1. Introduction

Since the discovery of mirror symmetry in string theory by physicists, there have been tremendous activities on Calabi-Yau manifolds both by physicists and mathematicians. The reason that mirror symmetry has attracted a lot of mathematicians’ attention is that it predicts successfully the number \( n_k \) of rational curves of degree \( k \) in these manifolds. This so-called Mirror Conjecture was first solved recently by Lian, Liu and Yau in their celebrated work [3]. In this paper we shall study the geometry of distinguished class of Calabi-Yau manifolds

\[
X_s = \{(x_1 : \cdots : x_n) \in \mathbb{C}P^{n-1} : x_1^n + \cdots + x_n^n + 8x_1x_2\cdots x_n = 0\}.
\]

For \( n = 5 \), this class of Calabi-Yau 3-manifolds were studied in detail by Candelas, Ossn, Green and Parkers [1] by means of the period map. In particular, they observed that the modular group is not \( SL(2, \mathbb{Z}) \).

It is the purpose of this paper to find out the moduli and the modular group of this one-parameter family of Calabi-Yau manifolds in (1.1) for all \( n \geq 5 \). Our argument is uniform for all \( n \geq 5 \). We remark that \( n = 3 \) was treated by our previous paper [2] with different motivation.

The crucial contribution of our paper is the introduction of some special points in Calabi-Yau manifolds.

Let \( \rho_i, i = 1,2,\ldots,n \), be \( n \)-distinct roots of \( x^n = -1 \). It is clear that the following \( N = \frac{1}{2}n^2(n - 1) \) points \( Q_1,\ldots,Q_N \) of the form

\[
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\]
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0), where 1, \( \rho_i \) run over all possible 2-tuple positions of 1, 2, \ldots, \( n \), are on each Calabi-Yau manifold \( X_s \). We shall show in Proposition 2.1 that there are \( (n - 2) \) independent hyperplanes through \( Q_i \) in \( T_Q(X_s) \), the tangent plane of \( X_s \) at \( Q_i \), for which all the lines passing through \( Q_i \) in these \( (n - 2) \) independent hyperplanes have contact order \( n \) with \( X_s \) at \( Q_i \).

**Definition 1.1.** A point \( Q \) in a \( (n - 2) \)-dimensional Calabi-Yau manifold \( X \) is said to have \( C - Y \) property if there are \( (n - 2) \) independent hyperplanes through \( Q \) in \( T_Q(X) \) for which all the lines passing through \( Q \) in these \( (n - 2) \) independent hyperplanes have contact order at least \( n \) with \( X \) at \( Q \). Such point \( Q \) is called a \( C - Y \) point in \( X \).

**Theorem A.** For \( n \geq 5 \), \( s \neq 0 \) and \( s^n \neq (-n)^n \), the \( C - Y \) points on the Calabi-Yau manifolds

\[
X_s = \{(x_1 : \ldots : x_n) \in \mathbb{CP}^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \cdots x_n = 0\}
\]

are precisely \( Q_1, \ldots, Q_N, N = \frac{1}{2} n^2 (n - 1) \), of the form \((0, \ldots, 0, 1, 0, \ldots, 0, \rho_i, 0, \ldots, 0)\), where \( 1, \rho_i, 1 \leq i \leq n \), run over all possible 2-tuple positions of \( 1, 2, \ldots, n \) and \( \rho_i, 1 \leq i \leq n \), are the \( n \)-distinct roots of \( x^n = 1 \).

Using Theorem A, we can prove the following theorem.

**Theorem B.** For \( n \geq 5 \), \( t \neq s \), \( s^n \neq 0 \) and \( t^n \neq 0 \) and \( \neq (-n)^n \), the group \( G \) of biholomorphisms between

\[
X_t = \{(x_1 : \ldots : x_n) \in \mathbb{CP}^{n-1} : x_1^n + \cdots + x_n^n + tx_1 \cdots x_n = 0\}
\]

and

\[
X_s = \{(x_1 : \ldots : x_n) \in \mathbb{CP}^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \cdots x_n = 0\}
\]

consists of all projective nonsingular linear transformation \( B \in \text{PGL}(n, \mathbb{C}) \) of the following form:

\[
B = \begin{pmatrix}
0 & \cdots & 0 & a_{1i_1} & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{2i_2} & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\end{pmatrix}
\]

where \((i_1, \ldots, i_n)\) is a permutation of \((1, \ldots, n)\) and \( a_{1i_1}, \ldots, a_{ni_n} \) are \( n \)-th root of unity. Each such \( B \) induces a linear transformation on the
parameter space by sending $t$ to $t\alpha_{i_1}\ldots\alpha_{i_n}$. The group $G$ has order $n^{n-1}(n!)$. Let $N$ be the group of automorphisms of $X_t$. Then $N$ is a normal subgroup of $G$ of order $n^{n-2}(n!)$.

**Theorem C.** For $n \geq 5$, the modulus function of the one parameter family of Calabi-Yau manifolds

$$X_s = \{ (x_1 : \ldots : x_n) \in \mathbb{C}P^{n-1} : x_1^n + \cdots + x_n^n + sx_1\ldots x_n = 0 \}$$

is $s^n$, i.e. for any two parameters $t, s$, $X_t$ is biholomorphically equivalent to $X_s$ if and only if $t^n = s^n$.

2. Special points on Calabi-Yau manifolds

Let $X_s$ be the $(n-2)$-dimensional hypersurface defined by $x_1^n + \cdots + x_n^n + sx_1\ldots x_n = 0$ in $\mathbb{C}P^{n-1}$. It is easy to see that $X_s$ is a nonsingular manifold for $s^n \neq (-n)^n$. In fact, let

$$f(x_1, \ldots, x_n) = x_1^n + \cdots + x_n^n + sx_1\ldots x_n.$$  

Then $X_s$ is nonsingular if and only if there is no common solution to the $n$ equations

$$\frac{\partial f}{\partial x_i} = nx_i^{n-1} + sx_1\ldots x_i\ldots x_n = 0, \quad 1 \leq i \leq n$$

in $\mathbb{C}P^{n-1}$. These equations imply that

$$nx_1^n = nx_2^n = \cdots = nx_n^n = -sx_1\ldots x_n,$$

whence

$$(-n)^n \prod_{i=1}^{n} x_i^n = (s)^n \prod_{i=1}^{n} x_i^n.$$  

If $P = (p_1 : \ldots : p_n) \in \mathbb{C}P^{n-1}$ is a common solution of equations (2.2), then none of the $p_i$’s may be zero by (2.3). Hence $s^n = (-n)^n$. Conversely it is easy to see that $X_s$ is singular when $s^n = (-n)^n$.

**Proposition 2.1.** Let $p_j, j = 1, 2, \ldots, n$, be $n$ distinct roots of $x^n = -1$. For each $s$ with $s^n \neq (-n)^n$, let $Q_i$ be one of the $N = n^2(n-1)/2$ points of the form $(0, \ldots, 0, 1, 0, \ldots, 0, p_j, 0, \ldots, 0)$ on the Calabi-Yau manifold $X_s = \{(x_1 : \ldots : x_n) \in \mathbb{C}P^{n-1} : x_1^n + \cdots + x_n^n + sx_1\ldots x_n = 0 \}$.
Then $Q_i$ is a $C - Y$ point i.e., there are $(n - 2)$ independent hyperplanes through $Q_i$ in $T_{Q_i}(X_s)$ for which all the lines passing through $Q_i$ in these $(n - 2)$ independent hyperplanes have contact order at least $n$ with $X_s$ at $Q_i$.

Proof. Without loss of generality, we only check that $Q_1 = (1, \rho_1, 0, \ldots, 0)$ is a $C - Y$ point. It is clear that the tangent plane $T_{Q_1}(X_s)$ of $X_s$ at $Q_1$ has equation

$$x_1 + \rho_1^{n-1}x_2 = 0.$$  \hspace{1cm} (2.5)

Thus $T_{Q_1}(X_s) \cap X_s$ is defined by the equations

$$\begin{cases}
    x_1 + \rho_1^{n-1}x_2 = 0 \\
    x_1^n + \cdots + x_n^n + sx_1 \cdots x_n = 0
\end{cases} \hspace{1cm} (2.6)$$

We can think of $\left(T_{Q_1}(X_s)\right) \cap X_s$ as a hypersurface in $\mathbb{P}(T_{Q_1}(X_s))$ with $(x_2 : x_3 : \ldots : x_n)$ as homogeneous coordinates. Its defining equation is

$$x_3^n + \cdots + x_n^n - s\rho_1^{n-1}x_2^2x_3 \cdots x_n = 0 \hspace{1cm} (2.7)$$

Observe that $x_2$ coordinate of $Q_1$ is nonzero. Let $x'_3 = \frac{x_3}{x_2}, \ldots, x'_n = \frac{x_n}{x_2}$ be the inhomogeneous coordinates. Then the inhomogeneous form of the equation of $\left(T_{Q_1}(X_s)\right) \cap X_s$ at $Q_1$ is

$$(x'_3)^n + \cdots + (x'_n)^n - s\rho_1^{n-1}x'_3 \cdots x'_n = 0 \hspace{1cm} (2.8)$$

It is clear that all lines tangent to $X_s$ at $Q_1$ are parameterized by $\mathbb{P}(T_{Q_1}(X_s)) = \mathbb{CP}^{n-3}$. Among all these lines we would like to find those lines with contact order to $X_s$ at least $n$. We can write the equation of a line $L$ as

$$\begin{cases}
    x'_3 = \alpha_3 t \\
    \vdots \\
    x'_n = \alpha_n t
\end{cases} \hspace{1cm} (2.9)$$

where $(\alpha_3 : \ldots : \alpha_n) \in \mathbb{P}(T_{Q_1}(X_s)) = \mathbb{CP}^{n-3}$. If the line $L$ has contact order $n$ with $X_s$ at $Q_1$, the coefficients of $t^k$ for $k \leq n - 1$ have to be zero when (2.9) is substituted in (2.8). It is clear that $L$ has contact order $n$ with $X_s$ at $Q_1$ if and only if one of the $\alpha_i$ has to be zero. This means that there are $(n - 2)$ independent hyperplanes through $Q_i$ in $T_{Q_i}(X_s)$ for which all the lines passing through $Q_i$ in these $(n - 2)$ independent hyperplanes have contact order at least $n$ with $X_s$ at $Q_i$. q.e.d.
We shall show that all the $C - Y$ points on $X_s$ are exactly those
$N = n^2(n - 1)/2$ points listed in Proposition 2.1. For this purpose, we
need to prove the following lemma.

**Lemma 2.2.** Let $Q = (q_1, \ldots, q_n)$ be a $C - Y$ point in the Calabi-
Yau manifold

$$X_s = \{(x_1 : \ldots : x_n) \in \mathbb{C}P^{n - 1} : x_1^n + \cdots + x_n^n + sx_1 \cdots x_n = 0\}.$$  

Let $f = x_1^n + \cdots + x_n^n + sx_1 \cdots x_n$ and $\frac{\partial f}{\partial x_1}(Q) = b_1, \ldots, \frac{\partial f}{\partial x_n}(Q) = b_n$. Suppose $b_1 = \frac{\partial f}{\partial x_1}(Q) \neq 0$ and $q_2 \neq 0$. Denote $a_2 = \frac{b_2}{b_1}, \ldots, a_n = \frac{b_n}{b_1}$. Then all partial derivatives of $f(-a_2x_2 - \cdots - a_nx_n, x_2, \ldots, x_n)$ with respect to the variables $x_3, \ldots, x_n$ with order at most $n - 3$ are zero at $Q$.

**Proof.** We first make a general observation. Let $g(x_2, \ldots, x_n)$ be a homogeneous polynomial of degree $m$. Let

$$g'(x_3', \ldots, x_n') = g(1, x_3', \ldots, x_n')$$

be a homogeneous form of $g$ where $x_3' = \frac{x_3}{x_2}, \ldots, x_n' = \frac{x_n}{x_2}$. It is easy to see that

$$\frac{\partial^p g}{\partial x_{i_1} \cdots \partial x_{i_p}} = (x_2)^{n-p} \frac{\partial^p g'}{\partial x'_{i_1} \cdots \partial x'_{i_p}}, \quad i_1, \ldots, i_p \in \{3, \ldots, n\}.$$  

Thus in order to prove the lemma, it is enough to prove the following statement: For the inhomogeneous form $w(x'_3, \ldots, x'_n)$ of $f(-a_2x_2 - \cdots - a_nx_n, x_2, \ldots, x_n)$, where $x'_3 = \frac{x_3}{x_2}, \ldots, x'_n = \frac{x_n}{x_2}$,

$$\left.\frac{\partial^p w(x'_3, \ldots, x'_n)}{\partial x'_{i_1} \cdots \partial x'_{i_p}}\right|_Q = 0$$

for $p \leq n - 3$ and $i_1, \ldots, i_p \in \{3, \ldots, n\}$.

Consider the inhomogeneous coordinate $(q'_3, \ldots, q'_n)$ of $Q$ where $q'_3 = q_3, \ldots, q'_n = q_n$. Let $x''_3 = x'_3 - q'_3, \ldots, x''_n = x'_n - q'_n$. It is clear that (2.10) holds if and only if the following (2.11) holds

$$\left.\frac{\partial^p w(x''_3, \ldots, x''_n)}{\partial x''_{i_1} \cdots \partial x''_{i_p}}\right|_{(0, \ldots, 0)} = 0$$

if $p \leq n - 3$, $i_1, \ldots, i_p \in \{3, \ldots, n\}$. 


Notice that under the new coordinates \((x_3', \ldots, x_n')\), the point \(Q\) is \((0, \ldots, 0)\). Consider the \((n-2)\) hyperplanes in \(T_Q(X_s)\) with the special property in the Definition 1.1. Let \(L_1, \ldots, L_{n-2}\) be their defining equations. Then \(L_3, \ldots, L_n\) are linearly independent 1-forms in \(x_3', \ldots, x_n'\) variables. Write

\[
(2.12) \quad w(x_3'', \ldots, x_n'') = w_{\geq n} + w_{\leq n-1},
\]

where \(w_{\geq n}\) denotes the sum of monomials in \(w(x_3'', \ldots, x_n'')\) with degrees at least \(n\) while \(w_{\leq n-1}\) denotes the sum of monomials in \(w(x_3'', \ldots, x_n'')\) with degree at most \(n-1\). We shall prove that \(w_{\leq n-1}\) can be divided by \(L_3, \ldots, L_n\).

Since \(L_3, \ldots, L_n\) are linearly independent, we can take \(L_3, \ldots, L_n\) as new coordinates. If \(w_{\leq n-1}\) is not divisible by \(L_3\), then

\[
(2.13) \quad w_{\leq n-1} = L_3P + R,
\]

where \(P\) is a polynomial in \(L_3, \ldots, L_n\) and \(R\) is a polynomial in \(L_4, \ldots, L_n\). Let \(\alpha_4, \ldots, \alpha_n\) be such that \(R(\alpha_4, \ldots, \alpha_n) \neq 0\). Consider the line \(L\)

\[
\begin{cases}
L_3 = 0 \\
L_4 = \alpha_4t \\
\vdots \\
L_n = \alpha_nt.
\end{cases}
\]

Then \(w_{\leq n-1}(0, \alpha_4t, \ldots, \alpha_nt)\) is a polynomial of \(t\) with degree less than or equal to \(n-1\). Thus the line \(L\) cannot have contact order \(n\) with \(w = 0\) at \(Q\). This is a contradiction.

From the above argument, we have proved that \(w(x_3'', \ldots, x_n'')\) as polynomials of \(L_3, \ldots, L_n\), contains only monomials with degree at least \(n-2\). Since \(L_3, \ldots, L_n\) are linear in \(x_3', \ldots, x_n'\) variables, we conclude that \(w(x_3'', \ldots, x_n'')\) contains only monomials of \(x_3', \ldots, x_n'\) with degree at least \(n-2\). Thus (2.11) is proved. q.e.d.

The following theorem is the key theorem of this paper.

**Theorem 2.3.** For \(n \geq 5\), the set \(\{Q_1, \ldots, Q_N\}\) in Proposition 2.1 is precisely the set of all \(C-Y\) points in the Calabi-Yau manifold \(X_s = \{(x_1 \colon \ldots, x_n) \in \mathbb{C}P^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \ldots x_n = 0\}, s \neq 0\).

**Proof.** Let \(Q = (q_1, \ldots, q_n)\) be a \(C-Y\) point on \(X_s\). We need to show that \(Q \in \{Q_1, \ldots, Q_N\}\). We shall consider the local form of the equation of \((T_Q(X_s)) \cap X_s\) at \(Q\). Let \(f(x_1, \ldots, x_n) = x_1^n + \cdots +
Calabi-Yau $n$-dimensional manifolds

$x^a + sx_1 \cdots x_n$ and $b_1 = \frac{\partial f}{\partial x_1}(Q), \ldots, b_n = \frac{\partial f}{\partial x_n}(Q)$. Without loss of generality, we shall assume $b_1 \neq 0$. Let $a_2 = \frac{b_2}{b_1}, \ldots, a_n = \frac{b_n}{b_1}$. The defining equation of $(T_Q(X_s)) \cap X_s$ is

$$f(-a_2 x_2 - \cdots - a_n x_n, x_2, \ldots, x_n) = 0$$

with homogeneous coordinates $(x_2 : \ldots : x_n)$ on $P(T_Q(X_s))$. We assume also without loss of generality that $q_2 \neq 0$. In view of Lemma 2.2, we know that all $2^{nd}$ order partial derivatives of $f(-a_2 x_2 - \cdots - a_n x_n, x_2, \ldots, x_n)$ with respect to $x_i, x_j, i, j \in \{3, \ldots, n\}$ at $Q$ are zero because of $n \geq 5$. Hence, we have

$$\frac{\partial^2 f(-a_2 x_2 - \cdots - a_n x_n, x_2, \ldots, x_n)}{\partial x_i \partial x_j} \bigg|_Q = 0, \ i, j \geq 3.$$  

By chain rule, we get

$$a_i a_j \frac{\partial^2 f}{\partial x_1^2}(Q) - a_i \frac{\partial^2 f}{\partial x_1 \partial x_j}(Q) - a_j \frac{\partial^2 f}{\partial x_i \partial x_1}(Q) + \frac{\partial^2 f}{\partial x_i \partial x_j}(Q) = 0.$$  

Multiplying (2.16) with $b_1^2 = (\frac{\partial f}{\partial x_1}(Q))^2 \neq 0$, we get, for $i, j \geq 3$,

$$b_i b_j \frac{\partial^2 f}{\partial x_1^2}(Q) = b_i \frac{\partial^2 f}{\partial x_1 \partial x_j}(Q) + b_i b_j \frac{\partial^2 f}{\partial x_i \partial x_1}(Q)$$

$$- b_i^2 \frac{\partial^2 f}{\partial x_i \partial x_j}(Q),$$

which can be rewritten as, for $i, j \geq 3$

$$n(n-1)q_1^{n-2}(nq_i^{n-1} + sq_1 \cdots q_{i-1}q_{i+1} \cdots q_n)$$

$$\cdot (nq_j^{n-1} + sq_1 \cdots q_{j-1}q_{j+1} \cdots q_n)$$

$$= sq_2 \cdots q_{i-1}q_{i+1} \cdots q_{j-1}q_{j+1} \cdots q_n(nq_i^{n-1} + sq_2 \cdots q_n)$$

$$\cdot (nq_i^n + nq_j^n - nq_1^n + sq_1 \cdots q_n).$$

Now we only need to prove that $q_3 = \cdots = q_n = 0$ because these will imply that $Q \in \{Q_1, \ldots, Q_N\}$. There are two cases to be considered.

**Case 1.** $q_3, \ldots, q_n$ are nonzero. If $q_1 = 0$ in this case, we have

$$nq_i^n + nq_j^n - nq_1^n + sq_1 \cdots q_n = 0 \quad \forall \ i, j \geq 3$$
by (2.18). Thus \( q_i^n + q_j^n = 0 \) for any \( i, j \geq 3 \). In particular, if we

take \( i = j \geq 3 \), we get \( q_i^n = 0 \) and hence \( q_i = 0 \) for \( i \geq 3 \). This is a

contradiction.

On the other hand if \( q_i \neq 0 \), we shall consider (2.18) for \( i, j \geq 3 \) and

\( k, j \geq 3 \). By dividing these two equalities, we have

\[
\frac{nq_i^n + sq_1 \ldots q_n}{nq_k^n + sq_1 \ldots q_n} = \frac{nq_j^n + sq_1 \ldots q_n + nq_i^n - nq_1^n}{nq_k^n + sq_1 \ldots q_n + nq_j^n - nq_1^n},
\]

which implies

\[
\frac{nq_i^n + sq_1 \ldots q_n}{nq_k^n + sq_1 \ldots q_n} = \frac{n(q_j^n - q_i^n)}{n(q_j^n - q_i^n)} = 1.
\]

Hence we have \( x_i^n = x_k^n \) for \( i, k \geq 3 \). Similarly by exchanging the roles of the indices 2 and 3, (recall that \( q_3 \neq 0 \) is assumed), we have

\( q_2^n = q_3^n = \cdots = q_n^n \). If \( b_2 = \frac{\partial f}{\partial x_2}(Q) = nq_2^n - sq_1 q_3 \ldots q_n \neq 0 \), then by exchanging the roles of the indices of 1 and 2, (note \( q_1 \neq 0 \)), we have

\( q_1^n = q_2^n = \cdots = q_n^n \). Since \( (q_1, \ldots, q_n) \) satisfies the following equation

\[
q_1^n + \cdots + q_n^n + sq_1 \ldots q_n = 0
\]

we have \( q_2(nq_2^n - 1 + sq_1 q_2 \ldots q_n) = 0 \). This contradicts our assumption

that \( b_2 = nq_2^n - 1 + sq_1 q_3 \ldots q_n \neq 0 \). If

\[
b_2 = \frac{\partial f}{\partial x_2}(Q) = nq_2^n - 1 + sq_1 q_3 \ldots q_n = 0,
\]

then \( nq_2^n + sq_1 q_2 \ldots q_n = 0 \). (2.22) implies \( q_1^n + (n-1)q_2^n + sq_1 \ldots q_n = 0 \).

By adding \( q_3^n \) in both sides of this equation, we get \( q_3^n = q_2^n \). Thus

\( q_1^n = q_2^n = \cdots = q_n^n \). (2.22) implies \( nq_i^n + sq_1 \ldots q_n = 0 \) for \( 1 \leq i \leq n \). This

implies that \( Q \) is a singular point of \( X_s \), a contradiction.

**Case 2.** At least one of \( q_3, \ldots, q_n \) is equal to zero. Without loss of generality, we shall assume \( q_3 = 0 \). Since

\[
b_1 = \frac{\partial f}{\partial x_1}(Q) = nq_1^n - sq_2 \ldots q_n \neq 0
\]

is assumed, we have \( q_1 \neq 0 \). Consider (2.18) for \( i = j \geq 4 \). The

right-hand side of (2.18) becomes zero because of our assumption that

at least one of \( q_3, \ldots, q_n \) is equal to zero. It follows that \( nq_i^{n-1} + sq_1 \ldots q_{i-1}q_{i+1} \ldots q_n = 0, 4 \leq i \leq n \). These \( n - 3 \) equations together
with the assumption that at least one of \( q_3, \ldots, q_n \) is zero imply \( q_i = q_5 = \cdots = q_n = 0 \). If \( q_3 \) is nonzero, then at least one of \( q_2, q_4, \ldots, q_n \) is zero. By considering (2.18) with \( i = j = 3 \), we have

\[
n(n-1)q_1^{n-2}(nq_3^{n-1} + sq_2q_4 \cdots q_n)^2 = sq_2q_4 \cdots q_n(nq_3^{n-1} + sq_2 \cdots q_n)(2nq_3^n - nq_1^n + sq_1 \cdots q_n) = 0,
\]

which implies \( q_3 = 0 \). Thus we have shown \( q_3 = q_4 = \cdots = q_n = 0 \) and \( Q \) has to be in \( \{ Q_1, Q_2, \ldots, Q_n \} \). q.e.d.

### 3. Moduli and modular group of Calabi-Yau manifolds

We shall use Theorem 2.3 to study the moduli and modular group of Calabi-Yau manifolds.

**Theorem 3.1.** For \( n \geq 5 \) and any nonzero \( t \neq s \), the biholomorphism between \( X_t = \{(x_1 : \ldots : x_n) \in \mathbb{CP}^{n-1} : x_1^t + \cdots + x_n^t + tx_1 \cdots x_n = 0, t^n \neq (-n)^n \} \) and \( X_s = \{(x_1 : \ldots : x_n) \in \mathbb{CP}^{n-1} : x_1^s + \cdots + x_n^s + sx_1 \cdots x_n = 0, s^n \neq (-n)^n \} \) is induced by a projective nonsingular linear transformation \( B \in PGL(n, \mathbb{C}) \) on coordinates with only one nonzero entry in each row and each column. Moreover, these entries is \( B \) are \( n \)-th roots of unity. Conversely any matrix \( B \) of the above form will send \( X_t \) to \( X_s \) where \( s = tc_1c_2\ldots c_n \), being \( c_1, \ldots, c_n \) the nonzero entries of \( B \).

**Proof.** It is well known that any biholomorphism between \( X_t \) and \( X_s \) is induced by a projective nonsingular linear transformation \( B = (b_{ij}), 1 \leq i, j \leq n, \) in \( PGL(n, \mathbb{C}) \). For any \( C - Y \) point \( Q \) in \( X_t \), it is clear that \( B(Q) \), the image of \( Q \) under \( B \), is also a \( C - Y \) point on \( X_s \). In view of Theorem 2.3, we have \( \{ B(Q_1), \ldots, B(Q_N) \} = \{ Q_1, \ldots, Q_N \} \) where \( N = \frac{1}{2}n^2(n-1) \).

We now consider the set of first coordinates of the points \( B(Q_1), \ldots, B(Q_N) \). This set consists of \( N = \frac{1}{2}n^2(n-1) \) elements of the form \( a_{ij} + \rho a_{ij} \), with \( 1 \leq i < j \leq n, 1 \leq m \leq n \). We know that there are \( \frac{1}{2}n(n-1)(n-2) \) of \( N \) first coordinates of those points equal to zero. Hence there are \( \frac{1}{2}n(n-1)(n-2) \) of \( a_{ij} + \rho a_{ij} \), with \( 1 \leq i < j \leq n, 1 \leq m \leq n \), equal to zero. Suppose that \( k \) of \( n \) numbers \( a_{11}, \ldots, a_{1n} \) are zero. Notice that for nonzero complex numbers \( c \) and
d, there is at most one zero among \( n \) complex numbers \( c + \rho_m d \). We also note that if precisely only one of \( c, d \) is zero, then \( c + \rho_m d \) can never be zero for \( 1 \leq m \leq n \). Thus among \( N \) complex numbers \( a_{ij} + \rho_m a_{ij}, 1 \leq i < j \leq n, 1 \leq m \leq n \), there are at most \( \frac{1}{2}nk(k-1) + \frac{1}{2}(n-k)(n-k-1) \) of them are zero. It follows that we have the following inequality

\[
\frac{nk(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \geq \frac{n(n-1)(n-2)}{2}.
\]

(3.1) implies \( k > 0 \). It follows that \( nk \geq n - k \) because \( k \) is a positive integer. Thus, in view of (3.1) we have

\[
\frac{nk(n-2)}{2} = \frac{nk(k-1)}{2} + \frac{nk(n-k-1)}{2} \geq \frac{n(n-1)(n-2)}{2}.
\]

(3.2) implies \( k \geq n - 1 \). Since \( B \) is a nonsingular matrix, we have \( k = n - 1 \). Therefore we have proved that there is only one nonzero entry in the first row. Similarly, we can prove that there is only one nonzero entry in each row. Since \( B \) is nonsingular, there is only one nonzero entry in each column.

Let \( a_{1i}, a_{2i}, \ldots, a_{ni} \) be the nonzero entries of the 1st row, 2nd row, \ldots, and \( n \)th row of the matrix \( B \) respectively. Consider the action of \( B \) on the point \( P = (0, \ldots, 0, \rho_m, 0, \ldots, 0, 1, 0, \ldots, 0) \) where \( 1 \leq m \leq n, \rho_m \) is the \( i_1 \)-coordinate of \( P \) while 1 is the \( i_2 \)-coordinate of \( P \). Clearly \( B(P) = (a_{1i}, \rho_m, a_{2i}, 0, \ldots, 0) \) is a \( C - Y \) point. In view of Theorem 2.3, we have \( \rho_m a_{1i}/a_{2i} \in \{ \rho_1, \ldots, \rho_n \} \). This implies \( a_{1i}/a_{2i} \) is a \( n \)th root of unity. Similarly we can show that all ratios between \( a_{1i_1}, a_{2i_2}, \ldots, a_{ni_n} \) are \( n \)th root of unity. The first part of Theorem 3.1 follows immediately.

Conversely, suppose that \( B \) is a nonsingular matrix given by

\[
B : (x_1, x_2, \ldots, x_n) \mapsto (a_{1i}, x_{i_1}, a_{2i}, x_{i_2}, \ldots, a_{ni}, x_{i_n}),
\]

where \( a_{1i}, a_{2i}, \ldots, a_{ni} \) are \( n \)th roots of unity and \( (i_1, i_2, \ldots, i_n) \) is a permutation of \( (1, 2, \ldots, n) \). Then clearly \( X_t : x_{i_1}^n + \cdots + x_{i_n}^n + tx_{i_1} \cdots x_{i_n} = 0 \) is sent to \( X_s : x_{i_1}^n + \cdots + x_{i_n}^n + sx_{i_1} \cdots x_{i_n} = 0 \) where \( s = ta_{1i_1} \cdots a_{ni} \).

**Corollary 3.2.** For \( n \geq 5, t \neq s, s^n \) and \( t^n \neq 0 \) and \( (-n)^n \), the group \( G \) of biholomorphisms between \( X_t = \{(x_{i_1} : \cdots : x_{i_n}) \in \mathbb{CP}^{n-1} : \}


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