J. DIFFERENTIAL GEOMETRY
90 (2012) 251-266

KOHN–ROSSI COHOMOLOGY AND ITS APPLICATION TO THE COMPLEX PLATEAU PROBLEM, III

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Abstract

Let X be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$ in $\mathbb{C}^N$. It has been an interesting question to find an intrinsic smoothness criteria for the complex Plateau problem. For $n \geq 3$ and $N = n+1$, Yau found a necessary and sufficient condition for the interior regularity of the Harvey–Lawson solution to the complex Plateau problem by means of Kohn–Rossi cohomology groups on X in 1981. For $n = 2$ and $N \geq n+1$, the problem has been open for over 30 years. In this paper we introduce a new CR invariant $g^{(1,1)}(X)$ of X. The vanishing of this invariant will give the interior regularity of the Harvey–Lawson solution up to normalization. In the case $n = 2$ and $N = 3$, the vanishing of this invariant is enough to give the interior regularity.

Dedicated to Professor Blaine Lawson on the occasion of his 68$^{\text{th}}$ Birthday.

1. Introduction

One of the natural fundamental questions of complex geometry is to study the boundaries of complex varieties. For example, the famous classical complex Plateau problem asks which odd-dimensional real submanifolds of $\mathbb{C}^N$ are boundaries of complex sub-manifolds in $\mathbb{C}^N$. In their beautiful seminal paper, Harvey and Lawson [Ha-La] proved that for any compact connected CR manifold X of real dimension $2n-1$, $n \geq 2$, in $\mathbb{C}^N$, there is a unique complex variety V in $\mathbb{C}^N$ such that the boundary of V is X. In fact, Harvey and Lawson proved the following theorem.

Theorem (Harvey–Lawson [Ha-La1, Ha-La2]) Let X be an embeddable strongly pseudoconvex CR manifold. Then X can be CR embedded in some $\mathbb{C}^{\widetilde{N}}$ and X bounds a Stein variety $V \subseteq \mathbb{C}^{\widetilde{N}}$ with at most isolated singularities.

Received 4/28/2010.

The above theorem is one of the deepest theorems in complex geometry. It relates the theory of strongly pseudoconvex CR manifolds on the one hand and the theory of isolated normal singularities on the other hand.

The next fundamental question is to determine when X is a boundary of a complex sub-manifold in $\mathbb{C}^N$, i.e., when V is smooth. In 1981, Yau [Ya] solved this problem for the case $n \geq 3$ by calculation of Kohn–Rossi cohomology groups $H^{p,q}_{KR}(X)$. More precisely, suppose X is a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$, $n \geq 3$, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^{n+1}$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if Kohn–Rossi cohomology groups $H^{p,q}_{KR}(X)$ are zeros for $1 \leq q \leq n-2$ (see Theorem 4.1).

Kohn–Rossi cohomology introduced by Kohn and Rossi [Ko-Ro] in 1965 is a fundamental invariant of CR manifold. In the recent work of Huang, Luk, and Yau [H-L-Y], it was shown that the Kohn–Rossi cohomology plays an important role in the simultaneous CR embedding of a family of strongly pseudoconvex CR manifolds of dimension at least 5.

For $n = 2$, i.e., X is a 3-dimensional CR manifold, the intrinsic smoothness criteria for the complex Plateau problem remains unsolved for over a quarter of a century even for the hypersurface case. The main difficulty is that the Kohn–Rossi cohomology groups are infinite-dimensional in this case. Let V be a complex variety with X as its boundary. Then the singularities of V are surface singularities. In [Lu-Ya2], the holomorphic De Rham cohomology, which is derived form Kohn–Rossi cohomology, is considered to determine what kind of singularities can happen in V . In fact, in [Ta], Tanaka introduced a spectral sequence $E^{p,q}_r(X)$ with $E^{p,q}_1(X)$ being the Kohn–Rossi cohomology group and $E^{k,0}_2(X)$ being the holomorphic De Rham cohomology denoted by $H^k_h(X)$. So consideration of De Rham cohomology is natural in the case of $n = 2$. Motivated by the deep work of Siu [Si], Luk and Yau introduced the Siu complex and s-invariant (see Definition 3.2, below) for isolated singularity $(V, 0)$ and proved a theorem in [Lu-Ya2] that if $(V, 0)$ is a Gorenstein surface singularity with vanishing s-invariant, then $(V, 0)$ is a quasihomogeneous singularity whose link is rational homology sphere. In [Lu-Ya2], they proved that if X is a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 3 contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$ and the holomorphic De Rham cohomology $H^2_h(X)$ vanishes, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing s-invariant (see Theorem 4.4). As a corollary of this theorem, they get that if $N = 3$, the variety V bounded by X has only isolated quasi-homogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star shaped and all the curves are rational (see Corollary 4.5). Even though one cannot judge when X is a boundary of a complex manifold with the vanishing of $H_h^2(X)$, it is a fundamental step toward the solution of the regularity of the complex Plateau problem. In this paper, we introduce a new CR invariant $g^{(1,1)}(X)$ which has independent interest besides its application to the complex Plateau problem. Roughly speaking, our new invariant $g^{(1,1)}(X)$ is the number of independent holomorphic 2-forms on X which cannot be written as a linear combination of those elements of the form holomorphic 1-form wedge with holomorphic 1-form on X . This new invariant will allow us to solve the intrinsic smoothness criteria up to normalization for the classical complex Plateau problem for $n = 2$.

Theorem A *Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension* 3*. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$ with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold up to normalization $V \subset D - X$ with boundary regularity if and only if $g^{(1,1)}(X) = 0$.*

Thus, the interior regularity of the complex Plateau problem is solved up to normalization. As a corollary of Theorem A, we have solved the interior regularity of the complex Plateau problem in case X is of real codimension 3 in $\mathbb{C}^3$.

Theorem B *Let X be a strongly pseudoconvex compact CR manifold of dimension* 3*. Suppose that X is contained in the boundary of a strongly pseudo-convex bounded domain D in $\mathbb{C}^3$ with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.*

In Section 2, we shall recall the definition of holomorphic De Rham cohomology for a CR manifold. In Section 3, after recalling several local invariants of isolated singularity, we introduce some new invariants of singularities and new CR invariants for CR manifolds. In Section 4, we prove the main theorem of this paper.

Finally, we would like to thank Professor Lawrence Ein and Professor Anatoly Libgober for helpful discussions.

Acknowledgments. R. Du's research was supported by the National Natural Science Foundation of China and the Innovation Foundation of East China Normal University. S. Yau's research was partially supported by NSF and Department of Mathematical Sciences, Tsinghua University, Beijing, P.R. China.

2. Preliminaries

Kohn–Rossi cohomology was first introduced by Kohn–Rossi. Following Tanaka [Ta], we reformulate the definition in a way independent of the interior manifold.

Definition 2.1. *Let X be a connected orientable manifold of real dimension $2n-1$. A CR structure on X is an $(n-1)$-dimensional sub-bundle S of $\mathbb{C}T(X)$ (complexified tangent bundle) such that:*

1 . $S \bigcap \bar{S} = \{0\}$.
2 . *If L, L' are local sections of S, then so is $[L, L']$.*

Such a manifold with a CR structure is called a CR manifold. There is a unique sub-bundle $\mathcal{H}$ of $T(X)$ such that $\mathbb{C}\mathcal{H} = S \bigoplus \bar{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^2 = -1$ and $S = \{v - iJv : v \in \mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of the CR structure.

Let X be a CR manifold with structure S. For a complex valued C^∞ function u defined on X, the section $\bar{\partial}_b u \in \Gamma(\bar{S}^*)$ is defined by

$$\bar{\partial}_b u(\bar{L}) = \bar{L}(u), L \in S.$$

The differential operator $\bar{\partial}_b$ is called the (tangential) Cauchy–Riemann operator, and a solution u of the equation $\bar{\partial}_b u = 0$ is called a holomorphic function.

Definition 2.2. *A complex vector bundle E over X is said to be holomorphic if there is a differential operator*

$$\bar{\partial}_E : \Gamma(E) \longrightarrow \Gamma(E \otimes \bar{S}^*)$$

satisfying the following conditions:

1 . $\bar{\partial}_E(fu)(\bar{L}_1) = (\bar{\partial}_b f)(\bar{L}_1)u + f(\bar{\partial}_E u)(\bar{L}_1) = (\bar{L}_1 f)u + f(\bar{\partial}_E u)(\bar{L}_1)$.
2. $(\bar{\partial}_E u)[\bar{L}_1, \bar{L}_2] = \bar{\partial}_E(\bar{\partial}_E u(\bar{L}_2))(\bar{L}_1) - \bar{\partial}_E(\bar{\partial}_E u(\bar{L}_1))(\bar{L}_2)$, *where* $u \in \Gamma(E)$, $f \in C^\infty(X)$, *and* L_1, $L_2 \in \Gamma(S)$.

The operator $\bar{\partial}_E$ is called the Cauchy–Riemann operator and a solution u of the equation $\bar{\partial}_E u = 0$ is called a holomorphic cross section.

A basic holomorphic vector bundle over a CR manifold X is the vector bundle $\widehat{T}(X) = \mathbb{C}T(X)/\bar{S}$. The corresponding operator $\bar{\partial} = \bar{\partial}_{\widehat{T}(X)}$ is defined as follows. Let p be the projection from $\mathbb{C}T(X)$ to $\widehat{T}(X)$. Take any $u \in \Gamma(\widehat{T}(X))$ and express it as $u = p(Z)$, $Z \in \Gamma(\mathbb{C}T(X))$. For any $L \in \Gamma(S)$, define a cross section $(\bar{\partial}u)(\bar{L})$ of $\widehat{T}(X)$ by $(\bar{\partial}u)(\bar{L}) = p([\bar{L}, Z])$. One can show that $(\bar{\partial}u)(\bar{L})$ does not depend on the choice of Z and that $\bar{\partial}u$ gives a cross section of $\widehat{T}(X) \otimes \bar{S}^*$. Furthermore, one can show that the operator $u \longmapsto \bar{\partial}u$ satisfies (1) and (2) of Definition 2.2, using the Jacobi identity in the Lie algebra $\Gamma(\mathbb{C}T(X))$. The resulting holomorphic vector bundle $\widehat{T}(X)$ is called the holomorphic tangent bundle of X.

If X is a real hypersurface in a complex manifold M, we may identify $\widehat{T}(X)$ with the holomorphic vector bundle of all $(1,0)$ tangent vectors to M and $\widehat{T}(X)$ with the restriction of $\widehat{T}(M)$ to X. In fact, since the structure S of X is the bundle of all $(1,0)$ tangent vectors to X, the inclusion map $\mathbb{C}T(X) \longrightarrow \mathbb{C}T(M)$ induces a natural map $\widehat{T}(X) \xrightarrow{\phi} \widehat{T}(M)|_X$ which is a bundle isomorphism satisfying $\bar{\partial}(\phi(u))(\bar{L}) = \phi(\bar{\partial}u(\bar{L}))$, $u \in \Gamma(\widehat{T}(X))$, $L \in S$.

For a holomorphic vector bundle E over X, set

$$C^q(X,E) = E \otimes \wedge^q \bar{S}^*, \mathscr{C}^q(X,E) = \Gamma(C^q(X,E))$$

and define a differential operator

$$\bar{\partial}_E^q : \mathscr{C}^q(X,E) \longrightarrow \mathscr{C}^{q+1}(X,E)$$

by

$$(\bar{\partial}_E^q \phi)(\bar{L}_1, \dots, \bar{L}_{q+1}) = \sum_i (-1)^{i+1} \bar{\partial}_E(\phi(\bar{L}_1, \dots, \widehat{\bar{L}}_i, \dots, \bar{L}_{q+1}))(\bar{L}_i)$$
$$+ \sum_{i<j} (-1)^{i+j} \phi([\bar{L}_i, \bar{L}_j], \bar{L}_1, \dots, \widehat{\bar{L}}_i, \dots, \bar{L}_{q+1})$$

for all $\phi \in \mathscr{C}^q(X,E)$ and $L_1, \dots, L_{q+1} \in \Gamma(S)$. One shows by standard arguments that $\bar{\partial}_E^q \phi$ gives an element of $\mathscr{C}^{q+1}(X,E)$ and that $\bar{\partial}_E^{q+1} \bar{\partial}_E^q = 0$. The cohomology groups of the resulting complex $\{\mathscr{C}^q(X,E), \bar{\partial}_E^q\}$ is denoted by $H^q(X,E)$.

Let $\{\mathscr{A}^k(X), d\}$ be the De Rham complex of X with complex coefficients, and let $H^k(X)$ be the De Rham cohomology groups. There is a natural filtration of the De Rham complex, as follows. For any integer p and k, put $A^k(X) = \wedge^k(\mathbb{C}T(X)^*)$ and denote by $F^p(A^k(X))$ the sub-bundle of $A^k(X)$ consisting of all $\phi \in A^k(X)$ which satisfy the equality

$$\phi(Y_1, \dots, Y_{p-1}, \bar{Z}_1, \dots, \bar{Z}_{k-p+1}) = 0$$

for all $Y_1, \dots, Y_{p-1} \in \mathbb{C}T(X)_0$ and $Z_1, \dots, Z_{k-p+1} \in S_0$, 0 being the origin of ϕ. Then

$$A^k(X) = F^0(A^k(X)) \supset F^1(A^k(X)) \supset \cdots$$
$$\supset F^k(A^k(X)) \supset F^{k+1}(A^k(X)) = 0.$$

Setting $F^p(\mathscr{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we have

$$\mathscr{A}^k(X) = F^0(\mathscr{A}^k(X)) \supset F^1(\mathscr{A}^k(X)) \supset \cdots$$
$$\supset F^k(\mathscr{A}^k(X)) \supset F^{k+1}(\mathscr{A}^k(X)) = 0.$$

Since clearly $dF^p(\mathscr{A}^k(X)) \subseteq F^p(\mathscr{A}^{k+1}(X))$, the collection $\{F^p(\mathscr{A}^k(X))\}$ gives a filtration of the De Rham complex.

We denote by $H^{p,q}_{KR}(X)$ the groups $E^{p,q}_1(X)$ of the spectral sequence $\{E^{p,q}_r(X)\}$ associated with the filtration $\{F^p(\mathscr{A}^k(X))\}$. We call $H^{p,q}_{KR}(X)$ the Kohn–Rossi cohomology group of type (p,q). More explicitly, let

$$A^{p,q}(X) = F^p(A^{p+q}(X)), \mathscr{A}^{p,q}(X) = \Gamma(A^{p,q}(X)),$$

$$C^{p,q}(X) = A^{p,q}(X)/A^{p+1,q-1}(X), \mathscr{C}^{p,q}(X) = \Gamma(C^{p,q}(X)).$$

Since $d : \mathscr{A}^{p,q}(X) \longrightarrow \mathscr{A}^{p,q+1}(X)$ maps $\mathscr{A}^{p+1,q-1}(X)$ into $\mathscr{A}^{p+1,q}(X)$, it induces an operator $d'' : \mathscr{C}^{p,q}(X) \longrightarrow \mathscr{C}^{p,q+1}(X)$. $H^{p,q}_{KR}(X)$ are then the cohomology groups of the complex $\{\mathscr{C}^{p,q}(X), d''\}$.

Alternatively, $H^{p,q}_{KR}(X)$ may be described in terms of the vector bundle $E^p = \wedge^p(\widehat{T}(X)^*)$. If for $\phi \in \Gamma(E^p)$, $u_1, \ldots, u_p \in \Gamma(\widehat{T}(X))$, $Y \in S$, we define $(\bar\partial_{E^p}\phi)(\bar Y) = \bar Y\phi$ by

$$\bar Y\phi(u_1, \ldots, u_p) = \bar Y(\phi(u_1, \ldots, u_p)) + \sum_i (-1)^i \phi(\bar Y u_i, u_1, \ldots, \widehat{u_i}, \ldots, u_p)$$

where $\bar Y u_i = (\bar\partial_{\widehat{T}(X)} u_i)(\bar Y)$, then we easily verify that E^p with $\bar\partial_{E^p}$ is a holomorphic vector bundle. Tanaka [Ta] proves that $C^{p,q}(X)$ may be identified with $C^q(X, E^p)$ in a natural manner such that

$$d''\phi = (-1)^p \bar\partial_{E^p}\phi, \phi \in \mathscr{C}^{p,q}(X).$$

Thus, $H^{p,q}_{KR}(X)$ may be identified with $H^q(X, E^p)$.

We denote by $H^k_h(X)$ the groups $E^{k,0}_2(X)$ of the spectral sequence $\{E^{p,q}_r(X)\}$ associated with the filtration $\{F^p(\mathscr{A}^k(X))\}$. We call $H^k_h(X)$ the holomorphic De Rham cohomology groups. The groups $H^k_h(X)$ are the cohomology groups of the complex $\{\mathscr{S}^k(X), d\}$, where we put $\mathscr{S}^k(X) = E^{k,0}_1(X)$ and $d = d_1 : E^{k,0}_1 \longrightarrow E^{k+1,0}_1$. Recall that $\mathscr{S}^k(X)$ is the kernel of the following mapping:

$$\begin{aligned} d_0 : E^{k,0}_0 &= F^k\mathscr{A}^k = \mathscr{A}^{k,0}(X) \\ &\to E^{k,1}_0 = F^k\mathscr{A}^{k+1}/F^{k+1}\mathscr{A}^{k+1} = \mathscr{A}^{k,1}(X)/\mathscr{A}^{k+1,0}. \end{aligned}$$

Note that $\mathscr{S}$ may be characterized as the space of holomorphic k-forms, namely holomorphic cross sections of E^k. Thus, the complex $\{\mathscr{S}^k(X), d\}$ (respectively, the groups $H^k_h(X)$) will be called the holomorphic De Rham complex (respectively, the holomorphic De Rham cohomology groups).

Definition 2.3. *Let $L_1, \ldots, L_{n-1}$ be a local frame of the CR structure S on X so that $\bar L_1, \ldots, \bar L_{n-1}$ is a local frame of $\bar S$. Since $S \oplus \bar S$ has complex codimension 1 in $\mathbb{C}T(X)$, we may choose a local section N of $\mathbb{C}T(X)$ such that $L_1, \ldots, L_{n-1}, \bar L_1, \ldots, \bar L_{n-1}$, N span $\mathbb{C}T(X)$. We may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by*

$$[L_i, \bar L_j] = \sum_k a^k_{i,j} L_k + \sum_k b^k_{i,j} \bar L_k + c_{i,j} N$$

is Hermitian, and it is called the Levi form of X.

Proposition 2.4. The number of nonzero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \ldots, L_{n-1}, N$.

Definition 2.5. *X is said to be strongly pseudoconvex if the Levi form is positive definite at each point of X.*

Definition 2.6. *Let X be a CR manifold of real dimension $2n-1$. X is said to be Calabi–Yau if there exists a nowhere vanishing holomorphic section in $\Gamma(\wedge^n \widehat{T}(X)^*)$, where $\widehat{T}(X)$ is the holomorphic tangent bundle of X.*

Remark:

1 . Let X be a CR manifold of real dimension $2n-1$ in $\mathbb{C}^n$. Then X is a Calabi–Yau CR manifold.

2 . Let X be a strongly pseudoconvex CR manifold of real dimension $2n-1$ contained in the boundary of bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then X is a Calabi–Yau CR manifold.

3. Invariants of singularities and CR-invariants

Let V be a n-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities. In [Ya2], Yau considered four kinds of sheaves of germs of holomorphic p-forms:

1 . $\bar{\Omega}^p_V := \pi_* \Omega^p_M$, where $\pi : M \longrightarrow V$ is a resolution of singularities of V.

2 . $\bar{\bar{\Omega}}^p_V := \theta_* \Omega^p_{V \setminus V_{sing}}$ where $\theta : V \setminus V_{sing} \longrightarrow V$ is the inclusion map and V_{sing} is the singular set of V.

3 . $\Omega^p_V := \Omega^p_{\mathbb{C}^N} / \mathscr{K}^p$, where $\mathscr{K}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega^p_{\mathbb{C}^N}; \beta \in \Omega^{p-1}_{\mathbb{C}^N}; f, g \in \mathscr{I}\}$ and $\mathscr{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.

4 . $\widetilde{\Omega}^p_V := \Omega^p_{\mathbb{C}^N} / \mathscr{H}^p$, where $\mathscr{H}^p = \{\omega \in \Omega^p_{\mathbb{C}^N} : \omega|_{V \setminus V_{sing}} = 0\}$.

Clearly Ω^p_V, $\widetilde{\Omega}^p_V$ are coherent. $\bar{\Omega}^p_V$ is a coherent sheaf because π is a proper map. $\bar{\bar{\Omega}}^p_V$ is also a coherent sheaf by a theorem of Siu (see Theorem A of [Si]). If V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}^n_V$.

Definition 3.1. *The Siu complex is a complex of coherent sheaves $J^\bullet$ supported on the singular points of V which is defined by the following exact sequence:*

$$0 \longrightarrow \bar{\Omega}^\bullet \longrightarrow \bar{\bar{\Omega}}^\bullet \longrightarrow J^\bullet \longrightarrow 0. \tag{3.1}$$

Definition 3.2. *Let V be a n-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. The geometric genus p_g and the irregularity q of the singularity are defined as follows (see [Ya2, St-St]):*

$$p_g := dim \Gamma(M \setminus A, \Omega^n) / \Gamma(M, \Omega^n), \tag{3.2}$$

$$q := dim\Gamma(M\backslash A, \Omega^{n-1})/\Gamma(M, \Omega^{n-1}), \tag{3.3}$$

$$g^{(p)} := dim\Gamma(M, \Omega^p_M)/\pi^*\Gamma(V, \Omega^p_V). \tag{3.4}$$

The s-invariant of the singularity is defined as follows:

$$s := dim\Gamma(M\backslash A, \Omega^n)/[\Gamma(M, \Omega^n) + d\Gamma(M\backslash A, \Omega^{n-1})]. \tag{3.5}$$

Lemma 3.3. *([Lu-Ya2])* Let V be a n-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Let $J^\bullet$ be the Siu complex of coherent sheaves supported on 0. Then:

1 . $dimJ^n = p_g$.
2 . $dimJ^{n-1} = q$.
3 . $dimJ^i = 0$, for $1 \le i \le n-2$.

Proposition 3.4. *([Lu-Ya2])* Let V be a n-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Let $J^\bullet$ be the Siu complex of coherent sheaves supported on 0. Then the s-invariant is given by

$$s := dimH^n(J^\bullet) = p_g - q \tag{3.6}$$

and

$$dimH^{n-1}(J^\bullet) = 0. \tag{3.7}$$

Let X be a compact connected strongly pseudoconvex CR manifold of real dimension 3, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^N$. By Harvey and Lawson [Ha-La], there is a unique complex variety V in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \le i \le k$, as exceptional sets. Then the s-invariant defined in Definition 3.2 is CR invariant, which is also called $s(X)$.

In order to solve the classical complex Plateau problem, we need to find some CR-invariant which can be calculated directly from the boundary X and the vanishing of this invariant will give the regularity of Harvey–Lawson solution to the complex Plateau problem.

For this purpose, we define a new sheaf $\bar{\bar{\Omega}}^{1,1}_V$.

Definition 3.5. *Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Define a sheaf of germs $\bar{\bar{\Omega}}^{1,1}_V$ by the sheaf associated to the presheaf*

$$U \mapsto < \Gamma(U, \bar{\bar{\Omega}}^1_V) \wedge \Gamma(U, \bar{\bar{\Omega}}^1_V) >,$$

where U is an open set of V.

Lemma 3.6. Let V be a 2-dimensional Stein space with 0 as its only singular point in $\mathbb{C}^N$. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Then $\bar{\bar{\Omega}}_V^{1,1}$ is coherent and there is a short exact sequence

$$0 \longrightarrow \bar{\bar{\Omega}}_V^{1,1} \longrightarrow \bar{\bar{\Omega}}_V^2 \longrightarrow \mathscr{G}^{(1,1)} \longrightarrow 0 \tag{3.8}$$

where $\mathscr{G}^{(1,1)}$ is a sheaf supported on the singular point of V. Let

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) >; \tag{3.9}$$

then $dim\mathscr{G}_0^{(1,1)} = dimG^{(1,1)}(M\backslash A)$.

Proof. Since the sheaf of germ $\bar{\bar{\Omega}}_V^1$ is coherent by a theorem of Siu (see Theorem A of [Si]), for any point $w \in V$ there exists an open neighborhood U of w in V such that $\Gamma(U, \bar{\bar{\Omega}}_V^1)$ is finitely generated over $\Gamma(U, \mathscr{O}_V)$. So $\Gamma(U, \bar{\bar{\Omega}}_V^1) \wedge \Gamma(U, \bar{\bar{\Omega}}_V^1)$ is finitely generated over $\Gamma(U, \mathscr{O}_V)$, which means $\Gamma(U, \bar{\bar{\Omega}}_V^{1,1})$ is finitely generated over $\Gamma(U, \mathscr{O}_V)$ – i.e., $\bar{\bar{\Omega}}_V^{1,1}$ is a sheaf of finite type. It is obvious that $\bar{\bar{\Omega}}_V^{1,1}$ is a subsheaf of $\bar{\bar{\Omega}}_V^2$ which is also coherent. So $\bar{\bar{\Omega}}_V^{1,1}$ is coherent.

Notice that the stalk of $\bar{\bar{\Omega}}_V^{1,1}$ and $\bar{\bar{\Omega}}_V^2$ coincide at each point different from the singular point 0, so $\mathscr{G}^{(1,1)}$ is supported at 0. And from Cartan Theorem B

$$dim\mathscr{G}_0^{(1,1)} = dim\Gamma(V, \bar{\bar{\Omega}}_V^2)/\Gamma(V, \bar{\bar{\Omega}}_V^{1,1}) = dimG^{(1,1)}(M\backslash A).$$

q.e.d.

Thus, from Lemma 3.6, we can define a local invariant of a singularity which is independent of resolution.

Definition 3.7. *Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Let*

$$g^{(1,1)}(0) := dim\mathscr{G}_0^{(1,1)} = dimG^{(1,1)}(M\backslash A). \tag{3.10}$$

We will omit 0 in $g^{(1,1)}(0)$ if there is no confusion from the context.

Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \le i \le k$, as exceptional sets. In this case, we still let

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) > .$$

Definition 3.8. *If X is a compact connected strongly pseudoconvex CR manifold of real dimension 3 which is in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^N$. Suppose V in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : (M, A = \bigcup_i A_i) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \le i \le k$, as exceptional sets. Let*

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) > \tag{3.11}$$

and

$$G^{(1,1)}(X) := \mathscr{S}^2(X)/ < \mathscr{S}^1(X) \wedge \mathscr{S}^1(X) > \tag{3.12}$$

where $\mathscr{S}^p$ are holomorphic cross sections of $\wedge^p(\widehat{T}(X)^*)$. Then we set

$$g^{(1,1)}(M\backslash A) := dim G^{(1,1)}(M\backslash A), \tag{3.13}$$

$$g^{(1,1)}(X) := dim G^{(1,1)}(X). \tag{3.14}$$

Lemma 3.9. Let X be a compact connected strongly pseudoconvex CR manifold of real dimension 3 which bounds a bounded strongly pseudoconvex variety V with only isolated singularities $\{0_1, \cdots, 0_k\}$ in $\mathbb{C}^N$. Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Then $g^{(1,1)}(X) = g^{(1,1)}(M\backslash A)$, where $A = \cup A_i$, $1 \leq i \leq k$.

Proof. Take a one-convex exhausting function ϕ on M such that $\phi \geq 0$ on M and $\phi(y) = 0$ if and only if $y \in A$. Set $M_r = \{y \in M, \phi(y) \geq r\}$. Since $X = \partial M$ is strictly pseudoconvex, any holomorphic p-form $\theta \in \mathscr{S}^p(X)$ can be extended to a one-sided neighborhood of X in M. Hence, θ can be thought of as holomorphic p-form on M_r– i.e., an element in $\Gamma(M_r, \Omega^p_{M_r})$. By Andreotti and Grauert ([An-Gr]), $\Gamma(M_r, \Omega^p_{M_r})$ is isomorphic to $\Gamma(M\backslash A, \Omega^p_M)$. So $g^{(1,1)}(X) = g^{(1,1)}(M\backslash A)$. q.e.d.

By Lemma 3.9 and the proof of Lemma 3.6, we can get the following lemma easily.

Lemma 3.10. Let X be a compact connected strongly pseudoconvex CR manifold of real dimension 3, which bounds a bounded strongly pseudoconvex variety V with only isolated singularities $\{0_1, \cdots, 0_k\}$ in $\mathbb{C}^N$. Then $g^{(1,1)}(X) = \sum_i g^{(1,1)}(0_i) = \sum_i dim \mathscr{G}^{(1,1)}_{0_i}$.

The following proposition is to show that $g^{(1,1)}$ is bounded above.

Proposition 3.11. Let V be a 2-dimensional Stein space with 0 as its only singular point. Then $g^{(1,1)} \leq p_g + g^{(2)}$.

Proof. Since

$$g^{(1,1)} = dim\Gamma(M\backslash A, \Omega^2_M)/ < \Gamma(M\backslash A, \Omega^1_M) \wedge \Gamma(M\backslash A, \Omega^1_M) >,$$
$$p_g = dim\Gamma(M\backslash A, \Omega^2_M)/\Gamma(M, \Omega^2_M),$$
$$g^{(2)} := dim\Gamma(M, \Omega^2)/\pi^*\Gamma(V, \Omega^2_V),$$

and

$$\begin{aligned} \pi^*\Gamma(V, \Omega^2_V) &= < \pi^*\Gamma(V, \Omega^1_V) \wedge \pi^*\Gamma(V, \Omega^1_V) > \\ &\subseteq \Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M) \\ &\subseteq \Gamma(M\backslash A, \Omega^1_M) \wedge \Gamma(M\backslash A, \Omega^1_M), \end{aligned} \tag{3.15}$$

the result follows. q.e.d.

The following theorem is the crucial part for the classical complex Plateau problem.

Theorem 3.12. Let V be a 2-dimensional Stein space with 0 as its only normal singular point with $\mathbb{C}^*$-action. Let $\pi : (M, A) \to (V, 0)$ be a minimal good resolution of the singularity with A as exceptional set, then $g^{(1,1)} \geq 1$.

Proof. If $dim \Gamma(M \backslash A, \Omega_M^2)/\Gamma(M, \Omega_M^2) > 0$, then there exists

$$\omega_0 \in \Gamma(M \backslash A, \Omega_M^2) \backslash \Gamma(M, \Omega_M^2).$$

So ω_0 must have pole along some irreducible component A_k of A. Suppose ω has the highest order of pole along A_k and $\omega \in \Gamma(M \backslash A, \Omega_M^2)$. Denote $Ord_{A_k} \omega = r < 0$. Let $z_1, \cdots, z_m$ be coordinate functions of $\mathbb{C}^m$. Choose a point b in A_k which is a smooth point of A. Let(x_1, x_2) be a coordinate system centered at b such that A_k is given locally by $x_1 = 0$ at b. Take the power series expansion of $\pi^*(z_j)$ around b:

$$\pi^*(z_j) = x_1^{r_j} f_j, 1 \leq j \leq m, \tag{3.16}$$

where f_j is holomorphic function such that $f_j(0, x_2) \neq 0$. So by the choice of ω, $min\{r_1, \ldots, r_m\} > 0 > r$.

Let $\xi_V \in \Gamma(V, \Theta_V)$, where $\Theta_V := \mathscr{H}om_{\mathscr{O}_V}(\Omega_V^1, \mathscr{O}_V)$, denote the generating vector field of the $\mathbb{C}^*$-action and i_{ξ_V} be the contraction map. For some $\alpha \in \Gamma(V, \bar{\bar{\Omega}}_V^1)$, write α as a sum $\sum \alpha^j$ of quasi-homogeneous elements where α^j is a quasi-homogeous element of degree $l_j > 0$. Let $L_{\xi_V} = i_{\xi_V} d + d i_{\xi_V}$ be the Lie derivation. Then

$$l_j \alpha^j = L_{\xi_V} \alpha^j = i_{\xi_V} d(\alpha^j) + d i_{\xi_V}(\alpha^j).$$

So

$$\Gamma(V, \bar{\bar{\Omega}}_V^1) = d(\Gamma(V, \mathscr{O}_V)) + i_{\xi_V}(\Gamma(V, \bar{\bar{\Omega}}_V^2)). \tag{3.17}$$

For minimal good resolution, we have $\pi_* \Theta_M = \Theta_V$ (see [Bu-Wa]), where Θ_M is the vector field on M. Thus, there exists ξ_M which is a lift of ξ_V – i.e., $\pi_* \xi_M = \xi_V$. We know that ξ_M is tangential to the exceptional set, so

$$\xi_M \mathrel{\hat{=}} x_1^{a_1} p \frac{\partial}{\partial x_1} + x_1^{a_2} q \frac{\partial}{\partial x_2}, a_1 \geq 1, a_2 \geq 0$$

where p and q are holomorphic functions.

Let $i_{\xi_M} : \Gamma(M \backslash A, \Omega_M^2) \longrightarrow \Gamma(M \backslash A, \Omega_M^1)$ be the contraction map corresponding to i_{ξ_V}. If $\zeta \in \Gamma(M \backslash A, \Omega_M^2)$ and $\zeta \mathrel{\hat{=}} x_1^u g dx_1 \wedge dx_2$, then

$$i_{\xi_M}(\zeta) \mathrel{\hat{=}} i_{\xi_M}(x_1^u g dx_1 \wedge dx_2) = -x_1^{u+a_2} q g dx_1 + x_1^{u+a_1} p g dx_2.$$

From (3.17),

$$\Gamma(M \backslash A, \Omega_M^1) = d(\Gamma(M \backslash A, \mathscr{O}_M)) + i_{\xi_M}(\Gamma(M \backslash A, \Omega_M^2)).$$

Since V is normal , $g^{(0)} = 0$ – i.e., $\Gamma(M, \mathscr{O}_M) = \pi^*(\Gamma(V, \mathscr{O}_V))$. Moreover, by the normality of $(V, 0)$, $\Gamma(M, \mathscr{O}_M) = \Gamma(M \backslash A, \mathscr{O}_M)$.

We now prove that ω is not contained in $< \Gamma(M\backslash A, \Omega^1_M) \wedge \Gamma(M\backslash A, \Omega^1_M) >$. Consider $\eta, \varphi \in \Gamma(M\backslash A, \Omega^1_M)$ locally around b

Suppose $\eta = \eta_1 + \eta_2$ and $\varphi = \varphi_1 + \varphi_2$, where η_1, $\varphi_1 \in d(\Gamma(M, \mathscr{O}_M))$, η_2, $\varphi_2 \in i_{\xi_M}(\Gamma(M\backslash A, \Omega^2_M))$. Let

$$\eta_2 = i_{\xi_M}(\zeta), \qquad \zeta \mathrel{\hat{=}} x_1^u g dx_1 \wedge dx_2, \qquad g(0, x_2) \neq 0$$

and

$$\varphi_2 = i_{\xi_M}(\varsigma), \qquad \varsigma \mathrel{\hat{=}} x_1^v h dx_1 \wedge dx_2, \qquad h(0, x_2) \neq 0.$$

So u and v are bounded lower by r.

Then

$$\eta \wedge \varphi = \eta_1 \wedge \varphi_1 + (\eta_1 \wedge \varphi_2 + \eta_2 \wedge \varphi_1) + \eta_2 \wedge \varphi_2.$$

Since

$$d\pi^*(z_i) \wedge d\pi^*(z_j) = (r_i x_1^{r_i + r_j - 1} f_i \frac{\partial f_j}{\partial x_2} - r_j x_1^{r_i + r_j - 1} f_j \frac{\partial f_i}{\partial x_2}) dx_1 \wedge dx_2,$$

$Ord_{A_k} \eta_1 \wedge \varphi_1 \geq 2 \cdot min\{r_1, \ldots, r_m\} - 1 > r$.

Write η_2 and φ_2 locally around b:

$$\eta_2 \mathrel{\hat{=}} -x_1^{u+a_2} q g dx_1 + x_1^{u+a_1} p g dx_2,$$
$$\varphi_2 \mathrel{\hat{=}} -x_1^{v+a_2} q h dx_1 + x_1^{v+a_1} p h dx_2.$$

So $\eta_2 \wedge \varphi_2 = \mathrel{\hat{=}} 0$.

Also notice that

$$d\pi^*(z_j) = r_j x_1^{r_j - 1} f_j dx_1 + x_1^{r_j} \frac{\partial f_j}{\partial x_2} dx_2.$$

So

$$Ord_{A_k} \eta_1 \wedge \varphi_2 \geq min\{r_1, \ldots, r_m\} + v > r$$

and

$$Ord_{A_k} \eta_2 \wedge \varphi_1 \geq min\{r_1, \ldots, r_m\} + u > r.$$

From the discussion above, we can get $Ord_{A_k} \eta \wedge \varphi > r$.

Therefore, ω is not a linear combination of elements in $< \Gamma(M\backslash A, \Omega^1_M) \wedge \Gamma(M\backslash A, \Omega^1_M) >$.

If $dim \Gamma(M\backslash A, \Omega^2_M)/\Gamma(M, \Omega^2_M) = 0$, the singularity is rational. So irregularity $q = 0$ (see [Ya4]). Then

$$\frac{\Gamma(M\backslash A, \Omega^2_M)}{< \Gamma(M\backslash A, \Omega^1_M) \wedge \Gamma(M\backslash A, \Omega^1_M) >} = \frac{\Gamma(M, \Omega^2_M)}{< \Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M) >},$$

$$g^{(1,1)} = dim \frac{\Gamma(M, \Omega^2_M)}{< \Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M) >}.$$

From [Ya3], the canonical bundle K_M is generated by its global sections in a neighborhood of the exceptional set. So there exists $\omega \in \Gamma(M, \Omega^2_M)$ such that ω does not vanish along some irreducible component A_k of

A. The rest of the argument is same as those arguments above with $r = 0$ – i.e., we can get ω is not a linear combination of elements in $< \Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M) >$. So

$$g^{(1,1)} = dim \frac{\Gamma(M, \Omega^2_M)}{< \Gamma(M, \Omega^1_M) \wedge \Gamma(M, \Omega^1_M) >} \geq 1.$$

q.e.d.

4. The classical complex Plateau problem

In 1981, Yau [Ya] solved the classical complex Plateau problem for the case $n \geq 3$.

Theorem 4.1. *([Ya])* Let X be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$, $n \geq 3$, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^{n+1}$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if Kohn–Rossi cohomology groups $H^{p,q}_{KR}(X)$ are zeros for $1 \leq q \leq n-2$

Next, we want to use our new invariants introduced in § 3 to solve the classical complex Plateau problem for the case $n = 2$.

First, we present some known results from the paper [Lu-Ya2].

Theorem 4.2. *([Lu-Ya2])* Let X be a compact connected $(2n-1)$-dimensional $(n \geq 2)$ strongly pseudoconvex CR manifold. Suppose that X is the boundary of an n-dimensional strongly pseudoconvex manifold M which is a modification of a Stein space V with only isolated singularities $\{0_1, \ldots, 0_m\}$. Let A be the maximal compact analytic set in M which can be blown down to $\{0_1, \ldots, 0_m\}$. Then:

1 . $H^q_h(X) \cong H^q_h(M \backslash A) \cong H^q_h(M), \qquad 1 \leq q \leq n-1.$
2 . $H^n_h(X) \cong H^n_h(M \backslash A), dim H^n_h(M \backslash A) = dim H^n_h(M) + s$, where $s = s_1 + \cdots + s_m$, s_i is the s-invariant of the singularity $(V, 0_i)$.

Theorem 4.3. *([Lu-Ya2])* Let $(V, 0)$ be a Gorenstein surface singularity. Let $\pi : M \to V$ be a good resolution with $A = \pi^{-1}(0)$ as exceptional set. Assume that M is contractible to A. If $s = 0$, then $(V, 0)$ is a quasi-homogeneous singularity, $H^1(A, \mathbb{C}) = 0$, $dim H^1(M, \Omega^1) = dim H^2(A, \mathbb{C}) + dim H^1(M, \mathscr{O})$, and $H^1_h(M) = H^2_h(M) = 0$. Conversely, if $(V, 0)$ is a 2-dimensional quasi-homogeneous Gorenstein singularity and $H^1(A, \mathbb{C}) = 0$, then the s-invariant vanishes.

Theorem 4.4. *([Lu-Ya2])* Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. If the holomorphic De Rham cohomology $H^2_h(X) = 0$, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing s-invariant.

Corollary 4.5. *([Lu-Ya2])* Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^3$. If the holomorphic De Rham cohomology $H_h^2(X) = 0$, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated quasi-homogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star shaped and all the curves are rational.

So from several theorems above we can see that, in the paper [Lu-Ya2], Luk and Yau give a sufficient condition $H_h^2(X) = 0$ to determine when X can bound some special singularities. However, even if both $H_h^1(X)$ and $H_h^2(X)$ vanish, V still can be singular.

We use CR invariants given in the last section to get sufficient and necessary conditions for the variety bounded by X being smooth after normalization.

Theorem 4.6. Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is a boundary of the complex variety $V \subset D - X$ with boundary regularity and the variety is smooth after normalization if and only if s-invariant and $g^{(1,1)}(X)$ vanish.

Proof. $(\Rightarrow)$: Since V is smooth after normalization, $g^{(1,1)}(X) = 0$ follows from Lemma 3.10.

$(\Leftarrow)$: It is well known that X is a boundary of a variety V in D with boundary regularity ([Lu-Ya, Ha-La2]). Since $s = 0$, X is a boundary of the complex sub-manifold $V \subset D - X$ with only isolated Gorenstein quasi-homogeneous singularities $\{0_1, \cdots, 0_k\}$ after normalization. Let $\pi_i : M_i \to V_i$ be the minimal good resolution of a sufficiently small neighborhood V_i of 0_i in V, $1 \le i \le k$. From Theorem 3.12, $dimG^{(1,1)}(M_i) > 0$, which contradicts $g^{(1,1)}(X) = 0$. So V is smooth. q.e.d.

Corollary 4.7. Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^3$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if s-invariant and $g^{(1,1)}(X)$ vanish.

From Theorem 4.2, we know that if $H_h^2(X) = 0$, and then $s = 0$. So we can get a necessary and sufficient condition in terms of boundary X, with $H_h^2(X) = 0$, to determine when X is a boundary of a manifold up to normalization.

Corollary 4.8. Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$ with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold up to normalization $V \subset D - X$ with boundary regularity if and only if $g^{(1,1)}(X) = 0$.

Corollary 4.9. Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^3$ with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.

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