha < \beta^* < \beta$. There exists an open neighborhood W'' of t^0 in W such that

$$(\dagger) \qquad \tilde{X}(W'')_{[0,a']} \cap F^{-1}\bigl(\overline{\Delta^k(\beta^*)}\bigr) \subset F^{-1}\bigl(\Delta^k(\alpha)\bigr)\ .$$

Since $\tilde{X} \cap F^{-}\bigl(\overline{\Delta^k(B^*)}\bigr)$ is disjoint from $(\partial\beta)(t^0)$, there exists an open neighborhood W of t^0 in W'' such that

$$(\dagger\dagger) \qquad \tilde{X} \cap F^{-1}\bigl(\overline{\Delta^k(\beta^*)}\bigr) \text{ is disjoint from } (\partial B)(W)\ .$$

Let V' be the union of

$$\tilde{X} \cap B(W) \cap F^{-1}\bigl(\Delta^k(\beta^*)\bigr)$$

and

$$V \cap B(W)_{(a',b)} \cap F^{-1}\bigl(\mathbf{C}^k - \overline{\Delta^k(\alpha)}\bigr)\ .$$

As the union of two locally closed subvarieties in $B(W)$, V' is a local subvariety in $B(W)$. Take $\alpha < \alpha' < \beta' < \beta^*$. Since X is the intersection of $\tilde{X}$ and

$$B(W')_{(a',b')} \cap F^{-1}\bigl(G^k(\alpha, \beta)\bigr)\ ,$$

it follows from (ii) and ($\dagger$) that V' is the union of

$$\tilde{X} \cap B(W) \cap F^{-1}\bigl(\overline{\Delta^k(\beta')}\bigr)$$

and

$$V \cap B(W)_{[a'b)} \cap F^{-1}\bigl(\mathbf{C}^k - \Delta^k(\alpha')\bigr)$$

which are both closed subsets of $B(W)$. Hence V' is a subvariety of $B(W)$.

Let

$$V^* = V' \cup \bigl((V - B)(W)\bigr)\ .$$

We claim that V^* is a subvariety of $W \times \Delta^N(b)$. Take $x \in (\partial B)(W)$ arbitrarily. Because of (i), (iii) and ($\dagger\dagger$), we can choose an open neighborhood Q of x in U which is disjoint from $W \times \overline{\Delta^N(a')}$,

$$U \cap V_{[a',b)} \cap F^{-1}\bigl(\overline{\Delta^k(\beta)}\bigr)\ ,$$

and $\tilde{X} \cap F^{-1}\bigl(\overline{\Delta^k(\beta^*)}\bigr)$. Then $Q \cap V^* = Q \cap V$. It follows that V^* is a subvariety of $W \times \Delta^N(b)$. The claim is proved. Since $V^*_{(b',b)} = V(W)_{(b',b)}$, by Lemma 6.1 (b), $V^*_{(a,b)} = V(W)$. Hence we have proved that for every $t \in A'$ there exists an open neighborhood W of t in D such that $V(W)$ can be extended to a subvariety in $W \times \Delta^N(b)$.

(b) Let D' be the largest open subset of D such that $V(D')$ can be extended to a subvariety in $D' \times \Delta^N(b)$. It follows from (a) that, under the following additional assumptions:

$$\begin{cases} V \text{ is of pure dimension } d, \\ \dim V(t) \leqq d - n \text{ for } t \in D, \end{cases}$$

D' is a nonempty closed subset of D. Hence $D' = D$.

(c) Let $\pi: V \to D$ be induced by the natural projection $D \times G^N(a, b) \to D$. Let S be the closure of the set of points of V where the rank of π is $<n$. Take $a < a' < b' < b$. Then $\pi(S_{[a',b']})$ is a closed thin set in D. Let $D' = D - \pi(S_{[a',b']})$.

Let $V = \bigcup_{i \in I} V^{(i)}$ be the decomposition of V into pure-dimensional components. Let C be the set of all $t \in D$ such that, for some $i \neq j$, some nonempty open subset of $V^{(i)}(t)$ is contained in $V^{(j)}$. Then C is thin in D. This is a consequence of the following proposition (appendix of Chapter 2 of [29]):

PROPOSITION 6.4. *Suppose D is an open subset of $\mathbf{C}^k$, G is an open subset of $\mathbf{C}^l$, and V, W are subvarieties of $D \times G$ such that V is irreducible. Suppose A is a thick set in D and for every $t \in A$ some nonempty open subset of $V \cap (\{t\} \times G)$ is contained in W. Then $V \subset W$.*

For $t \in A - C$, $V^{(i)}(t)$ can be extended to a subvariety of $\{t\} \times \Delta^N(b)$ for $i \in I$. By applying (b) to the subvariety $V^{(i)}(D')_{(a',b')}$ of $D' \times G^N(a', b')$ and to the thick set $A - \pi(S_{[a',b']}) - C$ in D', we conclude that $V^{(i)}(D')_{(a',b')}$ can be extended to a subvariety in $D' \times \Delta^N(b')$ for $i \in I$. By Lemma 6.1 (b), $V^{(i)}(D')$ can be extended to a subvariety in $D' \times \Delta^N(b)$ for $i \in I$.

Let L be an arbitrary relatively compact open subset of D. Then there exists an N-dimensional plane T in $\mathbf{C}^n \times \mathbf{C}^N$ such that for some nonempty open subset Q of D' and for some open neighborhood R of L in D we have

(i) $(Q + T) \cap (\mathbf{C}^n \times \Delta^N(b)) \subset D' \times \Delta^N(b)$,

(ii) $L \times \Delta^N(b) \subset (R + T) \cap (\mathbf{C}^n \times \Delta^N(b)) \subset D \times \Delta^N(b)$,

(iii) $\dim (x + T) \cap V^{(i)} \leqq \dim V^{(i)} - n$ for $x \in V^{(i)}$ and $i \in I$, where

$$Q + T = \{x + y : x \in Q, y \in T\},$$
$$x + T = \{x + y : y \in T\}.$$

This follows from the next lemma which can be found in [29].

LEMMA 6.5. *Suppose A is a subvariety of dimension $\leqq k$ in an open subset Ω of $\mathbf{C}^n$ and $1 \leqq l \leqq k$. Let $G_{n-l}(\mathbf{C}^n)$ be the Grassmannian of all $(n-l)$-dimensional planes in $\mathbf{C}^n$ passing through* 0. *Let R be the set of all $T \in G_{n-l}(\mathbf{C}^n)$ such that* $\dim A \cap (x + T) \geqq k - l + 1$ *for some $x \in A$. Then R is thin in $G_{n-l}(\mathbf{C}^n)$.*

By (b) there exists a subvariety $\tilde{V}^{(i)}$ in

$$(R + T) \cap (\mathbf{C}^n \times \Delta^N(b))$$

such that

$$\tilde{V}^{(i)} \cap (R + T) \cap (\mathbf{C}^n \times G^N(a, b)) = V^{(i)} \cap (R + T) \cap (\mathbf{C}^n \times G^N(a, b)) \quad (i \in I)\ .$$

$\tilde{V}^{(i)} \cap (L \times \Delta^N(b))$ is a subvariety in $L \times \Delta^N(b)$ extending $V^{(i)}(L)$ for $i \in I$. Since L is a subvariety of a relatively compact open subset of D, V can be extended to a subvariety in $D \times \Delta^N(b)$. Q.E.D.

COROLLARY 6.6. *Suppose $0 \leqq a < b$ in $\mathbf{R}^N$, D is a connected open subset of $\mathbf{C}^n$, D' is a nonempty open subset of D, and V is a subvariety in $(D \times G^N(a, b)) \cup (D' \times \Delta^N(b))$ whose every branch has dimension $\geqq n + 1$. Then V can be extended uniquely to a subvariety in $D \times \Delta^N(b)$ whose every branch has dimension $\geqq n + 1$.*

We shall employ the following notation in the remainder of our discussion. Let

$$\varphi_\sigma = \left(1 + \frac{1}{r^2}\right) z_\sigma \bar{z}_\sigma - \sum_1^n z_j \bar{z}_j - 1\ ; \qquad r > 0;\ \sigma = 1, \cdots, s$$

and $\mathfrak{A} = \bigcap_{\sigma=1}^s (\varphi_\sigma < 0)$; $\mathfrak{R} = \partial\mathfrak{A}$; $\mathfrak{T} = \bigcup (\varphi_\sigma > 0)$. Further let p be a point of $\mathfrak{R}$ and U be a neighborhood of p.

The following two theorems on the local extension of analytic sets which we are going to use are slight variation of Lemma 6.1 and Theorem 6.2 (cf. [26], page 125, Theorem 2 and Theorem D).

THEOREM 6.7. *Let M be a pure k-dimensional analytic subset in $U \cap \mathfrak{A}$ and $k \geqq s + 1$. Then M admits unique extension across p. That is there exist arbitrarily small neighborhoods V of p and a pure k-dimensional analytic set M_V in V with the following properties:*

(1) *$M_V \cap \mathfrak{A} = M \cap V$.*

(2) *The germs $(M_V)_p$ are all the same.*

(3) *M_V is the union of finitely many components which all contain p.*

(4) *$M_V \cap \mathfrak{A}$ has only finitely many components.*

(5) *If m represents a prime germ of M_V at a point $q \in \mathfrak{R}$, then $m \cap \mathfrak{A} \neq \emptyset$; hence $m \in \overline{M \cap V \cap \mathfrak{A}}$.*

In addition: every holomorphic (meromorphic) function on the normalization of M has unique extension to the normalization of M_V.

The uniqueness of the extension follows directly from the next theorem.

THEOREM 6.8. *Let* M_1, M_2 *be a pure k-dimensional analytic set in U and* $k \geqq s$. *Suppose* $M_1 \cap \mathfrak{A} = M_2 \cap \mathfrak{A}$. *Then the germs of* M_1, *and* M_2 *are equal at p. Further, for each neighborhood V of p*: $M_1 \cap V \cap \mathfrak{A} \neq \varnothing$, $M_2 \cap V \cap \mathfrak{A} \neq \varnothing$.

An important consequence of Theorem 6.7 is the following theorem.

THEOREM 6.9. *Let M be an analytic and pure k-dimensional set in U with* $k \geqq s+1$. *Further let M be irreducible at p. Then there are arbitrary small neighborhoods V at p so that* $M \cap V \cap \mathfrak{A}$ *is also irreducible.*

Proof. Because of (4) of Theorem 6.7, we can assume that $M \cap V \cap \mathfrak{A}$ has only finitely many components. Then V can also be chosen so that each of these components has p as boundary point. Let M_1 and M_2 be such components. Each of them is an analytic set in $V \cap \mathfrak{A}$ and by Theorem 6.7 has a uniquely determined extension at p which is contained in M. The germs generated by the extension at p must coincide with germs of M since M is irreducible at p. Then M_1 and M_2 must be equal.

Following Rothstein, by an analytic surface-piece (or surface) F of dimension k over $\mathbf{C}^n$, we mean F is a normalization of a local analytic set in $\mathbf{C}^n$. We shall use the following convention. By a real analytic point set R, we mean a closed set with the property: each point has a neighborhood U, so that $U \cap R$ is described by finitely many real analytic equations and inequalities. In this case then each point p of R has an arbitrary small open connected neighborhood U so that 1) $U \cap R$ is path-connected and $\bar{U} \cap R = \overline{U \cap R}$. Here we use $\bar{M}$ for the closure of M. 2) $U - R$ has finitely many connected components. Each of them has p as a boundary point.

Connectedness always means path-connectedness unless otherwise specified.

We now prove the following extension theorem of analytic surface pieces which will play a fundamental role in the sequel.

Let F be an analytic surface-piece over $\mathbf{C}^n$. Then $j: F \to \mathbf{C}^n$ is an embedding which associates at each $p \in F$ its coordinates in $\mathbf{C}^n$.

THEOREM 6.10. *Assumptions*: (1) F, F' *are subspaces of a k-dimensional analytic surface* $\tilde{F}$ *over* $\mathbf{C}^n$; $k \geqq s+1$.

(2) $F' \subseteq \tilde{F}$.

(3) $\{\overline{F' \cap \mathfrak{A}} \cap \mathfrak{R}\} \supset \{\bar{F}' \cap \mathfrak{R}\}$.

(4) $\overline{\partial F \cap \mathfrak{A}} \subseteq F'$.

(5) $j(F) \cap \mathfrak{T} = \varnothing$.

(6) *If p_m is a sequence of points of F without a limiting point in $F \cup F'$, then the limiting points of $j(p_m)$ are in $\mathfrak{R}$.*

Conclusion: there exists a surface-piece F^* with the following properties:

1) $F^* \cap \mathfrak{A} = (F' \cup F) \cap \mathfrak{A}$.

2) $F \subset F^*$.

3) *For* $R := F^* \dashrightarrow \partial F'$, *then* $j(R) \subset \mathfrak{T}$.

4) $\overline{F^* \cap \mathfrak{A} \cap \mathfrak{R}}$ *contains* $\bar{F}^* \cap \mathfrak{R}$.

Moreover, each holomorphic (meromorphic) function on $F \cup F'$ can be extended to a holomorphic (meromorphic) function on $F \cup F' \cup F^*$.

In fact, we can write $F^* = F \cup F' \cup M^*$ where M^* is a neighborhood of those boundary points of F which do not lie in F'. All boundary points of $M^* \cap (\mathfrak{A} \cup \mathfrak{R})$ lie inside $F \cup F'$. So $R = \partial M^* - (\partial M^* \cap (F \cup F'))$. M^* can be made arbitrarily small.

Remark 6.12. For surfaces over C^n, we write $F \cap \mathfrak{A}$ instead of $j^{-1}(j(F) \cap \mathfrak{A})$. $\mathfrak{T}$ is the complement of $\mathfrak{A} \cup \partial\mathfrak{A}$.

Before proving Theorem 6.10, we first prove the following lemma.

LEMMA 6.13. *Under the assumption of Theorem* 6.10: *If $p_m \in F$ and $j(p_m)$ converges to $q \in \mathfrak{R}$, then there exists a neighborhood U of q and an analytic set M in $\bar{U}$ (that is, M analytic in $U^* \supset U$) with the properties:*

(1) $\{U \cap j(F \cup F')\} \supset \{M \cap U \cap \mathfrak{A}\} \supset \{U \cap j(F)\}$;

(2) *M has only finitely many components and each of them contains q.*

(3) *If m is a representative of a prime germ of $M \cap \bar{U}$ at $p \in \mathfrak{R}$, then m has points in $\mathfrak{A}$.*

(4) *If m is a prime germ of $M \cap \bar{U}$ at $p \in R$, then m is a boundary point of F or m is an interior point of F' or else both are true.*

Proof. Since $\partial F \cap \mathfrak{A}$ lies completely in F', there exist at most finitely many q_i in $\overline{\partial F \cap \mathfrak{A}}$ with $j(q_i) = q$. Therefore there exist a neighborhood V_* of q and a neighborhood U_i of q_i with the properties:

(α) $U_i \subset F'$;

(β) $j(U_i)$ is an analytic set in V_*;

(γ) If $p \in \partial F \cap \mathfrak{A}$ and $j(p) \in V_*$, then p lies in one of the U_i.

(α) through (γ) are always realized because of the assumptions of Theorem 6.10. We claim:

(*) *The union N of all $j(U_i) \cap \mathfrak{A}$ with $j(F) \cap V_*$ is an analytic set in $V_* \cap \mathfrak{A}$.*

For the proof, let s be a point in $V_* \cap \mathfrak{A}$. The boundary points of F situated above s are interior points of F' (Assumption (4)). Then above s, there exist only finitely many interior points p_k and finitely many boundary points γ_l of F. The γ_l belong to the union of U_i because of (γ). In a neighborhood of s, N is therefore the union of finitely many analytic subsets. It follows that N is locally analytic there. It follows further that N is closed in $V_* \cap \mathfrak{A}$. Hence (*) follows.

Since both F and F' are purely k-dimensional, N is purely k-dimensional. By the construction we have further,

(**) $$\{j(F \cup F') \cap V_*\} \supset N \supset \{j(F) \cap V_*\}\,.$$

Because of Theorem 6.7, N extends across q. There are arbitrarily small neighborhoods V of q and an analytic subset M_V of V with the properties in Theorem 6.7. We choose U so that $\bar{U} \subset V$ and set $M := M_V$. Then M is the set we want. (1) is clear. (2) and (3) are given in Theorem 6.7. Finally (4) follows, because if $m \in M \cap \bar{U} \cap \mathfrak{R}$, then m lies on the boundary of F or in $\bar{F}'$ because of (3). In the second case, it follows that $m \in \bigcup j(U_i)$, i.e., $m \in F'$ and $m \notin \partial F'$.

Proof of Theorem 6.10. Let H be the set of limit points q of $j(F)$ in $\mathfrak{R}$. H is closed. For each q given as in Lemma 6.13, there exist a neighborhood $U(q)$ and an analytic set $M(q)$ in $U(q)$ with the properties (1)-(4). Let $m(q)$ be the union of an analytic surface-piece above $M(q)$ whose points are the prime germs of $M(q)$. Further, let $F(q) := F \cup F' \cup m(q)$. Obviously $F(q)$ is an analytic surface-piece because of properties (1)-(3) of Lemma 6.13. Condition (1) is satisfied: $F(q) \cap \mathfrak{A} \cap U(q) = (F \cup F') \cap \mathfrak{A} \cap U(q)$. Moreover, the following is obvious:

(*) For each neighborhood $\hat{U}(q) \Subset U(q)$, the part of F lying over $\hat{U}$ is relative compact in $F(q)$.

Under $\hat{U}(q)$, $q \in H$, there exist finitely many $\hat{U}(q_i)$ with the associated $U(q_i)$, $m(q_i)$ and $F(q_i) := F \cup F' \cup M(q_i)$, so that $H \Subset \bigcup \hat{U}(q_i)$. Now let F^* be the union of the $F(q_i)$. This F^* satisfies the claim. Because of (*), the part F_1 of F lying over $\bigcup \hat{U}(q_i)$ is relatively compact in F^*. $F - F_1$ is a compact subset of $(F \cup F') \cap \mathfrak{A}$ because of assumption (6). Hence 2), which states $F \Subset F^*$, follows. From the construction, 1) follows immediately. From Lemma 6.13, 4), we have: if $r \in \mathfrak{R}$ is a boundary point of some $m(q_i)$, then r is an interior point of F' or boundary point of F. In both cases, r is an interior point of F^*. Since the boundary points of $m(q_i)$ lying over $\mathfrak{A}$ are interior points of $F \cup F'$, 3) follows. Finally to obtain 4), let p be a point of $\bar{F}^* \cap R$. It is required to show: $p \in S := \overline{F^* \cap \mathfrak{A}} \cap \mathfrak{R}$. If p is a boundary point of F^*, then p must be a boundary point of F' because of 3). By assumption (3), it follows that $p \in S$. Thus let p be an interior point of F^*. If $p \in F'$, then $p \in S$. If p is not in F', then p is an interior point of $m(q_i)$. Then by Lemma 6.13, 3), it follows that $p \in S$. Therefore the existence of F^* is proved.

At the same time because of Theorem 6.7, the function extends to F^*.

THEOREM 6.14. *Let $\tilde{F}$ be an analytic k-dimensional surface; $k \geqq s + 1$. Further let M be a closed, connected* (i.e., *if $M = M_1 \cup M_2$; M_1, M_2 closed and $M_1 \cap M_2 = \emptyset$, then $M_1 = M$ or $M_2 = M$*) *set of interior points of $\tilde{F}$ and $M \subset \tilde{F} \cap \mathfrak{R}$. Finally let $U(M) \subset \tilde{F}$ be a connected open neighborhood of M, which satisfies the following condition: Each connected component of $U(M) \cap \mathfrak{A}$ has a boundary point of M. Then $U(M) \cap \mathfrak{A}$ is connected.*

Proof. Let the U_i with $\bar{U}_i \cap M = R_i$ be the connected components of $U(M) \cap \mathfrak{A}$. Then $M = \bigcup \overline{R_i}$; $R_i = \overline{R_i}$. Suppose that there are several U_i. Since M is connected, there exist U_1, U_2 and a point $r \in M$ so that $r \in R_1 \cap R_2$. By Theorem 6.9, r has an arbitrary small neighborhood $V(r) \subset \tilde{F}$, for which $V(r) \cap \mathfrak{A}$ is connected. U_1 and U_2 must therefore be connected to each other; so $U_1 = U_2$.

Before we can continue, we have to introduce the new concepts of cycles and arcs on an analytic surface.

Following the terminology of Rothstein and Sperling, by a cycle Z of the analytic surface F, we mean here that Z is a closed, connected set of the interior of F, which satisfies the following conditions.

a) There exists an arbitrarily small neighborhood U of Z in F such that Z separates U into two open parts U^+, U^- with

1) $U^+ \cap U^- = \emptyset$, 2) $U = U^+ \cup Z \cup U^-$, 3) $Z = \partial U^+ \cap \partial U^-$.

b) If p, q are points in $U - Z$ and $w \subset U$ is any path from p to q, then there exists in every neighborhood of w, a path $\tilde{w}$ from p to q, which is divided by the cycle Z into finitely many parts, belonging alternatively to U^+ and U^-. (Hence the path goes from U^+ through the intersection point $\tilde{w} \cap Z$ towards U^- or vice versa.)

c) Z is "piecewise smooth"; i.e. Z is the union of finitely many Z_i in such a way that $j(Z_i)$ is a closed real analytic point set of topological dimension $2k - 1$ and such that the embedding $j: Z_i \to j(Z_i)$ is topological.

Remark 6.15. We do not demand that U^+ or U^- be connected. This assumption will not simplify the proof. In case F is a manifold, one can naturally assume that U^+ and U^- are connected. From c) it follows that every intersection of Z with a real-analytic point set has only finitely many connected components. This we shall often use without special note.

Definition 6.16. Let F be an analytic surface over $\mathbf{P}^n$; $\dim F = k$. The point set N in F is called bounded, if there exists an analytic plane E of dimension q in $\mathbf{P}^n$ with $q + k = n + 1$, so that $E \cap \bar{N} = \emptyset$.

Following Rothstein and Sperling, we now introduce the notion of arc. We first fix our notation. The set $\mathfrak{A}(r)$: $= \bigcap_1^s (\varphi_\sigma < 0)$, $s = n - q$, $q + k = n + 1$, with $\varphi_\sigma := z_\sigma \bar{z}_\sigma (1 + 1/r^2) - \sum_1^n z_j \bar{z}_j - 1$ is a neighborhood of the plane $z_1 = \cdots = z_{n-q} = 0$ in $\mathbf{P}^n$. Further, let $\mathfrak{R}(r) = \partial \mathfrak{A}(r)$ and $\mathfrak{I}(r)$ be the complement of $\mathfrak{A}(r) \cup \mathfrak{R}(r)$. We write also $\mathfrak{A}$ instead of $\mathfrak{A}(r)$ if there is no confusion. Let Z be a bounded cycle in F, $\dim F = k \geqq s + 1$, so that $Z \cap \mathfrak{A}(r_0) = \emptyset$ for some r_0.

There exists an ω such that $Z \cap \mathfrak{A}(\omega) = \emptyset$ and $Z \cap \mathfrak{R}(\omega) \neq \emptyset$. If $Z \cap \mathfrak{R}(r) \neq \emptyset$, then $Z \cap \mathfrak{A}(r)$ decomposes into finitely many connected components $A(r)$, called "A-arcs". Also $Z \cap \mathfrak{R}(r)$ decomposes into finitely many connected components K_j, called "K-arcs". The latter ones are closed; on the other hand A-arcs are not. All boundary points of A-arcs lie on K-arcs. But there exist K-arcs which contain no boundary points of A-arcs. This is the case for all $K(\omega)$. These K-arcs, which contain no boundary points of A-arcs, are called T-arcs and will be denoted by $T(r)$.

LEMMA 6.17 (embedding of K-arcs in strips). *For each K-arc K, there exists an arbitrarily small neighborhood $S(K) \subset F$ with the properties:*

1) *$S(K)$ is open and connected; $K \subset S(K)$;*

2) *$S(K) \cap \mathfrak{A}$ is connected.*

Moreover

3) a) *$S(K) \cap \mathfrak{R}$ is connected,* b) $\overline{S(K)} \cap \mathfrak{R} = \overline{S(K) \cap \mathfrak{R}}$.

4) a) *$Z \cap S(K)$ is connected,* b) $\overline{S(K)} \cap Z = \overline{S(K) \cap Z}$.

c) $K = \mathfrak{R} \cap \overline{S(K) \cap Z} = \mathfrak{R} \cap S(K) \cap Z$.

5) *$\overline{S(K) \cap \mathfrak{A}} \cap \mathfrak{R}$ contains $\overline{S(K)} \cap \mathfrak{R}$.*

Remark 6.18. Neighborhoods with these properties are called strips. That strips with property 2) exist is a function theoretic result.

Proof of Lemma 6.17. Since $\mathfrak{R}$, F, Z, K are real-analytic sets, then at each point p in K, there exists an arbitrarily small open neighborhood $U(p) \subset F$, so that 3) and 4) a), b) with $U(p)$ in place of $S(K)$ are fulfilled and in place of 4) c) we have: $U(p) \cap K$ is connected and $\overline{U(p) \cap K} = \mathfrak{R} \cap \overline{U(p) \cap Z} = \mathfrak{R} \cap \overline{U(p)} \cap Z$. Further, every component of $U(p) \cap \mathfrak{A}$ has p as a boundary point. Let $S(K) := \bigcup U(p_i)$ be a finite covering of K. Then for $S(K)$, 1), 3) and 4) hold. Also 5) is fulfilled. For by Theorem 6.8, it follows $p \in \overline{S(K) \cap \mathfrak{A}}$ if $p \in S(K) \cap \mathfrak{R}$. Now $p \in \overline{S(K)} \cap \mathfrak{R}$, so by 3) b), also $p \in \overline{S(K) \cap \mathfrak{R}}$.

Finally we assert that 2) is also fulfilled. By Theorem 6.14 it suffices to show that every component S of $S(K) \cap \mathfrak{A}$ has boundary points on K. This is now clear since every component of $U(p_i) \cap \mathfrak{A}$ should have the boundary point p_i.

Now we are going to deduce some consequences for T-arcs. $T := T(r)$ is embedded into a strip $S(T)$ according to Lemma 6.17. As T by definition contains no boundary points of A-arcs and $Z \cap S(T)$ is connected by 4), Lemma 6.17, so $Z \cap S(T) \cap \mathfrak{A}$ is empty. However $S(T) \cap \mathfrak{A}$ is not empty (Theorem 6.8). We can assume $S(T) \subset U$, where U is a neighborhood of Z introduced before Remark 6.15. Let $S^+ := U^+ \cap S(T)$ and $S^- := U^- \cap S(T)$. As $Z \cap S(T) \cap \mathfrak{A}$ is empty and $S(T) \cap \mathfrak{A}$ is connected, exactly one of the following two cases occurs: $(+)$ $S^- \cap \mathfrak{A} \neq \emptyset$; $S^+ \cap \mathfrak{A} = \emptyset$ or $(-)$ $S^+ \cap \mathfrak{A} \neq \emptyset$; $S^- \cap \mathfrak{A} = \emptyset$. Assume $(+)$ holds. We claim that then $(*)$ $T = \overline{S^+} \cap \mathfrak{R}$.

For the proof, let t be a point of T. Then t lies in the interior of $S(T)$ and on Z. Hence t is also in $\overline{S^+} \cap \mathfrak{R}$. Secondly, let $t \in \overline{S^+} \cap R$. Since $S^+ \cap \mathfrak{A} = \emptyset$, then $S(T) \cap \mathfrak{A} = \overline{S^-} \cap \mathfrak{A}$. Because of 5), Lemma 6.17, t is then in $\overline{S^-} \cap \mathfrak{R}$. Hence t is in Z. It follows that $t \in \overline{S(T)} \cap Z$. According to 4), c), Lemma 6.17, it follows finally that $t \in T$.

Now let R be the set of these boundary points of S^+ which do not lie in Z. Then $j(R) \Subset \mathfrak{f}$.

Suppose $r \in \bar{R}$, then $r \in \partial S(T)$; hence r does not lie in T. Because of $(*)$, r does not lie in $\overline{S^+} \cap \mathfrak{R}$. As $S^+ \cap \mathfrak{A}$ is empty it follows that $j(R) \Subset \mathfrak{f}$. Therefore the following lemma is proved.

LEMMA 6.19. *For every T-arc, there exists a strip $S(T)$ such that either for the piece of surface $F(T) := S^+$ or for $F(T) := S^-$, we have*

a) $F(T) \subset \mathfrak{f}$,

b) $Z \cap S(T) \subset \partial F(T)$,

c) $\partial F(T) - (Z \cap S(T)) \Subset \mathfrak{f}$.

Next we are going to deduce two important results for A-arcs. In the following, let V be a neighborhood of Z, which is separated into the parts V^+, V^- by Z. Further, let the set of $A(r')$-arcs α_i, which satisfy the following conditions, be given:

(1) To each α_i, we associate a piece of surface f_i (not necessarily connected) so that

1.1 $\alpha_i \subset \partial f_i$ and every connected component of f_i has boundary point in α_i.

1.2 α_i lies neither on the boundary nor in the interior of $f_k \neq f_i$.

1.3 f_i lies either on the positive or on the negative bank of α_i. This means that one of the following statements hold.

(+) For every $p \in \alpha_i$, there exists a neighborhood $U(p) \subset V$, so that $f_i \supset U(p) \cap V^+$; $\bigcup(f_i \cap U(p) \cap V^-) = \emptyset$ or

(−) For every $p \in \alpha_i$ there exists a neighborhood $U(p) \subset V$ so that $f_i \supset U(p) \cap V^-$; $\bigcup(f_i \cap U(p) \cap V^+) = \emptyset$.

(2) $j(f_i) \subset \mathfrak{A}(r')$. The boundary points of f_i lie either on $\bigcup \alpha_i$ or above $\mathfrak{R}(r')$.

Remark 6.20. The α_i's are naturally disjoint. Now because of (2) $\partial f_i \cap \mathfrak{A} \subset \bigcup \alpha_i$, it follows from 1.1 and 1.2 that either $f_i = f_k$ or $f_i \cap f_k = \emptyset$. It is explicitly allowed that different α can be assigned to the same f. In the proof of Theorem 6.24, we shall show in p. 108 (d) that the f are uniquely determined by α. We shall not use this temporarily.

Definition 6.21 (Statement E). If α_i, f_i satisfy the conditions (1), (2), we shall say: "$\bigcup \alpha_i$ bounds $\bigcup f_i$ in $\mathfrak{A}$".

Assumption for Lemma 6.22 *and Lemma* 6.23. Let $\hat{A}_\lambda$ be a connected component of $Z \cap \{\mathfrak{A}(r') \cap \mathfrak{R}(r')\}$. Then $\bigcup \hat{A}_\lambda = \bigcup \alpha_i \cup \bigcup K_j$ with α_i an A-arc and K_j a K-arc. Suppose $\hat{A}_\lambda \cap \mathfrak{A}(r') \neq \emptyset$ so that on each K there are boundary points of α. No K is also a T-arc. For each α_i, let f_i be assigned as above so that for α_i, f_i, Statement E holds. Finally we assume here that $\bigcup f_i \cup \bigcup \hat{A}_\lambda$ is connected.

Now the following lemma is of decisive significance.

LEMMA 6.22. *Let* $\hat{A} = \bigcup \alpha'_i \cup \bigcup k'_j$ *be a connected component of* $Z \cap \{\mathfrak{A}(r') \cup \mathfrak{R}(r')\}$. *Then either for all these* α_i*'s the statement* (+) *is satisfied or for all these* α_i*'s the statement* (−) *is correct. Suppose* (+) *holds. Then for those* K'_j *there exist arbitrarily small strips* $S(K)$ (cf. Lemma 6.17), *so that* $S(K) \cap Z \cap \mathfrak{A} \subset \bigcup \alpha'_i$, $S^-(K) \cap (\bigcup f_i) = \emptyset$, $S^+(K) \cap \mathfrak{A} \subset \bigcup f_i$ *and* $S^+(K) \cap (\bigcup f_i) \neq \emptyset$.

Proof. (1) We show that if $\alpha_1 \cup K \cup \alpha_2$ is connected, then f_1, f_2 lie on the same side of Z. Embed K according to Lemma 6.17 in a strip $S(K) \subset V$ in such a way that $S(K) \cap \mathfrak{A}$ is connected. $\bigcup \alpha_i$ separates $S(K) \cap \mathfrak{A}$ into connected subspaces of a surface-piece of V. There exist points p_1 in $f_1 \cap S(K) \cap \mathfrak{A}$ and p_2 in $f_2 \cap S(K) \cap \mathfrak{A}$. There exists also a path w from p_1 to p_2 with the properties: 1) $w \cup S(K) \cap \mathfrak{A}$; 2) w intersects $\bigcup \alpha_i$ only in finitely many points s_λ and runs from V^+ through s_λ to V^- or vice versa (cf. b) in the proof of Theorem 6.14). It follows now that as the boundary points of f_i lie either on $\bigcup \alpha_i$ or above $\mathfrak{R}$, the open piece of the path w_λ between s_λ and $s_{\lambda+1}$ either lies completely in f_i or no points of it lie in f_i. Further it follows from the assumption that if $w_\lambda \subset \bigcup f_i$, then $w_{\lambda+1} \cap (\bigcup f_i) = \emptyset$ and conversely that if $w_\lambda \cap (\bigcup f_i) = \emptyset$, then $w_{\lambda+1} \subset \bigcup f_i$. So we have finally that if $p_1 \in V^+$, then also $p_2 \in V^+$. Therefore the first part is proved.

(2) Suppose (+) holds. Then obviously $S^+(K) \cap (\bigcup f_i) \neq \emptyset$. By Lemma 6.17 $S(K) \cap Z$ is connected and $S(K) \cap Z \cap \mathfrak{R} = K$. Thence $p \in S(K) \cap Z \cap \mathfrak{A}$ can be connected to K through a path in $\mathfrak{A} \cap S(K) \cap Z$. Therefore $S(K) \cap Z \cap \mathfrak{A} \subset \bigcup \alpha_i'$. Further, since $S(K) \cap \mathfrak{A}$ is connected, it follows as in (1) that $S^+(K) \cap \mathfrak{A} \subset \bigcup f_i$ and $S^-(K) \cap \bigcup f_i = \emptyset$.

We repeat the assumption for Lemma 6.23: The $\hat{A}_\lambda$ are connected components of $Z \cap \{\mathfrak{A}(r') \cup \mathfrak{R}(r')\}$; $\bigcup \hat{A}_\lambda = \bigcup \alpha_i \cup \bigcup K_j$. On each of the K there lies a boundary point of α. The α_i, f_i satisfy Statement E. Finally let $\bigcup f_i \cup \bigcup \hat{A}_\lambda$ be connected.

LEMMA 6.23. *There exist a neighborhood F' of $\bigcup \hat{A}_\lambda$, $r^* > r'$ and a piece of surface F^* so that for all $\tilde{r} \leqq r^*$, the following is true: the components A_ρ, B_σ of $F' \cap Z \cap \mathfrak{A}(\tilde{r})$ together with the components of $F^* \cap \mathfrak{A}(\tilde{r})$ satisfy the Statement* E. *The A_ρ arise from $\hat{A}_\lambda$ "by continuation". The B_σ are the other components.*

Proof. (1) Take a component $\hat{A}_\lambda = \bigcup \alpha_i' \cup \bigcup K_j'$ and use Lemma 6.22. For the α_i', suppose (+) is valid. Then there exist arbitrarily small disjoint strips $S(K_j') \subset V$ with the properties: $\{S(K') \cap Z \cap \mathfrak{A}\} \subset \bigcup \alpha_i'$; $S^-(K') \cap (\bigcup f_i) = \emptyset$; $\{S^+(K') \cap \mathfrak{A}\} \subset \bigcup f_i$ and $S^+(K') \cap (\bigcup f_i) \neq \emptyset$. For the union S_λ of $S(K_j')$ the corresponding statements hold.

To $\alpha_i' - (S_\lambda \cap \alpha_i')$, we associate a neighborhood $U_i \subset V$ with $j(U_i) \subset \mathfrak{A}(r')$ so that $U_i \cap V^+$ is contained in $\bigcup f_i$. Now let $\hat{S}_\lambda := S_\lambda \cup \bigcup U_i$. In our case, we have again $\hat{S}_\lambda^- \cap (\bigcup f_i) = \emptyset$ and $\hat{S}_\lambda^+ \cap \mathfrak{A}(r') \subset \bigcup f_i$.

We can do this for all components of $\hat{A}$ and require at the same time that the closure of $\hat{S}_\lambda$ is disjoint. At first we must assume that both case (+) and case (−) can occur for $\hat{A}$. Therefore we establish that $S_\lambda^* = \hat{S}_\lambda^-$ when (+) occurs, and $S_\lambda^* := \hat{S}_\lambda^-$ when (−) occurs.

(2) Now let $F' := \bigcup \hat{S}_\lambda$. For a fixed f_i, we set $F := f_i$ and then F' satisfies the assumptions of Theorem 6.10 (we set $\tilde{F} := V \cup \bigcup f_i$). Consequently there exists a piece of surface F_i with the properties

1) $f_i \subset F_i$;
2) $(F' \cup f_i) \cap \mathfrak{A}(r')$ is equal to $F_i \cap \mathfrak{A}(r')$;

3) $\overline{F_i \cap \mathfrak{A}(r')} \cap \mathfrak{R}(r')$ contains $\bar{F}_i \cap \mathfrak{R}(r')$;

4) $F_i = F' \cup f_i \cup M_i$ where M_i is a neighborhood (which can be chosen arbitrarily small) of those boundary points of f_i which do not lie in F';

5) Let $R_i := (\partial M_i - \partial M_i \cap (F' \cup f_i))$. Then $j(R_i) \Subset \mathfrak{T}(r')$.

(3) Let M_i be so chosen that $\overline{M}_i \cap \overline{F' \cap Z} = \emptyset$ and $\overline{M}_i \cap (\overline{F' - \bigcup S_\lambda^*}) = \emptyset$. For $\overline{F' \cap Z} \cap \mathfrak{R} = \bigcup K$ consists of only interior points of F'. For $L_i := \partial f_i - (\partial f_i \cap F')$, then $L_i \cap \overline{F' \cap Z} = \emptyset$. Hence also $\overline{M}_i \cap \overline{F' \cap Z} = \emptyset$, if M_i is sufficiently small. Now assume $p \in L_i \cap (\overline{F' - \bigcup S_\lambda^*})$; then $f_i \cap (F' - \bigcup S_\lambda^*) \neq \emptyset$. Because p must be on $\mathfrak{R}$ and an interior point of M_i, there is a neighborhood $U(p)$ with connected $U(p) \cap \mathfrak{A}$. The whole $U(p) \cap \mathfrak{A}$ must therefore belong to f_i. Then $f_i \cap (F' - \bigcup S_\lambda^*) \neq \emptyset$ since $\{\overline{F' \cap \mathfrak{A}} \cap \mathfrak{R}\} \supset \{\overline{F' \cap \mathfrak{T}} \cap \mathfrak{R}\}$ and also $(F' - \bigcup S_\lambda^*) \cap U(p) \cap \mathfrak{A}$ cannot be empty. This contradicts however the definition of S_λ^*. For example, $S^* = S^+$ is precisely true when $S^- \cap (\bigcup f_i) = \emptyset$. So it follows that $L_i \cap (F' - \bigcup S_\lambda^*)$ is empty. If we take M_i sufficiently small, then $\overline{M}_i \cap (\overline{F' - \bigcup S_\lambda^*})$ is also empty.

(4) Finally let $\hat{F} := \bigcup S_\lambda^* \cup \bigcup f_i \cup \bigcup M_i$ and L be the set of those boundary points of $\hat{F}$, which do not lie on $\overline{F' \cap Z}$. We claim that $j(L) \Subset \mathfrak{T}(r')$. For the proof, notice that $j(L \cap \overline{M}_i) \Subset \mathfrak{T}(r')$ (cf. (2), 5) above). Further, since the boundary points of f_i are either in M_i or on $\bigcup \alpha_i$ or lie in F_i, we still need only to investigate $L \cap \bigcup S_\lambda^* := L^*$. Assume there exists a point p in $L^* \cap \mathfrak{R}$. Then there is a K so that $p \in S(k) \cap \mathfrak{R}$, because of 5), Lemma 6.17. Therefore p is also in $\overline{S(k) \cap \mathfrak{A}}$. Further $\{S^*(K) \cap \mathfrak{A}\} \subset \bigcup f_i$ and by construction $\hat{F} \cap (F' - \bigcup S_\lambda^*) = \emptyset$ (cf. (3)). Consequently p must be a boundary point of an f_i. Since p does not lie in $\overline{F' \cap Z}$, it follows that $p \in \bigcup M_i$. This means p is not a boundary point of $\hat{F}$. Contradiction! On $\mathfrak{A}$ there cannot be any point of L^*. For by construction, $\bigcup S_\lambda^* \cap \mathfrak{A} \subset \bigcup f_i$. Hence $j(L) \Subset \mathfrak{T}(r')$ is proved.

(5) Now fix $r^* > r'$ so small that $j(L) \Subset \mathfrak{T}(r^*)$. Define $F^* := \hat{F} \cap \mathfrak{A}(r^*)$. By construction the boundary points of F^* either lie above $\mathfrak{R}(r^*)$ or they are boundary points of $\hat{F}$, whicn are contained in $\overline{F' \cap Z}$. The latter form the arcs A_ρ, B_σ. The A_ρ contain A_λ. The B_σ are now arcs.

By construction, it follows at once that A_ρ, B_σ and the component of F^* satisfy Statement E, if to each arc α_i there is associated the surface-piece V_i consisting of the union of all the components C_γ of F where α_i meets ∂C_γ.

1.1 $\alpha_i \subset \partial V_i$ and each component of V_i has a boundary point of α_i (clear).

1.2 α_i lies neither on the boundary nor in the interior of $V_k \neq V_i$. This is also clear from the construction of S_λ^* and M_i (it is $\overline{M}_i \cap$

$\overline{F' \cap Z} = \emptyset$).

1.3 V_i lies either on the positive side of α_i or on the negative side of α_i. This is obviously true by construction for $V_i \cap (\bigcup S_i^*)$ and for $V_i \cap (\bigcup f_i)$. By (3) it is also true for $V_i \cap (\bigcup M_i)$. Hence it is true for all of V_i.

2. $j(V_i) \subset \mathfrak{A}(r^*)$. The boundary points of V_i lie either on α_i or above $\mathfrak{R}(r^*)$. This is clear.

(6) That the corresponding statement remains true for all $\tilde{r} \leqq r^*$ is obvious. Therefore Lemma 6.23 is completely proved.

Finally we come to prove the main theorem of this section.

THEOREM 6.24. *Let Z be a bounded cycle on a surface $\widetilde{F}$; $\dim \widetilde{F} = k$, over $\mathbf{P}^n$ $(k \geqq 2)$ and A be an $A(c)$-arc, hence a connected component of $Z \cap \mathfrak{A}(c)$. Then there exists a connected piece of surface F, $j(F) \subset \mathfrak{A}(c)$ so that $A = \partial F \cap \mathfrak{A}(c)$, $j(\partial F - A) \subset \mathfrak{R}(c)$.*

Remark 6.25. In particular the following is valid: For every $\rho \in A$ there exists a neighborhood $U(p)$, so that (let V be a neighborhood of Z introduced in a), the proof of Theorem 6.14) either

(+) $U(p) \cap V^+ \subset F$; $U(p) \cap V^- \cap F = \emptyset$ for all p or

(−) $U(p) \cap V^- \subset F$; $U(p) \cap V^+ \cap F = \emptyset$ for all p.

Proof. Let N be the set of all r, for which the following weaker statement holds. Let A_i be a connected component of $A \cap \mathfrak{A}(r)$, hence an $A(r)$-curve. To each A_i, a unique (not necessarily connected) piece of surface f_i is assigned, so that for A_i, f_i, Statement E is valid; that is, $\bigcup A_i = \partial(\bigcup f_i) \cap \mathfrak{A}(r)$.

(a) N is not empty.

For the proof, let ω be so defined that $A \cap \mathfrak{A}(\omega)$ is empty; however $A \cap \mathfrak{R}(\omega)$ is not empty. This is possible because Z is bounded. Then $A \cap \mathfrak{R}(\omega)$ consists of finitely many $T(\omega)$-arcs T. Every T can be embedded into a strip $S(T)$ by Lemma 6.19 so that either for $F(T) := S^-$ or for $F(T) := S^+$, the following hold:

i) $F(T) \subset \mathfrak{T}(\omega)$,

ii) $Z \cap S(T) \subset \partial F(T)$,

iii) $\{\partial F(T) - (Z \cap S(T))\} \subset \mathfrak{T}(\omega)$.

Take $S(T)$ sufficiently small, so that they are disjoint. We now fix $r' > \omega$ so small that the boundary points not situated on A of every $F(T)$ lie over $\mathfrak{T}(r')$. Thereupon, set $F^*(T) := F(T) \cap \mathfrak{A}(r')$. By construction $F^*(T)$ lies either entirely in V^+ or entirely in V^-. We denote now the component of $A \cap \mathfrak{A}(r')$ by A_i' and assign f_i' to A_i', where f_i' is the union of all those components of $F^*(T)$ which have boundary points on A'_i. Hence Statement E holds for A'_i, f'_i; i.e., $\bigcup A'_i = \partial(\bigcup f'_i) \cap \mathfrak{A}(r')$. Therefore (a) is proved.

(b) If $r' > r''$ and $r' \in N$, then also $r'' \in N$. Because if A'_i, f'_i are the arcs and surfaces corresponding to r', then we have to take, for r'', only the components of $A'_i \cap \mathfrak{A}(r'')$ and $f'_i \cap \mathfrak{A}(r'')$.

(c) N is open in $\omega < r \leqq c$.

Proof. (1) Let $r' < c$ and $r' \in N$. Further let α_i be the component of $A \cap \mathfrak{A}(r')$ and f_i the associated surface-piece, so that $\bigcup \alpha_i = \partial(\bigcup f_i) \cap \mathfrak{A}(r')$. The closure of $(A \cap \mathfrak{A}(r')) \cup \bigcup f_i$ would in general not be connected. We consider one of its connected components L_1 and denote the arcs and surfaces on it again with α_i, f_i. For these and components $\hat{A}_\lambda$ of $L_1 \cap A \cap (\mathfrak{A}(r') \cup \mathfrak{R}(r'))$, the assumptions of Lemma 6.23 are satisfied. Hence there exist a neighborhood U_1 of $\bigcup \hat{A}_\lambda$, further an $r^* > r$ and a surface-piece f_k^*, so that for the component A_k^* (the B's of Lemma 6.23 belong here as well) of $U_1 \cup A \cap \mathfrak{A}(r^*)$ and the f_k^*, Statement E is valid: $\bigcup A_k^* = \partial(\bigcup f_k^*) \cap \mathfrak{A}(r^*)$. For the other L_σ, the corresponding statement holds. We choose U_σ to be pairwise disjoint.

(2) If now $A \cap \mathfrak{A}(r^*)$ is equal to the union of components of $U_\sigma \cap A \cap \mathfrak{A}(r^*)$, then we will be done. This is the case, if $A \cap \mathfrak{R}(r')$ does not contain any T-arcs and r^* is sufficiently near r'.

Suppose however that $A \cap \mathfrak{R}(r')$ contains T-arcs T_j; then by Lemma 6.19 there are strips $S(T_j)$; furthermore there exist $r_0 > r'$ and surface-pieces F_j, so that $S(T_j) \cap A \cap \mathfrak{A}(r_0)$ bounds the surface-pieces $F_j \cap \mathfrak{A}(r_0)$ in $\mathfrak{A}(r_0)$ (more precisely: $S(T_j) \cap A \cap \mathfrak{A}(r_0)$ is equal to $\partial F_j \cap \mathfrak{A}(r_0)$).

We can now assume: $r^* = r_0$, because instead of r^* we can take any $r < r^*$ and instead of r_0 we can take any $r < r_0$. Finally if r^* is sufficiently near r', then $A \cap \mathfrak{A}(r^*)$ is the union of the components of $\bigcup U_\sigma \cap A \cap \mathfrak{A}(r^*)$ and the components of $\bigcup S(T_j) \cap A \cap \mathfrak{A}(r^*)$.

(3) Case 1. T_j does not lie in $\bigcup L_\sigma$. Choose U_σ different from $S(T_j)$. Now add the components of $S(T_j) \cap A \cap \mathfrak{A}(r^*)$ and the components of $F_j \cap \mathfrak{A}(r^*)$ to the ones already at hand.

Case 2. T_0 lies in L_1. Obviously T_0 must lie in the boundary of some f_j. Let $f_j \subset f_j^*$ (cf. (1)). First of all we keep r^* fixed and make r_0 so small that by (2) the surface-piece F_0 associated to T_0 and r_0 is contained entirely in f_j^*. Let r^* approach r_0; it follows that for $r^* = r_0$, $F_0 \subset f_j^*$.

Set $f_j := f_j^* - F_0$. The boundary of $\tilde{f}_j$ differs from the boundary of f_j^* precisely by the facts that 1) the new components of $W := S(T) \cap A \cap \mathfrak{A}(r^*)$ occur in $\tilde{f}_j$, and 2) the boundary points of F_0 which are not contained in W

(and because of this lie over $\mathfrak{R}(r^*)$) disappear when one forms $\tilde{f}_j$. W lies neither in the interior nor on the boundary of f_k^* different from f_i^*, because the f_j^* are pairwise disjoint.

Perform the above process for all T from $\bigcup L_\sigma$ and replace the f_j^*, which contains T, by $\tilde{f}_j$. We can always take $r_0 = r^*$. Add the new curves and surfaces from the T arcs, which do not lie in $\bigcup L_\sigma$ (case 1); then we get a set of $A(r^*)$-arcs and the associated surfaces which satisfy Statement E. The set of these $A(r^*)$-arcs is by construction equal to $A \cap \mathfrak{A}(r^*)$. Consequently r^* belongs to N and then also all $r \leqq r^*$. Therefore c) is proved.

(d) N is closed in $\omega < r \leqq c$.

Next we establish the following statement: For a fixed r, if Statement E holds for a set of curves A_i and surfaces f_i, then it cannot hold for the same A_i and other surface F_i. The surfaces F_i are therefore uniquely determined by the arcs. It suffices to show this for the connected components of $\bigcup \bar{f}_i$. Suppose therefore that $\bigcup \bar{f}_i$ is connected. Suppose for some i, $f_i \neq F_i$. Then $f_i \cap F_i = \emptyset$ and A_i consists of interior points of $\bar{f}_i \cup \bar{F}_i$, as we can see immediately. Further it follows that $f_k \neq F_k$ for all A_k which are points of boundary of f_i. Since $\bigcup \bar{f}_i$ is connected, for all j, $f_j \neq F_j$ and A_j lies in the interior of $\bigcup(\bar{f}_i \cup \bar{F}_i)$. Then $\bigcup(\bar{f}_i \cup \bar{F}_i)$ is a piece of surface G over $\mathfrak{A}$, whose boundary lies over $\mathfrak{R}$. It follows that there exists an $\tilde{r}$ with the property: $G \cap \mathfrak{A}(\tilde{r}) = \emptyset$ and $G \cap \mathfrak{R}(r) \neq \emptyset$. This contradicts Theorem 6.8.

Let N_0 = maximal open set in N and r_0 its upper bound. Furthermore for r', r'' in N let $r' < r''$; finally A_i', f_i' and A_j'', f_j'' are the corresponding A and f. By the proof above, the components of A_i', f_i' are components of $\bigcup(A_j'' \cap \mathfrak{A}(r''))$ and $\bigcup(f_j'' \cap \mathfrak{A}(r'))$. We take now for r_0 the A_k^0 as components of $\bigcup_{r'<r_0} A_i'$ and the components of f_k^0 as components of $\bigcup_{r'<r_0} f_i'$. For the A_k^0, f_k^0 thus defined, Statement E obviously holds. Thus $r_0 \in N$.

(e) From a)-b), c is in N. But this is simply the statement of the theorem.

An immediate consequence is the following.

THEOREM 6.26. *Every bounded cycle Z on $\tilde{F}$ bounds. In other words: There exists a surface-piece $F(Z)$, so that $Z = \partial F(Z)$. Moreover $F(Z)$ is also bounded. All holomorphic (meromorphic) functions on $U(Z)$ extend to holomorphic (meromorphic) functions on $F(Z)$.*

($\tilde{F}$ is assumed to be a surface over $\mathbf{P}^n$ with $\dim \tilde{F} = k \geqq 2$. The set M is said to be bounded if there exists an analytic plane $E \subset \mathbf{P}^n$, $\dim E = n - k + 1$ and a neighborhood $U(E)$ of E such that $M \cap U(E)$ is empty.)

Proof. It follows from Theorem 6.24 that for every positive r, the components of $Z \cap \mathfrak{A}(r)$ are boundaries of bounded surface-pieces. Now only the points $p_1 = (1, 0, \cdots, 0)$; $\cdots$; $p_s = (0, \cdots, 1, 0, \cdots, 0)$ (homogeneous coordinates) stay outside all $\mathfrak{A}(r)$. We obtain a surface-piece F such that Z lies on the boundary of F. The critical points p which lie on Z are obviously harmless and those other "boundary points" all lie above $p_1, \cdots, p_s$. For each i, the set $j(F)$ is analytic in $U(p_i)$ with the exception of point p_i alone. Then the closure of $j(F)$ is also analytic at p_i. It follows that $Z = \partial\bar{F}$. This is what we have proved. Moreover, all the functions holomorphic (meromorphic) on Z extend to $F(Z)$.

Finally let us again use the notation we had before in Theorem 5.12. Take a cycle Z in W and a surface-piece $F(Z)$ such that $\partial F(Z) = Z$. By the proofs of Lemmas 6.17, 6.19, 6.22 and 6.23, Theorems 6.24 and 6.26, we have V equal to the image of V' in $\mathbf{C}^N$, which is a subspace of the analytic surface-piece $W \cup F(Z)$.

HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS
UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE

REFERENCES

[1] A. ANDREOTTI and H. GRAUERT, Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France **90** (1962), 193-259.

[2] A. ANDREOTTI and D. HILL, E. E. Levi Convexity and Hans Lewy Problems, Parts I and II, Ann. Sc. Nor. Sup. Pisa **26** (1972), 325-363, 767-806.

[3] M. AUSLANDER and D. A. BUCHSBAUM, Codimension and multiplicity, Ann. of Math. **68** (1958), 625-657.

[4] S. BOCHNER, Analytic and meromorphic continuation by means of Green's formula, Ann. of Math. **44** (1943), 652-673.

[5] D. BURNS, On rational singularities in dimension >2, Math. Ann. **211** (1974), 237-244.

[6] F. DONNELLY, Eta invariants and the boundaries of Hermitian manifolds, Amer. J. Math. **99** (1968), 879-900.

[7] G. B. FOLLAND and J. J. Kohn, *The Neumann Problem for the Cauchy-Riemann complex*, Ann. of Math. Studies, No. 75, Princeton Univ. Press, 1972.

[8] T. GAMELIN, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

[9] R. C. GUNNING and H. ROSSI, *Analytic Functions of Several Complex Variables*, Prentice-Hall, 1965.

[10] TOR. GULLIKSON and L. GERSON, *Homology of Local Rings*, Queen's papers in Pure and Applied Math., No. 20, Queen's Univ., Kingston, Ontario, 1969.

[11] R. HARVEY, Holomorphic chains and their boundaries, Proc. Sym. in Pure Math., Vol. 30, Part I, (1977), 309-382.

[12] R. HARVEY and B. LAWSON, On boundaries of complex analytic varieties I, Ann. of Math. **102** (1975), 233-290.

[13] R. HARVEY, The theory of hyperfunctions on totally real subsets of a complex manifold with applications to extension problems, Amer. J. Math. **91** (1969), 853-873.

[14] L. HÖRMANDER, L^2-estimates and existence theorems for the $\bar{\partial}$-operators, Acta Math. **113** (1965), 89-152.

[15] ———, *An Introduction to Complex Analysis in Several Complex Variables*, Van Nostrand, Princeton, 1966.

[16] J. J. KOHN, Harmonic integrals on strongly pseudoconvex manifolds I, Ann. of Math. **78** (1963), 112-148; II, ibid. **79** (1964), 450-472.
[17] ———, Boundaries of complex manifolds, Proc. Conf. on Complex Manifolds (Minneapolis), Springer-Verlag, New York, 1965.
[18] J. J. KOHN and H. ROSSI, On the extension of holomorphic functions from the boundary of a complex manifold, Ann. of Math. **81** (1965), 451-472.
[19] M. KURANISHI, Application of $\bar{\partial}_b$ to deformation of isolated singularities, Proc. Sym. in Pure Math., Vol. 30, Part I, (1977), 97-106.
[20] H. LAUFER, On rational singularities, Amer. J. Math. **94** (1972), 597-608.
[21] H. LEWY, On the local character of the solution of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables, Ann. of Math. **64** (1956), 514-522.
[22] H. ROSSI, Attaching analytic spaces to an analytic space along a pseudoconcave boundary, Proc. Conf. on Complex Analysis, Minneapolis, Springer-Verlag, 1964.
[23] W. ROTHSTEIN, Bemerkungen zur theorie komplexon Raüme, Math. Ann. **137** (1959), 304-315.
[24] ———, Die Fortsetzung vier- und hoherdimensionaler analytischer Flachen des R_{2n} ($n \geqq 3$), Math. Ann. **121** (1950), 340-355.
[25] ———, Uber die Fortsetzung analytic Flacher, Math. Ann. **122** (1953), 424-434.
[26] ———, Zur Theorie der analytischen Mannigfaltigkeiten im Raüme von n komplexen Veranderlichen, Math. Ann. **129** (1955), 96-138; **133** (1957), 271-280 und 400-409.
[27] W. ROTHSTEIN and H. SPERLING, Einsetzen und analytischer Flächenstücke in Zyklen auf komplexen Raümen, Festschrift zur Gedächtnisfeier für Karl Weierstrass 1815-1865 (Behnke and Kopfermann, eds.), Westdeutscher Verlag, Köln, 1965, 531-554.
[28] J.-P. SERRE, *Algèbre Locale Multiplicités*, Lecture Notes in Math., Vol. 11, Springer-Verlag, Berlin-Heidelberg-New York, 1965.
[29] Y.-T. SIU, *Techniques of Extension of Analytic Objects*, Lecture Notes in Pure and Applied Mathematics, Vol. 8, Marcel Dekker, Inc. New York, 1974.
[30] Y.-T. SIU and G. TRAUTMANN, *Gap-sheaves and the Extension of Coherent Analytic Subsheaves*, Lecture notes in Math., Vol. 172, Springer-Verlag, Berlin-Heidelberg-New York, 1971.
[31] H. SPERLING, Zur Fortsetzung analytischer Flächen, Dissertation, Marburg, 1957.
[32] N. TANAKA, *A Differential Geometric Study on Strongly Pseudoconvex Manifolds*. Lectures in Mathematics, Dept. of Math., Kyoto University 9, Kinokuniya Book-Store Co., Ltd.
[33] J. WERMER, The hull of a curve in C^n, Ann. of Math. **68** (1958), 550-561.
[34] STEPHEN S.-T. YAU, The signature of Milnor fibers and duality theorems for strongly pseudoconvex manifolds, Invent. Math. **46** (1978), 81-97.
[35] ———, Noether formula for strongly pesudoconvex manifolds and various numerical invariants for isolated singularities, preprint.
[36] ———, Kohn-Rossi cohomology and its application to the complex Plateau problem, II (in preparation).

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