ON A CR FAMILY OF COMPACT STRONGLY
PSEUDOCONVEX CR MANIFOLDS

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Abstract

We study the simultaneous filling and embedding problem for a CR family of compact strongly pseudoconvex CR manifolds of dimension at least 5. We also derive, as a consequence, the normality of the Stein fibers of the filled-in Stein space under the constant dimensionality assumption of the first Kohn-Rossi cohomology group of the fiber CR manifolds. Two main ingredients for our approach are the work of Catlin on the solution of the $\bar{\partial}$-equation with mixed boundary conditions and the work of Siu and Ling on the study of the Grauert direct image theory for a (1,1)-convex-concave family of complex spaces.

1. Introduction

In this paper, we are concerned with a Cauchy-Riemann type of deformation for a compact strongly pseudoconvex manifold of real dimension at least 5. We will address the simultaneous embedding and filling problem of the family, as well as their applications in the deformation theory of isolated complex singularities. To start with, we introduce the following notion: (For more definitions, see §2).

**Definition 1.1.** Write $\Delta := \{ t \in \mathbb{C}, \ |t| < 1 \}$ and $\Delta_r := \{ t \in \mathbb{C}, \ |t| < r \}$ for $r > 0$. Assume that $\{ M_t \}_{t \in \Delta_r}$ is a parameterized family of connected compact $C^\infty$-smooth strongly pseudoconvex CR manifolds of (real) dimension $2n - 1$. The family $\{ M_t \}_{t \in \Delta_r}$ is said to be a CR family, or $M_{t_1}$ is said to be a CR deformation of $M_{t_2}$ for any $t_1, t_2 \in \Delta_r$, if there is a $C^\infty$-smooth strongly pseudoconvex CR manifold $X_r$ of real dimension $2n + 1$ and a $C^\infty$ CR map $\pi : X_r \rightarrow \Delta_r$ such that

(I) $\pi$ is a proper submersion;
(II) for any $t \in \Delta_r$, $M_t = \pi^{-1}(t)$ and $M_t$ is a $C^\infty$-smooth CR submanifold of $X_r$.

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In what follows, we simply say \((X_r, \pi, \Delta_r)\) or \(\pi : X_r \to \Delta_r\) is a CR family of compact strongly pseudoconvex CR manifolds.

As compact CR manifolds often come as the smooth boundaries of complex spaces with isolated singularities, the above definition is modeled by the following typical example of the holomorphic deformation of the complex structure of isolated singularities: Let \((\mathcal{V}_r, \pi, \Delta_r)\) be a small deformation of the complex space \(V_0 = \pi^{-1}(t_0)\) with an isolated singularity at \(p_0 \in V_0\). Assume that \(\mathcal{V}_r\) is embedded in \(\mathbb{C}^N\). For a positive \(\epsilon\), write \(S_{\epsilon}(p_0)\) for the sphere centered at \(p_0\) with radius \(\epsilon\). When it cuts \(V_r\) only at smooth points and CR-transversally, then \((V_r \cap S_{\epsilon}(p_0), \pi, \Delta_r)\) gives a CR deformation of the strongly pseudoconvex CR manifold \(V_0 \cap S_{\epsilon}(p_0)\), as defined above. It will be seen later in this paper that a CR deformation defined above can be generically realized in such a concrete manner when \(2n - 1 \geq 5\). Hence, the study of the deformation of isolated singularities is closely related to the study of the above notion of CR deformation of compact strongly pseudoconvex CR manifolds.

A compact strongly pseudoconvex CR manifold \(M\) of dimension at least 5 can be CR diffeomorphically mapped to the smooth boundary of a certain Stein space with at most isolated singularities embedded in some complex Euclidean space \(\mathbb{C}^N\), by the work of Boutet de Monvel \([8]\) and Harvey-Lawson \([16]\). (See §2 for the basic definitions and notations.) In general, \(N\) well depends on the intrinsic CR structure of \(M\). For a smooth family of strongly pseudoconvex CR manifolds, Tanaka addressed the simultaneous embedding problem under the assumption that the first Kohn-Rossi cohomology group of each fiber has a fixed dimension \([35]\). (Namely, \(\dim H^{(0,1)}_{KR}(M_t)\) is independent of \(t\). For the precise definition of \(H^{(0,1)}_{KR}(M_t)\), the reader is referred to \([35]\), \([13]\) or the first paragraph of §4 of the present paper.)

However, the methods in \([35]\) cannot be used to deal with the CR dependence on the parameter for the CR families, which turns out to be crucial for many studies in the deformation theory of the complex structure of isolated singularities.

In this paper, we will study the simultaneous embedding and lifting problems for a CR family of CR manifolds. We will also give applications to problems concerning the deformation of complex isolated singularities.

Before we give our main result, we briefly recall some basic definitions and results.

Suppose \(M\) is a finitely generated module over a local ring \((R, m)\) where \(m\) is the maximal ideal in \(R\). Suppose that \(\{f_1, \ldots, f_k\}\) in \(m\) is a
sequence such that $f_1$ is not a zero divisor for $M$ and $f_j$ is not a zero-divisor for $M \sum_{i=1}^{j-1} f_i M$ for $2 \leq j \leq k$ if $k \geq 2$. We then call $\{f_1, \ldots, f_k\}$ an $M$-sequence. Any permutation of an $M$-sequence is still an $M$-sequence. An $M$-sequence is called maximal if it is not contained in a longer $M$-sequence. All maximal $M$-sequences have the same length. This common length is called the homological codimension of $M$ over $R$, denoted by $\text{codh}_R M$ or simply by $\text{codh} M$. (We say that the homological codimension of $M$ over $R$ is zero if there is no $M$-sequence.)

**Definition 1.2** ([33]). Let $(X, \mathcal{O})$ be a complex space, $\mathcal{F}$ an analytic sheaf of $X$, and $p$ a non-negative integer. The $p$-th absolute gap-sheaf of $\mathcal{F}$, denoted by $\mathcal{F}^{[p]}$, is the analytic sheaf over $X$ defined by the following pre-sheaf: Suppose $U \subset V$ are open subsets of $X$. Then

$$\mathcal{F}^{[p]}(U) = \text{direct limit}_{A \in \mathcal{U}(U)} \Gamma(U - A, \mathcal{F}),$$

where $\mathcal{U}$ is the directed set of all subvarieties in $U$ of dimension $\leq p$ directed by inclusion. The map $\mathcal{F}^{[p]}(V) \to \mathcal{F}^{[p]}(U)$ is induced by the restriction map.

Let $\mathcal{F}$ be a coherent analytic sheaf over $X$. Set

$$S_k(\mathcal{F}) = \{x \in X : \text{codh}_{\mathcal{O}_x} \mathcal{F}_x \leq k\}.$$  

Then $S_k(\mathcal{F})$ is a subvariety of dimension $\leq k$ in $X$ ([29, Satz 5, p. 81]). The following proposition gives a relation for the above introduced objects:

**Proposition 1.3** ([34, 3.13]). Let $\mathcal{F}$ be a coherent analytic sheaf on $X$. Then $\mathcal{F}^{[p]} = \mathcal{F}$ if and only if $\dim S_{k+2}(\mathcal{F}) \leq k$ for $-1 \leq k < p$.

The following definition was given in Andreotti and Siu [3].

**Definition 1.4** ([3]). Let $(X, \mathcal{O})$ be a complex space. We say that $X$ is $p$-normal at $x \in X$ if $\mathcal{O}_x^{[p]} = \mathcal{O}_x$. We say that $X$ is $p$-normal if $\mathcal{O}^{[p]} = \mathcal{O}$.

**Main Theorem.** Let $\pi : X \to \Delta$ be a CR family of compact strongly pseudoconvex CR manifolds of (real) dimension $2n - 1$ ($n \geq 3$). Let $M_t = \pi^{-1}(t)$ for $t \in \Delta$. Then there exists a unique (up to isomorphism) 2-normal Stein complex space $\hat{X}$, which has $X$ as part of its smooth boundary. The CR structure of $X$ coincides with the inherited CR structure from $\hat{X}$ and is strongly pseudoconvex with respect to the complex Stein space $\hat{X}$. Moreover, there is a holomorphic map $\hat{\pi} : \hat{X} \to \Delta$ such that the following hold

(I) For any $t \in \Delta$, $\hat{\pi}^{-1}(t) = \hat{M}_t$ is a Stein space with $M_t = \pi^{-1}(t)$ as its smooth strongly pseudoconvex boundary.
Here, we will be content to state the following corollaries:

(II) For \( \epsilon < 1 \), write \( X_\epsilon := \bigcup_{|t| \leq \epsilon} M_t \) and \( \overline{X}_\epsilon = \bigcup_{|t| \leq \epsilon} M_t \). Also write \( \widehat{X}_\epsilon = \hat{\pi}^{-1}(\Delta_\epsilon) \) and \( \overline{X}_\epsilon = \hat{\pi}^{-1}(\Delta_\epsilon) \cup \overline{X}_\epsilon \). Then there exists a smooth function \( \rho^\epsilon \) defined in \( \overline{X}_\epsilon \), such that

(a) \( \rho^\epsilon \) is strictly plurisubharmonic near \( \overline{X}_\epsilon \);
(b) \( c^* < \rho^\epsilon \leq 0 \) for some \( c^* < 0 \);
(c) \( \rho^\epsilon = 0 \) exactly on \( \overline{X}_\epsilon \) and \( d\rho^\epsilon|_{\overline{X}_\epsilon} \neq 0 \).

(III) \( \hat{\pi} \) extends smoothly up to \( X \). Denote its smooth extension over \( X \) by \( \hat{\pi}|_X \). Then \( \hat{\pi}|_X \equiv \pi \).

(IV) Assume that for a certain \( \epsilon_0 \in (0, 1) \), there is a complex manifold \( Z_{\epsilon_0} \) such that \( X_{\epsilon_0} \) can be CR embedded into \( Z_{\epsilon_0} \). Suppose that \( f \) is a smooth CR equivalence map from \( M_0 \) to a certain CR submanifold \( M'_0 \subset C^m \), that extends holomorphically to \( \hat{M}_0 \). Assume that \( M'_0 \) is the smooth boundary of a certain Stein space \( V'_0 \) embedded in \( C^m \). (In particular, \( V'_0 \) is assumed to have only smooth points in a small neighborhood of \( M'_0 \) in \( C^m \).) Then when \( \epsilon \ll 1 \), there is a CR embedding \( T : X_\epsilon \rightarrow C^m \times C \) such that the following holds:

**IV1** There is a Stein space \( \hat{X}'_\epsilon \subset C^m \times C \), which has \( X'_\epsilon := T(X_\epsilon) \) as part of its smooth boundary.

**IV2** Let \( \hat{\pi}' \) be the natural projection from \( C^m \times \Delta_\epsilon \) into \( \Delta_\epsilon \). Then \( \hat{\pi}'^{-1}(t) \cap \hat{X}'_\epsilon := \hat{M}'_t \) is a Stein subvariety of \( \hat{X}'_\epsilon \) with \( M'_t \) as its strongly pseudoconvex boundary. Moreover, \( M'_t = T(M_t) \).

**IV3** \( T \) extends to a proper holomorphic map, still denoted by \( T \), from \( \hat{X}_\epsilon \) into \( \hat{X}'_\epsilon \) such that \( T = (F, \hat{\pi}) \) with \( F|_{\hat{M}_0} = f \).

We call the triplet \((\hat{X}, \hat{\pi}, \Delta)\) the Siu-Ling completion of the CR family \((X, \pi, \Delta)\). By the theorems proved in \([8] , [16], [30] – [31] \) and Ling \([24] \), for many interesting families, we can always find the map \( f \) as in the Main Theorem (IV), provided that \( m \gg 1 \). This makes our Main Theorem usable in many applications. We will address this issue in \( \S 4 \). Here, we will be content to state the following corollaries:

**Corollary 1.5.** Let \((X, \pi, \Delta)\) be a CR family of compact strongly pseudoconvex CR manifolds \( \{M_t\} \). Suppose that \( X_{\epsilon_0}(= \pi^{-1}(\Delta_{\epsilon_0})) \) for a certain \( \epsilon_0 \in (0, 1) \) can be CR embedded into a complex manifold. Assume that the real dimension of \( M_t \) is at least 5 and \( \dim H_{KR}^{(0,1)}(M_t) \) is constant. Then any \( \hat{M}_t = \hat{\pi}^{-1}(t) \) with \( t \in \Delta_{\epsilon_0} \), in the Siu-Ling completion of \((X, \pi, \Delta)\), is a normal Stein space.

**Corollary 1.6.** Let \((X, \pi, \Delta)\) be a CR deformation of a compact strongly pseudoconvex CR manifold \( M_0 \). Suppose that \( X_{\epsilon_0}(= \pi^{-1}(\Delta_{\epsilon_0})) \) for a certain \( \epsilon_0 \in (0, 1) \) can be CR embedded into a complex manifold. Assume that the real dimension of \( M_0 \) is at least 5 and \( \dim H_{KR}^{(0,1)}(M_0) \) is constant. If \( M_0 \) can be CR embedded into \( C^m \) by the smooth CR diffeomorphism \( f_0 \), then when \( 0 < \epsilon \ll \epsilon_0 \), there is a CR embedding...
\[ \Psi : X_{\epsilon} \to \mathbb{C}^m \times \mathbb{C} \text{ such that } \Psi|_{M_t} \text{ CR embeds } M_t \text{ into } \mathbb{C}^m \times \{t\}. \]

Moreover, we can make \( \Psi|_{M_0} = (f_0, 0) \).

**Remark 1.7.** Let \( X \) and \( X_{\epsilon} \) be defined as above. By a result to be proved in §2 of this paper, \( X_{\epsilon} \) (for any \( \epsilon \in (0, 1) \)) can always be CR embedded into a complex manifold, when the CR structure over \( X \) is real analytic. Here, we recall that the CR structure over \( X \) is said to be real analytic if \( X \) is a real analytic manifold and the bundle \( T^{(1,0)} X \) is a real analytic bundle over \( X \). Hence, when the total space \( X \) has a real analytic CR structure, the assumption that \( X_{\epsilon_0} \) can be CR embedded into a complex manifold for a certain \( \epsilon_0 \) is redundant in Main Theorem (IV), Corollary 1.5 and Corollary 1.6. By a very deep result of Catlin [10, Theorem 1.1], one also notices that for any \( 0 < \epsilon_0 < 1 \), \( X_{\epsilon_0} \) can be CR embedded into a complex manifold, even when \( X \) is merely assumed to be \((C^\infty)\) smooth. (See a detailed discussion on this matter in Remark 2.3 of §2.) We should mention that in many applications of the theory on CR manifolds to the study of complex singularities, the total space \( X \) comes as the smooth link of complex singularities and thus is naturally embedded in a complex manifold.

A special case of Corollary 1.5 was obtained by a different method by Fujiki [14] when \( \dim H_{KR}^{(0,1)}(M_t) \equiv 0 \). Corollary 1.6 can be viewed as a Cauchy-Riemann strengthening property for the CR family (or, for a holomorphic family of Stein spaces, respectively) along the parameter space. Corollary 1.6 has an immediate application to the study of the simultaneous blowing-down problem for strongly pseudoconvex complex manifolds, which will be addressed in Corollary 4.4 in §4. Also, the constant dimensionality of \( H_{KR}^{(0,1)}(M_t) \) in Corollary 4.6 (and thus Corollaries 1.5–1.6) seems to be important for the results to hold by the work of Knorr–Schneider and Riemenschneider [20], [28] on the conditions for the simultaneous blowing-down problem of a holomorphic family of the exceptional sets.

The key step for the proof of these results is to obtain a CR extension theorem for CR functions from the submanifold \( M_0 \) to \( X \). In our argument, it is important to have the dimension of \( M_t \) at least 5. However, it is not clear to us whether the Main Theorem still holds when the real dimension of \( M_t \) is 3, assuming that each fiber is fillable by complex spaces with isolated singularities, which we state as an open question. We do not know if we can also have some version of Corollary 1.6 when each \( M_t \) has real dimension 3, and each \( M_t \) is assumed to be globally embeddable. Apparently, by the work of Rossi and Jacobowitz-Treves [19], it cannot be true in general if one just considers the real analytic family.

The basic ingredients for the proof of the Main Theorem and Corollary 1.6 include the work on the embedding of the CR structures and
holomorphic completion of the so-called $(1,1)$-convex and concave space. 
(See the papers by Androitti-Siu [3], Siu [30] and Ling [24], Kuranishi [21], Akahori [2], Webster [37], Catlin [10], etc.) Especially, the work of Catlin [10] for solving the $\bar{\partial}$-equation with mixed boundary conditions and the work of Ling [24], Siu [30]–[31] on the generalization of the Grauert direct image theorem will be crucial to us. The interaction of the deformation of CR manifolds and the deformation of isolated normal singularities, which in some work is also referred to as the Kuranishi program, has attracted some attention in recent years. Related to this work, we would like to mention the long papers by Buchweitz-Millson [9] and Miyajima [25] and the references therein, to name a few. There has also been much work done on smooth families of CR manifolds, in conjunction with the embedding and related problems of three dimensional compact CR manifolds. Here, we refer the reader to the papers of Lempert [23] and Bland-Epstein [7], and the references therein.

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2. Simultaneous filling of a CR family

In this section, we first recall some definitions and notations. Then we turn to the Hartogs-Rossi type of holomorphic filling of complex manifolds by applying the work of Kuranishi-Akahori-Webster ([21], [2], [37]) and Ling [24].

Let $M$ be a $(C^\infty)$ smooth manifold of real dimension $(2n - 1)$. A smooth real 1-form $\theta$ over $M$ is called a contact form if the $(2n - 1)$-form $\theta \wedge d\theta \wedge \cdots \wedge d\theta$ vanishes nowhere on $M$. The complexified contact bundle $CS$ is then the subbundle of $CTM$ annihilated by $\theta$. A complex structure $J$ is a base point preserving smooth bundle isomorphism of $CS$ with $J^2 = -\text{id}$. $T^{(1,0)}M$ is defined to be the eigenspace of $i$, which is apparently a subbundle of $CS$ and $T^{(0,1)}M$ is defined to be the complex conjugate of $T^{(1,0)}M$. $J$ is also required to be integrable in the sense that the space of cross sections of $T^{(1,0)}M$ is closed under the Lie bracket operation. When $M$ is a real analytic manifold, we say the CR structure $J$ is real analytic if $T^{(1,0)}M$ is locally generated by real analytic complex vector fields. In this case, we also say $M$ is a real analytic CR manifold. Unless mentioned explicitly, all CR manifolds in this paper are assumed to $C^\infty$-smooth. The Levi form $\mathcal{L}$ associated to $\theta$ is a Hermitian form over $T^{(1,0)}M$ such that for any two cross sections, $L_1, L_2$, $\mathcal{L}(L_1, L_2) = i(\theta, [L_1, L_2])$. For an integrable $J$, $M$ equipped with
$J$ is called a pseudoconvex CR manifold if the Levi form defined above is semi-definite and is called strongly pseudoconvex CR manifold if the Levi-form is definite. When $M$ is part of the smooth boundary of a complex manifold $V$, then we adapt the standard meaning for the notion that $V$ lies on the pseudoconvex side when the CR structure inherited from $V$ is pseudoconvex.

Let $N \subset M$ be a smooth submanifold. If for any $p \in N$, $CS_p \cap CT_p N = J|_{CT_p N \cap S_p}(CS_p \cap CT_p N)$ and has complex codimension 1 in $CT_p N$, then $J$ naturally induces a CR structure (of hypersurface type) on $N$. $N$ equipped with such a CR structure is called a CR submanifold of $M$. An important class of functions over $M$ is the class of CR functions which is annihilated by any $(0,1)$-vector field along $M$. A map from $M$ into $\mathbb{C}^k$ is called a CR map if each of its components is a CR function.

A real hypersurface in $\mathbb{C}^{n+1}$ is strongly pseudoconvex with the inherited complex structure from its ambient space if it can be defined by a strongly plurisubharmonic function. A famous theorem of Kuranishi-Akahori-Webster states that any strongly pseudoconvex smooth CR manifold of real dimension $2n+1 \geq 7$ can be locally embedded as a real hypersurface in $\mathbb{C}^{n+1}$ through a CR diffeomorphism. More recently, in a very deep paper of Catlin [10, Theorem 1.1], one sees that any pseudoconvex manifold $X$ with at least three positive Levi eigenvalues can be realized as the smooth pseudoconvex boundary of some complex manifold $Z$. In the case that we are considering, the construction of $Z$ directly follows from the embedding theorem of Kuranishi-Akahori-Webster, which we will explain in details as follows. In the rest of this paper, all strongly pseudoconvex CR manifolds are assumed to be connected.

Notice that when a strongly pseudoconvex manifold $M$ is part of the smooth boundary of a certain complex manifold $V$, then there is a Lewy-type extension phenomenon for CR functions. Here we state the following one which can be easily proved by using the Baouendi-Treves approximation theorem and the so-called analytic disk argument (see [4]).

**Lemma 2.1.** Let $V$ be a domain in $\mathbb{C}^n$ with $M$ as part of its smooth strongly pseudoconvex boundary. For any subdomain $M' \subset M$ of $M$, there is a subdomain $V' \subset V$ such that any CR function over $M'$ can be holomorphically extended to $V'$. Here $V'$ is assumed to have $M'$ as part of its smooth boundary.

Let $(X, \pi, \Delta)$ be a CR deformation of the strongly pseudoconvex manifold $M_0 = \pi^{-1}(0)$ of dimension $2n - 1 \geq 5$, as defined in Definition 1.1. Since the total space $X$ has real dimension at least 7, by the above mentioned Kuranishi-Akahori-Webster’s embedding theorem, for each $p \in M_0$, there is a neighborhood $U_p$ of $p$ in $X$ and a CR diffeomorphism
\[ F_p : U_p \to \mathbf{C}^{n+1} \text{ such that } U^*_p = F_p(U_p) \text{ is a strongly pseudoconvex real hypersurface in } \mathbf{C}^{n+1}. \]

Let \( 0 < \epsilon < 1 \) be fixed. There is a finite set of such open pieces \( U_j \) of \( X \) which covers \( X_\epsilon := \pi^{-1}(\Delta_\epsilon) \). Each of them can be assumed to be connected. Next, we choose a finer finite cover \( \{V_j\} \) of \( X_\epsilon \), for which we can find another finite set of connected open subsets \( \{B_j\} \) of \( X \) with the following properties:

(a) \( V_j \subset B_j \);

(b) for each \( j \), there is a certain index \( L(j) \) such that \( B_k \subset U_{L(j)} \) for any \( k \) for which there is an \( l \) with \( B_l \cap B_j \neq \emptyset \) and \( B_k \cap B_l \neq \emptyset \).

Now, for each \( V_j \), let \( F_j \) be a CR diffeomorphism from \( U_{L(j)} \) to a certain strongly pseudoconvex hypersurface \( U^*_{L(j)} \subset \mathbf{C}^{n+1} \). Let \( D^*_j \) be a domain in the pseudoconvex side of \( U^*_{L(j)} \) with \( V^*_{L(j)} \) as part of its smooth boundary, and define \( \delta_j = \sup \{ \text{dist}(z, V^*_{L(j)}) : z \in D^*_j \} \), where we write \( B^*_{L(j)} = F_j(B_j) \) and \( V^*_{L(j)} = F_j(V_j) \). We assume that \( \delta_j \ll 1 \) so that for each \( j, k \), \( F_{jk} := F_j \circ F_k^{-1} \) extends holomorphically to \( D^*_k \) by Lemma 2.1; whenever there is an \( l \) with \( B_j \cap B_l \neq \emptyset \) and \( B_k \cap B_l \neq \emptyset \).

Now let \( Z^* \) be the disjoint union of the finite set \( \{V^*_{L(j)} \cup D^*_j\}_j \). We say that \( p \in V^*_{L(j)} \cup D^*_j \) and \( q \in V^*_{L(k)} \cup D^*_k \) are equivalent if (a): \( B_j \cap B_k \neq \emptyset \) and (b): \( p = F_{jk}(q) \). By the following Lemma 2.2, one sees that this equivalence relation is well-defined when \( \delta = \max \{\delta_j\} \) is sufficiently small. Hence we obtain the quotient space, which is denoted by \( \Omega_\epsilon \).

**Lemma 2.2.** Suppose \( \delta \ll 1 \). Then the above mentioned equivalence relation is well-defined. Moreover, the quotient space \( \Omega_\epsilon \) carries an integrable complex structure which has a smooth piece of its boundary CR-diffeomorphic to a neighborhood of \( X_\epsilon \) in \( X \).

**Proof of Lemma 2.2.** Let \( l_1, l_2, l_3 \) be such that \( B_{l_1} \cap B_{l_2} \neq \emptyset \), \( B_{l_2} \cap B_{l_3} \neq \emptyset \), but \( B_{l_1} \cap B_{l_3} = \emptyset \). We first claim that when \( \delta \) is sufficiently small, there are no points \( p \in D^*_{l_1} \) and \( q \in D^*_{l_3} \) such that \( p = F_{l_1, l_3}(q) \). Indeed, suppose not. There would be a sequence \( p_j \to p \in \overline{V}^{\mathbf{C}}_{l_1} \) and \( q_j \to q \in \overline{V}^{\mathbf{C}}_{l_3} \) such that \( p_j = F_{l_1, l_3}(q_j) \). Passing to the limit, it thus follows that \( F_{l_1}^{-1}(p) = F_{l_3}^{-1}(q) \in B_{l_1} \cap B_{l_3} \). This is a contradiction. From this claim, Lemma 2.1, as well as the simple fact that \( F_{l_1, l_3} = F_{l_1, l_2} \circ F_{l_2, l_3} \), it is easy to see that the equivalence relation is well-defined when \( \delta \ll 1 \).

Now, we assume \( \delta \) is sufficiently small so that the above claim holds. Denote the equivalence class of \( D^*_{l_1} \) by \( \overline{D}^*_{l_1} \). Then what we just obtained shows that \( \overline{D}^*_{l_1} \cap \overline{D}^*_{l_2} = \emptyset \) when \( B_{l_1} \cap B_{l_2} = \emptyset \). When \( B_{l_1} \cap B_{l_2} \neq \emptyset \), a similar argument shows that for

\[ \delta \ll 1, \quad \overline{D}^*_{l_1} \cap \overline{D}^*_{l_2} = \overline{D}^*_{l_1} \cap \overline{F}_{l_1, l_2}(D^*_{l_2}). \]
Now, we assign the topology to $\Omega_\epsilon$ so that $\widetilde{D}_{l_1}^*\epsilon$ is homeomorphic to $D_{l_1}^*$ with the inherited topology from $\mathbb{C}^{n+1}$. Then $\Omega_\epsilon$ can be easily seen to be a Hausdorff space. Indeed, for $p \in D_{l_1}^*$ and $q \in D_{l_2}^*$ with $\widetilde{p} = [p] \neq \widetilde{q} = [q]$, $\widetilde{p}$ and $\widetilde{q}$ are clearly separated by open subsets when $\widetilde{D}_{l_1}^* \cap \widetilde{D}_{l_2}^* = \emptyset$ or when $p \in F_{l_1,l_2}(D_{l_2}^*)$. In case $p \not\in F_{l_1,l_2}(D_{l_2}^*)$, let $U_q$ be a small open neighborhood of $q$ in $\widetilde{D}_{l_2}^*$ with $U_q \subset \subset F_{l_1,l_2}(\widetilde{D}_{l_2}^*)$; then $\widetilde{p}$ and $\widetilde{q}$ are separated by $\widetilde{U}_q$ and $\widetilde{D}_{l_2}^* \setminus \widetilde{U}_q$. Moreover, we can see that $\Omega_\epsilon$ with the local charts $\{\widetilde{D}_{l}^*\}$ is a complex manifold with holomorphic transition functions $F_{jk}$. This completes the proof of Lemma 2.2. q.e.d.

Clearly, the complex manifold $\Omega_\epsilon$ discussed above has a piece of smooth boundary which is diffeomorphic to a neighborhood of $X_\epsilon$ in $X$ by the way it was constructed. Next, by shrinking $\delta$ thus $\Omega_\epsilon$ if necessary, we can assume that $\Omega_\epsilon$ has a topological boundary, which can be decomposed into three pieces $Y_0$, $Y_1$ and $Y_2$, where $Y_0$ is CR diffeomorphic to $X_\epsilon$. Moreover, Lemma 2.1 can be used to see that $\pi$ can be extended to a holomorphic submersion $\hat{\pi}$ from $\Omega_\epsilon$ to $\Delta_\epsilon$. Also there is a strongly plurisubharmonic function $\rho' \in C^\infty(\overline{\Omega_\epsilon})$ over $\overline{\Omega_\epsilon}$ such that

(a) $Y_1 = \pi^{-1}(|t| = \epsilon)$,
(b) $\rho'|_{Y_0} = 0$, $d\rho'|_{Y_0} \neq 0$, $\rho' < 0$ over $\overline{\Omega_\epsilon} \setminus Y_0$, and
(c) $\rho'|_{Y_2} = -\epsilon_2$, with $\epsilon_2$ a sufficiently small positive constant.

Also write the naturally defined CR embedding from $Y_0$ to $X_\epsilon$ as $\Psi$. Then $\pi \circ \Psi = \hat{\pi}$. Hence, $(\Omega_\epsilon, \hat{\pi}, \Delta_\epsilon)$ is a $(1, 1)$ convex-concave complex space as defined in [24].

Notice that $\Omega_\epsilon$ is 2-normal when $n \geq 3$. Namely, for any complex analytic variety of dimension at most two $E \subset \Omega_\epsilon$, any holomorphic function in $\Omega_\epsilon \setminus E$ extends holomorphically to $\Omega_\epsilon$.

By the work of Siu [30] and Theorem (I)$_n$ in [24], $\Omega_\epsilon$ can be completed to a 2-normal Stein space $(\widetilde{X}_\epsilon, \widetilde{\pi}, \Delta_\epsilon)$. By the uniqueness part of the Ling theorem mentioned above, we can patch all those $(\widetilde{X}_\epsilon, \widetilde{\pi}, \Delta_\epsilon)$ into the required completion $(\widetilde{X}, \widetilde{\pi}, \Delta)$, which has Properties (I)–(III) as described in the Main Theorem if we identify in an obvious way $X_\epsilon$ with $Y_0$ defined above. Here, we only mention that the Steinness of $\widetilde{X}$ follows from the fact that $\widetilde{X}_{\epsilon_1}$ is a holomorphically convex subset of $\widetilde{X}_{\epsilon_2}$ for any $\epsilon_2 > \epsilon_1$. (See [32, §3].)

In what follows, we call such a 2-normal completion the Siu-Ling completion of the family $(X, \pi, \Delta)$.

Further assume that $X$ is a real analytic CR manifold. Then we can similarly define $D_j^{*0} = D_j^* \cup D_j^{*-} \cup V_{L(j)}$ such that $F_{jk}$ extends holomorphically to $D_j^{*0}$, by the reflection principle for strongly pseudoconvex hypersurfaces. (See [4], for instances.) Here $D_j^{*-}$ is a certain domain in
the pseudoconcave side of $U_{L(j)}^*$, which has $V_{L(j)}^*$ as part of its smooth boundary. We can then define $\delta_0^j = \sup\{\text{dist}(z, V_{L(j)}^*) : z \in D_j^{*0}\}$ and define the equivalence relation over the space $\cup D_j^{*0}$ by identifying points through $F_{jk}$. Then when $\delta_0^j \ll 1$, the quotient space $\Omega^0$ we get is also a complex manifold whose induced CR structure over $X_\epsilon$ coincides with the original one. (We also say $X_\epsilon$ is CR embedded into $\Omega^0$ as a CR submanifold). With the same discussions presented above, we conclude that there exists a complex space $\hat{Z}_\epsilon$, containing $\hat{X}_\epsilon$, such that

(i) $\hat{X}_\epsilon' \subset \subset \hat{Z}_\epsilon$ for any $\epsilon' < \epsilon$; and
(ii) the singular set of $\hat{Z}_\epsilon$ is contained in the singular set of $\hat{X}_\epsilon$.

**Remark 2.3.** We mention that if one uses [10, Theorem 1.1], one can conclude the existence of the aforementioned $\hat{Z}_\epsilon$ even when $X$ is purely smooth. (See already [11, Theorem 6].) Following the argument in the proof of [10, Theorem 1.3], we here indicate how the main result of Catlin [10, Theorem 1.1] can be used to construct $\hat{Z}_\epsilon$: One first extends $\hat{X}$ into $Z$ with the same dimension as that of $\Omega_\epsilon$, which is smooth away from the singular set $\text{Sing}(\hat{X})$ of $\hat{X}$ and has $X_\epsilon$ in its interior for $\epsilon < 1$. Let $\theta$ be the contact form which makes $X$ strongly pseudoconvex and let $T$ be a real vector field along $X$ such that $\langle \theta, T \rangle \equiv 1$. Notice that $X$ then has $n$-negative Levi eigenvalues with $n \geq 3$ with respect to $-\theta$.

Now, by [10, Theorem 1.1], one can find an integrable complex structure over $\Omega_\epsilon$ (after making $\delta \ll 1$), whose (complex conjugate) reflection to the other side of $X_\epsilon$ in $Z$ extends smoothly to $X_\epsilon$ for $\epsilon < 1$ and induces the same CR structure over $X_\epsilon$. Moreover, it is related with the original complex structure over $\Omega_\epsilon$ so that the formal uniqueness result in [10, Theorem 4.2] can be applied (with the map $G$ there side-preserving). Now, following the same argument as in the proof of [10, Theorem 1.3], one sees the existence of the aforementioned $\hat{Z}_\epsilon$ by modifying the complex structure in the pseudoconcave side of $X$.

For any CR family of strongly pseudoconvex family $(X, \pi, \Delta)$ which will appear in the rest of the paper, we always assume that $X$ is smooth with its Siu-Ling completion $(\hat{X}_\epsilon, \hat{\pi}, \Delta_\epsilon)$, for a certain $0 < \epsilon < 1$, being contained in a larger complex space $\hat{Z}_\epsilon$ as described above.

### 3. $\overline{\partial}$-equation on a lunar domain and extension of CR functions

We now let $(X, \pi, \Delta)$ be a smooth family of compact strongly pseudoconvex CR manifolds with the dimension of each fiber at least 5. We now proceed to the study of the simultaneous embedding problem of the CR family. For this, we need to study the solutions of a certain
\( \bar{\partial} \)-equation with good boundary behavior on \( \hat{X}_\epsilon \) constructed in §2. However, the non-smooth feature of \( \hat{X}_\epsilon \) makes a direct approach difficult. What we will do here is to remove a neighborhood of the singular set of \( \hat{X}_\epsilon \) so that we need only to work on a smooth manifold. But, this also brings the problem arising from its boundary. To deal with this, we use the work of Catlin [10] for solving the \( \bar{\partial} \)-equation with mixed boundary conditions.

As in §2, we first construct a smooth domain \( \Omega_\epsilon \subset \hat{X}_\epsilon \), which has three pieces of smooth boundaries, \( Y_0, Y_1, Y_2 \), that intersect CR-transversally at their intersections. Namely, they satisfy the following properties:

(a) \( Y_0 = X_\epsilon \), and thus \( Y_0 \) is strongly pseudoconvex with respect to \( \Omega_\epsilon \),
(b) \( Y_1 = \hat{\pi}^{-1}\{(|t| = \epsilon)\} \), and
(c) \( Y_2 = \{p \in \hat{X}_\epsilon \cap \hat{\pi}^{-1}(\Delta_{\epsilon_0}) : \rho^0 = -\epsilon_2 \} \) with \( \epsilon_2 \ll \epsilon \).

We use the notation set up before. For instance, we will write \( \hat{M}_\epsilon = \hat{\pi}^{-1}(t) \). Without loss of generality, we will assume, in this section, that \( \hat{X}_{\epsilon'} \subset \subset \hat{\mathcal{Z}}_{\epsilon_0} \) for any \( \epsilon' < \epsilon_0 \) with \( \epsilon_0 = 3/4 \). Here, as above, \( (\hat{\mathcal{Z}}_{\epsilon_0} \subset \hat{\mathcal{Z}}_{\epsilon_0}) \) is a complex space with the same singular set as that for \( \hat{X}_{\epsilon_0} \). We also fix \( \epsilon = 1/2 \) in this section. For simplicity, we write \( \rho \) for the \( \rho^0 \) constructed at the end of §2, which is defined over \( \overline{\hat{X}_{\epsilon_0}} \).

The main step is to prove the following extension theorem:

**Theorem 3.1.** Let \( \phi \) be a holomorphic function over \( \hat{M}_0 \), which is smooth up to its boundary \( M_0 \). Then \( \phi \) admits an extension that is holomorphic over \( \hat{X}_\epsilon \) and is smooth up to the strongly pseudoconvex manifold \( X_\epsilon \).

**Proof of Theorem 3.1.** Let \( \{U_j\}_{j=0}^m \) be a finite (open) covering of the compact space \( \overline{\hat{X}_\epsilon} \) and let \( \{\chi_j\} \) be a partition of unity with \( \text{Supp} \chi_j \subset \subset U_j \) for each \( j \). Here we let each \( U_j \) be a connected open subset of \( \overline{\hat{X}_{\epsilon_0}} \) such that \( U_j \cap \hat{X}_\epsilon \) is Stein for each \( j \). Make \( U_0 = \overline{\hat{X}_{\epsilon_0}} \setminus \{p(\approx X_\epsilon) \in \overline{\hat{X}_\epsilon} : \rho(p) \geq -\delta_0 \} \) with \( 0 < \delta_0 \ll 1 \) and \( \chi_0 \equiv 1 \) in a Stein neighborhood of the singular set of \( \hat{X}_{\epsilon_0} \). Assume that \( U_j \) for \( j \neq 0 \) does not cut the singularities of \( \hat{X}_\epsilon \). Without loss of generality, we can also assume that \( \phi \) admits a holomorphic extension \( \phi_j \) to \( U_j \cap \hat{X}_\epsilon \) when \( j \neq 0 \) and \( U_j \cap \hat{M}_0 \neq \emptyset \). Moreover, we can further assume that \( \phi_j \in C^\infty(U_j) \), when \( j \neq 0 \) and \( U_j \cap \hat{M}_0 \neq \emptyset \). (For instance, see [6]).

We let \( \phi_j \equiv 0 \) when \( j \neq 0 \) and \( U_j \cap \hat{M}_0 = \emptyset \).

Notice that by Cartan’s Theorem A and B, \( \phi \) extends to a holomorphic function \( \phi_0 \) in \( U_0 \). For \( p \in \overline{\hat{X}_\epsilon} \), write \( t(p) := \hat{\pi}(p) \). Choose \( \chi^*(t) = \chi^*(|t|) \) such that it is identically one for \( |t| \ll 1 \) and zero for
\[ |t| > \frac{1}{2}. \] Consider the following closed \((0, 1)\) form
\[
\omega(p) = \bar{\partial} \left( \chi^*(t) \frac{\sum \chi_j(p) \phi_j(p)}{\pi(p)} \right).
\]

Then, it can be easily verified that \(\omega\) is smooth over \(\overline{X_\epsilon} \setminus \text{Sing}(\hat{X}_\epsilon)\) and at smooth points in a small Stein neighborhood of \(\text{Sing}(\hat{X}_\epsilon)\). We denote \(\omega = \bar{\partial}(\chi^* \phi_0)\). Notice that \(\omega\) is compactly supported along the \(t\)-direction. For convenience of the reader, we say a few words on the smoothness of \(\omega\) along \(M_0\). The other cases can be done similarly.

Let \(p_0 \in M_0 \cap U_j\) for a certain \(j\). Let \(\{z_j\}_{j=1}^{n+1}\) be a set of holomorphic functions over \(U_j \cap \hat{X}_\epsilon\), that are smooth up to a certain small neighborhood of \(p_0\) in \(X\) and satisfy the condition: \(dz_1 \wedge \cdots \wedge dz_{n+1} \neq 0\) at \(p_0\). Assume that \(z_j(p_0) = 0\) for each \(j\). Since \(dt|_{p_0} \neq 0\), we can assume, without loss of generality, that \(z_1 = t\). Then the map \(\Psi = (z_1, \ldots, z_{n+1})\) diffeomorphically maps a small neighborhood of \(p_0\) in \(\overline{X}_\epsilon\) to a certain \(D_j\) in \(\mathbb{C}^{n+1}\). Certainly \(\Psi\) is holomorphic in the interior of \(\overline{X}_\epsilon\) and CR up to the boundary. For each \(l\) with \(p_0 \in U_l\), write the formal power series expansion of \(\phi_l \circ \Psi^{-1}\) at 0 as \(\sum_{k_1, \ldots, k_{n+1} \geq 0} a^l_{k_1, \ldots, k_{n+1}} z_1^{k_1} \cdots z_{n+1}^{k_{n+1}}\). Since \(\phi_l\) is the extension of \(\phi\), we see that \(a^l_{0k_2, \ldots, k_{n+1}}\) are independent of \(l\). Now, still write \(\omega\) for its push-forward form through \(\Psi\). Then at 0, we have the following formal expansion for \(\omega\):
\[
\omega = \sum_{l : p_0 \in U_l} \left( \bar{\partial} \chi_l \sum_{k_1, \ldots, k_{n+1} \geq 0, k_1 \geq 1} a^l_{k_1, \ldots, k_{n+1}} z_1^{k_1-1} \cdots z_{n+1}^{k_{n+1}} \right).
\]

Similarly, we have a formal Taylor series expansion for \(\omega\) at any nearby point of \(p_0\) in \(\overline{X}_\epsilon\). From this, the smoothness of \(\omega\) at \(p_0\) follows.

We can always find a smooth function \(u\) over \(\hat{X}_\epsilon\) which solves the equation \(\bar{\partial}(u) = \omega\) over \(\hat{X}_\epsilon \setminus \text{Sing}(\hat{X}_\epsilon)\) as follows: Take a Stein refinement \(\{V_l\}\) of \(\{U_j \cap \hat{X}_\epsilon\}\) such that \(V_0\) contains the singular set of \(\hat{X}_\epsilon\) with \(\omega = \bar{\partial} \chi^* \cdot \frac{\phi_0}{\pi}\) over \(V_0\) and any other does not cut the singularity of \(\hat{X}_\epsilon\). Notice that on \(V_j\) for \(j \neq 0\), there is a smooth solution \(u_j\) to the equation \(\bar{\partial} u_j = \omega\). Let \(u_0 = (\chi^* - 1) \frac{\phi_0}{\pi}\), which is smooth over \(V_0\). Then it is clear that \(u_j - u_l\) is holomorphic over \(V_j \cap V_l\). Since \(\hat{X}_\epsilon\) is Stein, we have \(h_k \in \text{Hol}(V_k)\) such that \(u_j - u_l = h_j - h_l\) over \(V_j \cap V_l\). Hence \(u := u_j - h_j\), which is smooth over \(\hat{X}_\epsilon\), solves the equation \(\bar{\partial} u = \omega\).

The solution produced from above may not have good behavior near \(X_\epsilon\). If we can find a solution \(u^*\) which is also smooth up to \(X_\epsilon\), then
\[
(3.1) \quad \phi^* := \chi^*(t) \sum \chi_j(p) \phi_j(p) - tu^*
\]
is holomorphic over \(\hat{X}_\epsilon\) and smooth up \(X_\epsilon\). Moreover, \(\phi^* \equiv \phi\) over \(\hat{M}_0\).
Indeed, if we can find \( u^* \) that is continuous over \( \hat{X}_\epsilon \cup X_\epsilon \) and whose restriction to \( X_\epsilon \) is smooth, then the restriction of \( \phi^* \) to \( X_\epsilon \) is a smooth CR function. Hence, by the strong pseudoconvexity of \( X \), it follows easily that the \( \phi^* \) in (3.1) must be smooth over \( \hat{X}_\epsilon \cup X_\epsilon \). Hence, the proof of Theorem 3.1 will be complete, if we can prove the following:

**Proposition 3.2.** Let \( \omega \) be as above. Then \( \overline{\partial} u = \omega \) has a solution \( u \) that is continuous up to \( X_\epsilon \) and whose restriction to \( X_\epsilon \) is smooth.

**Proof of Proposition 3.2.** As above, let \( \epsilon_0 = 3/4 \) and assume that the Stein space \( \hat{Z}_{\epsilon_0} \) has precisely the same singular set as that of \( \hat{X}_{\epsilon_0} \). Smoothly extend the \( \rho \)-function in the end of §2 to \( \hat{Z}_{\epsilon_0} \). Also smoothly extend \( t(p)(= \hat{\pi}(p)) \) to \( \hat{Z}_{\epsilon_0} \). Notice that for \( p \notin X_\epsilon \approx X_\epsilon \), \( \rho(p) > 0 \).

Define

\[
\Omega^- = \left\{ p(\approx X_\epsilon) \in \hat{Z}_{\epsilon_0} \setminus \hat{X}_{\epsilon_0} : \pi(p) \in \Delta, \; \sigma^4 > r(p) > 0, \; r(p) = \frac{\rho}{(\epsilon^2 - |t|^2)^{1/2}} \right\},
\]

where \( 0 < \sigma << 1 \). Then \( \Omega^- \) is a lunar domain which has the boundary component \( Y^- := \{ r(p) = \sigma^4 \} \) strongly pseudoconvex and the boundary component \( X_\epsilon \) strongly pseudoconcave with at least \( 3 \)-negative Levi eigenvalues.

Next we let \( \omega \) be as in Proposition 3.2 and we extend it smoothly to \( \Omega^0 := \Omega^- \cup \Omega_\epsilon \cup X_\epsilon \), where \( \Omega_\epsilon \) is as defined in §2 and \( \Omega_\epsilon = \hat{\pi}^{-1}(\{ |t| < \epsilon \}) \cap \Omega_\epsilon \). Still write \( \omega \) for its smooth extension to \( \Omega^0 \). Then \( \overline{\partial} \hat{\omega} \) vanishes to infinite order along \( X_\epsilon \). As mentioned before, we can assume that \( \omega \equiv 0 \) when \( |t| \) is sufficiently close to \( \epsilon \). Consider the \( \overline{\partial} \)-equation

\[
\overline{\partial} \omega = \overline{\partial} \omega
\]

over \( \Omega^- \) with the \( \overline{\partial} \)-Neumann boundary condition along \( Y^- \) and the Dirichlet boundary condition along \( X_\epsilon \).

More precisely, let \( M \) be a real hypersurface defined by \( r_0 = 0 \) in a complex manifold (or complex space) of dimension \( n \geq 2 \) with \( p \in M \).

Let \( \{ L_j \}_{j=1}^n \) be a smooth basis of the cross sections of \( T^{(1,0)}U_p \), where \( U_p \) is a small neighborhood of \( p \) in the ambient space. Let \( \{ \omega_j \} \) be its dual frame. Assume that \( L_j(r_0) \equiv 0 \) when restricted to \( M \) for \( j \neq n \).

For a \((0,q)\)-form

\[
A = \sum_{i_1 < i_2 < \ldots < i_q} a_{i_1 \ldots i_q} \overline{w_{i_1}} \wedge \cdots \wedge \overline{w_{i_q}}
\]

defined in a certain side of \( U_p \cap M \), that is continuous up to \( M \). We say \( A \) satisfies the \( \overline{\partial} \)-Neumann condition along \( M \) if \( a_I |_M = 0 \) whenever \( I = (i_1, \ldots, i_q) \) with \( i_q = n \). We say that \( A \) satisfies the Dirichlet condition along \( M \) if \( a_I |_M = 0 \) when \( i_q \neq n \).
Return to the domain $\Omega^-$. After making $\sigma \ll 1$ we can always find a globally defined $(1,0)$-type smooth vector field $L_{n+1}$ over $\Omega^-\rho$ such that $L_{n+1}(\rho) \equiv 1$. Fix a smooth Hermitian metric $\langle \cdot , \cdot \rangle_0$ over $\Omega^-\sigma$ and define a weighted metric $\langle \cdot , \cdot \rangle$ over $\Omega^-\sigma$ such that the following holds:

(i) For any $(1,0)$-type vectors $L_1, L_2 \in T^{(1,0)}\Omega^-\sigma$ with $L_1(r)(p) = L_2(r)(p) = 0$,

$$\langle L_1, L_2 \rangle = \sigma^{-1}(e^2 - |t|^2)^{-4}\langle L_1, L_2 \rangle_0;$$

(ii) $\langle L_{n+1}, L_{n+1} \rangle = \sigma^{-2}(e^2 - |t|^2)^{-8}\langle L_{n+1}, L_{n+1} \rangle_0$; and $\langle L_{n+1}, L \rangle = 0$ for any $L \in T^{(1,0)}\Omega^-\sigma$ with $L(r)(p) = 0$.

Following Catlin in [10], we write $\mathcal{E}_c^k(\Omega^-\sigma)$ for the collection of smooth $(0,k)$-forms over $\Omega^-\sigma$ that vanish when $|t|$ is sufficiently close to $\epsilon$. Write $\mathcal{B}_k^+(\Omega^-\sigma)$ for the subset of $\mathcal{E}_c^k(\Omega^-\sigma)$, whose elements satisfy the Dirichlet boundary condition along $X_\sigma$. Write $\mathcal{B}_k^-(\Omega^-\sigma)$ for the subset of $\mathcal{E}_c^k(\Omega^-\sigma)$, whose elements satisfy the $\bar{\partial}$-Neumann boundary condition along $Y^-\sigma$.

Define $\mathcal{B}_k^0(\Omega^-\sigma) := \mathcal{B}_k^-(\Omega^-\sigma) \cap \mathcal{B}_k^+(\Omega^-\sigma)$.

We define the formal adjoint $\bar{\partial}_k^*$ of the $\bar{\partial}$-operator acting on the $(0,k)$-form as in the standard way. We say that $U \in L^2_{q-1}(\Omega^-\sigma)$ is in the domain of the operator $\bar{\partial}_q^\text{mix}$, or $U \in \text{Dom}(\bar{\partial}_q^\text{mix})$, with $\bar{\partial}_q^\text{mix}(U) = F$ if for any $V \in \mathcal{B}_q^-(\Omega^-\sigma)$, we have $(U, \bar{\partial}_q^fV) = (F, V)$. We write $\bar{\partial}_q^\text{mix}*$ for the Hilbert space adjoint of $\bar{\partial}_q^\text{mix}$ by using the norm induced from the inner product defined above.

Then

$$\text{Dom}(\bar{\partial}_k^\text{mix}) \cap \mathcal{E}_c^k(\Omega^-\sigma) = \mathcal{B}_k^+(\Omega^-\sigma),$$

$$\text{Dom}(\bar{\partial}_k^\text{mix}) \cap \mathcal{E}_c^k(\Omega^-\sigma) = \mathcal{B}_k^-(\Omega^-\sigma),$$

and

$$\text{Dom}(\bar{\partial}_k^\text{mix}) \cap \text{Dom}(\bar{\partial}_k^\text{mix}) = \mathcal{B}_k(\Omega^-\sigma).$$

(See [10].) For $U, V \in \text{Dom}(\bar{\partial}_k^\text{mix}) \cap \text{Dom}(\bar{\partial}_k^\text{mix})$, define

$$Q_k(U, V) = \langle \bar{\partial}_k^\text{mix}(U), \bar{\partial}_k^\text{mix}(V) \rangle + \langle \bar{\partial}_k^\text{mix}(U), \bar{\partial}_k^\text{mix}(V) \rangle,$$

where the inner product on forms is induced from the above defined Hermitian metric on vectors. Then the following basic estimate is contained in the work of Catlin ([10]) (see [10, Theorem 7.1]):

**Theorem 3.3** (Catlin [10]). When $\sigma$ is sufficiently small, one has, for a certain constant $C$, that $Q_2(U, U) \geq C\|U\|^2$ for any $(0,2)$-form $U \in \text{Dom}(\bar{\partial}_2^\text{mix}) \cap \text{Dom}(\bar{\partial}_2^\text{mix})$.

Hence, by the standard Hilbert space theory argument as in [13], Theorem 3.3 shows that for $\omega$ introduced above, there is a unique $\alpha_2$ in
the domain of $Q_2$ such that

$$Q_2(\alpha_2, U) = (\bar{\partial}(\omega), U)$$

for any $U$ in the domain of $Q_2$.

By the sub-elliptic estimate established in [10, Theorem 9.2, Lemma 10.1], one concludes that $\alpha_2$ is smooth over $\Omega^-_\sigma \cup X_\epsilon \cup Y^-_\sigma$ with $\bar{\partial}^{\text{mix}}_2 \alpha_2 \in \text{Dom}(\bar{\partial}^{\text{mix}}_3)$ and $\bar{\partial}^{\text{mix}}_2(\alpha_2) \in \text{Dom}(\bar{\partial}^{\text{mix}}_1)$. Also, from [10, Theorems 10.3, 10.5], it follows that $\bar{\partial}^{\text{mix}}_3 \bar{\partial}^{\text{mix}}_2 \alpha_2 = 0$. Hence, write $\beta = \bar{\partial}^{\text{mix}}_2 \alpha_2$.

We thus obtain $\bar{\partial}^{\text{mix}}_1(\beta) = \bar{\partial}^{\text{mix}}_1(\beta) = \bar{\partial}(\omega)$ over $\Omega^-_\sigma$, where $\bar{\partial}_1$ is the regular $\bar{\partial}$-operator acting on $(0, 1)$-forms. Notice that $\beta$ satisfies the Dirichlet boundary condition along $X_\epsilon$.

Next, define $\tilde{\beta}_0(p)$ to be $-\beta(p)$, for $p \in \Omega^-_\sigma$; and to be $0$ for $p \in \hat{X}_\epsilon$. Define $\beta = \tilde{\beta}_0 + \omega$. $\beta$ is a (locally) $L^2$-integrable $(0, 1)$-form over $\hat{\Omega} \setminus \text{Sing}(\hat{X}_\epsilon)$ with $\hat{\Omega} := \Omega^-_\sigma \cup \hat{X}_\epsilon \cup X_\epsilon$. We also claim that $\bar{\partial}(\beta) = 0$ in the sense of distribution (over $\hat{\Omega} \setminus \text{Sing}(\hat{X}_\epsilon)$). Indeed, we need only to verify that for each $p \in X_\epsilon$ and a small neighborhood $U_p$ of $p \in \Omega^-_\sigma$, $\langle \beta, \bar{\partial}^{\text{mix}}_2 \chi \rangle = 0$ for any smooth $(0, 2)$-form $\chi$ compactly supported in $U_p$.

For this, we can assume without loss of generality that $U_p$ is an open subset in $\mathbb{C}^{n+1}$. Also assume that $\{L_j\}$ is a smooth orthonormal basis of $(0, 1)$-vector fields over $U_p$ with $L_j$ tangent to $X_\epsilon$ for $j \neq n + 1$. Also, we write $\{\omega_j\}$ for its dual basis. Write $\beta_0 = \sum_{j=1}^{n+1} b_j \omega_j$. Notice that $b_j \in C^\infty(U_p \setminus \hat{X}_\epsilon)$ and $b_j(p) = 0$ for $p \in \hat{X}_\epsilon$. By the Dirichlet condition of $\beta$ along $X_\epsilon$, $b_j = 0$ along $X_\epsilon$ for $j \neq n + 1$. Clearly, to prove the above statement, it suffices for us to verify that the distribution $\bar{\partial}(\beta)$ in $U_p$ coincides with the (locally) $L^2$-integrable function $\bar{\partial}^{\text{mix}}_0$, which is $\bar{\partial}(-\beta)$ for $p \in U_p \setminus \hat{X}_\epsilon$ and is 0 otherwise. Write $\chi = \sum_{j<l} \chi_{jl} \omega_j \wedge \overline{\omega_l}$ with $\chi_{jl} \in C^\infty_0(U_p)$. Then a direct verification shows that

$$\bar{\partial}^{\text{mix}}_1(\chi) = \sum_{j<l} L_l(\chi_{jl}) \overline{\omega_j} - \sum_{j<l} L_j(\chi_{jl}) \overline{\omega_l} + \sum_j K_j(\chi) \overline{\omega_j},$$

where $K$ only linearly involves the zeroth order terms in $\chi_{jl}$.

Hence,

$$\langle \tilde{\beta}_0, \bar{\partial}^{\text{mix}}_j(\chi) \rangle := -\sum_{j<l} \int \overline{b_l L_j(\chi_{jl})} + \sum_{j<l} \int \overline{b_j L_l(\chi_{jl})} - \sum_j \int b_j K_j(\chi).$$

When $l \neq n + 1$, we have $b_j, b_l = 0$ along $X_\epsilon$ and thus

$$\int \overline{b_l L_j(\chi_{jl})} = \int_{U_p \setminus \hat{X}_\epsilon} \overline{L_j^*(b_l) \chi_{jl}}, \quad \int \overline{b_j L_l(\chi_{jl})} = \int_{U_p \setminus \hat{X}_\epsilon} \overline{L_l^*(b_j) \chi_{jl}},$$

where $L_j^*$ and $L_l^*$ are the $\bar{\partial}$-adjoints of $L_j$ and $L_l$.
where $L^*_j, L^*_l$ are the formal adjoint of $L_j$ and $L_l$, respectively. When $l = n + 1$, since $j < n + 1$ and $L_j(\rho) = 0$ along $X_\epsilon$, we see also that in the integrals \( \int b_j \overline{L}_l(\chi_{jl}) \) and \( \int b_l \overline{L}_j(\chi_{jl}) \), there are no boundary integral terms after integrating by parts. Therefore, the distribution $\overline{\partial} \beta$ coincides with the (locally) $L^2$-integrable function defined above.

Finally, we consider the following $\overline{\partial}$-equation

\[
\overline{\partial} u = \beta \text{ over } \tilde{\Omega} \setminus \text{Sing} (\tilde{X}_\epsilon),
\]

with $u$ smooth in a neighborhood of $\text{Sing} (\tilde{X}_\epsilon)$.

Since $\tilde{\Omega}$ is also Stein, the same argument at the beginning of this section together with Hörmander's $L^2$-estimates for the $\overline{\partial}$-equation ([18]) shows that it has a solution $u^0$, that is in the $L^2_{\text{loc}}(\tilde{\Omega} \setminus \text{Sing}(\tilde{X}_\epsilon))$ space and is smooth in a neighborhood of the singular set of $\tilde{X}_\epsilon$ in $\tilde{\Omega}$. (The solution must also be smooth away from $X_\epsilon$.) Notice that $\overline{\partial} u^0 = \omega$ over $\tilde{X}_\epsilon \setminus \text{Sing}(\tilde{X}_\epsilon)$ and

\[
\phi^* := \chi^*(t) \sum \chi_j(p) \phi_j(p) - tu^0
\]

gives a holomorphic extension of $\phi$ to $\tilde{X}_\epsilon$. As remarked right before Proposition 3.2, the proof of Proposition 3.2 will be complete if we can prove the following:

**Lemma 3.4.** Let $u^0$ be as above. Then $u^0 \in C^0(\tilde{X}_\epsilon \cup X_\epsilon)$ and $u^0|_{X_\epsilon} \in C^\infty(X_\epsilon)$.

**Proof of Lemma 3.4.** By the construction of $\tilde{\beta}$, it suffices to prove that for any $p \in X_\epsilon$, there is a small neighborhood $U_p$ of $p$ in $\tilde{\Omega}$ such that $u^0 \in C^0(U_p \cap \overline{X}_\epsilon) \cap C^\infty(X_\epsilon)$. Since the problem under study is purely local, without loss of generality, we can assume that $U_p$ is the Euclidean ball $B_{n+1}(2) := \{ z \in \mathbb{C}^{n+1} : |z| < 2 \}$ and $p = 0$. Notice that $u^0$ is in the Sobolev $H^1(B_{n+1}(2))$-space. By the Bochner-Martinelli formula, we have the following:

\[
u(z) = \frac{1}{W(n+1)} \int_{|\xi| = 1} \frac{u^0(\xi) \eta(\xi - z) \wedge \hat{\omega}(\xi)}{|\xi - z|^{2(n+1)}} - \frac{1}{W(n+1)} \int_{|\xi| < 1} \frac{\tilde{\beta}(\xi) \wedge \eta(\xi - z) \wedge \hat{\omega}(\xi)}{|\xi - z|^{2(n+1)}},
\]

where $W(n+1)$ is a constant depending only on $n + 1$, $\hat{\omega}(z) = dz_1 \wedge \cdots \wedge dz_{n+1}$ and

\[
\eta(z) = \sum_{j=1}^{n+1} (-1)^{j+1} z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \cdots \wedge dz_{n+1}.
\]
Clearly, the first integral is $C^\infty$ for $|z| < 1$. We need only to explain that
\[
\int_{|\xi|<1} \frac{\tilde{\beta}(\xi)\eta(\xi - z)}{|\xi - z|^{2(n+1)}} \wedge \hat{\omega}(\xi)
\]
defines a continuous function over $B_{n+1}(1) \cap \overline{X_e}$ whose restriction to $B_{n+1}(1) \cap X_e$ is smooth.

From the way $\tilde{\beta}$ was constructed and after a smooth change of coordinates, it then suffices to prove the following fact:

**Fact.** Assume that $h(x)$ is a function defined over $\mathbb{R}^n$ with compact support. Suppose that $h$ is $C^\infty$-smooth for $x_n < 0$, and extends smoothly up to $x_n \leq 0$. Also suppose that $h$ is $C^\infty$-smooth for $x_n > 0$ and extends smoothly up to $x_n \geq 0$ from the upper half-space. Let
\[
J_h(x) = \int_{\mathbb{R}^n} \frac{h(\xi)(\xi_1 - x_1)}{|x - \xi|^n} d(Vol)(\xi).
\]
Then $J_h(x)$ is continuous over $\{x_n \leq 0\}$ and the boundary value of $J_h$ to the hyperplane defined by $\{x_n = 0\}$ from $\{x_n < 0\}$ is $C^\infty$-smooth.

Indeed, use the polar coordinates $(r, \tau)$ centered at $x \in \mathbb{R}^n$. Here $\tau = (\tau_1, \ldots, \tau_{n-1}, \tau_n)$ is in the unit sphere in $\mathbb{R}^n$, $r$ is the distance from $\xi$ to $x$ and $\xi - x = r \tau$. Write $dS(\tau)$ for the volume element of the unit sphere. Then we have
\[
J_h(x) = \int_0^\infty dr \int_{|\tau|=1} \frac{h(x + r \tau) r \tau_1 r^{n-1}}{r^n} dS(\tau) = \int_0^\infty dr \int_{|\tau|=1} h(x + r \tau) r \tau_1 dS(\tau).
\]
Hence, it follows easily that $J_h(x)$ is $C^\infty$-smooth if $h$ is smooth over $\mathbb{R}^n$. Also, under the assumptions in the Fact, it immediately implies that $J_h$ is smooth at any point with $x_n \neq 0$. Now, extend the function $h$ on the lower half space to an element $h \in C^\infty_0(\mathbb{R}^n)$. Considering $J_{h \tilde{x}_n}$ instead of $J_h$, we can assume without loss of generality that $h(x) = 0$ for $x = (x', x_n)$ with $x_n < 0$. Also, we can assume $x_n \approx 0$.

Next, for $x_n < 0$ with $-x_n \leq r$, write $\theta(x_n, r) \in [0, \pi/2]$ with $r \cos(\theta(x_n, r)) = -x_n$. Use the spherical coordinates
\[
\tau_n = \cos \theta_{n-1}, \quad \tau_{n-1} = \cos \theta_{n-2} \sin \theta_{n-1}, \ldots,
\]
\[
\tau_2 = \cos \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1}, \quad \tau_1 = \sin \theta_{n-1} \cdots \sin \theta_1,
\]
with $\theta_1 \in [0, 2\pi]$, $\theta_2, \ldots, \theta_{n-1} \in [0, \pi]$. Notice that $h(x + r \tau) = 0$ when $\theta_{n-1} \notin [0, \theta(x_n, r)]$ or when $r < -x_n$. Hence, we can easily see the following expression for $J_h$:
\[
J_h = \int_{-x_n}^\infty dr \int_0^{\theta(x_n, r)} G(\theta_{n-1}, r, x) d\theta_{n-1}.
\]
Here \( G(\theta_{n-1}, r, x) \) is computed by the iterated integral with respect to \( \theta_1, \ldots, \theta_{n-2} \) in the procedure of applying the Fubini theorem to the multiple integral \( \int_{|\tau|=1} h(x + r \tau) \tau_1 dS(\tau) \). Clearly, we can view \( G(\theta_{n-1}, r, x) \) as a smooth function in \( (\theta_{n-1}, r, x') \) with parameter \( x_n \) for 
\[
r \geq -x_n, |x'| \leq 1, \theta_{n-1} \in [0, \theta(x_n, r)].
\]
As \( x_n \to 0^- \), \( G(\theta_{n-1}, r, x) \) is uniformly bounded and approaches (uniformly on compacta) to a function that is smooth over the region given by 
\[
\theta_{n-1} \in [0, \pi/2], r \geq 0, |x'| \leq 1.
\]
Notice that the limit is \( G(\theta_{n-1}, r, (x_1, \ldots, x_{n-1}, 0)) \). Also \( G(\theta_{n-1}, r, x) = 0 \) when \( r \geq r_0 \gg 1 \). Therefore, we see that 
\[
\lim_{x_n < 0, x_n \to 0} J_h(x_1, \ldots, x_{n-1}, x_n)
\]
\[
= \int_0^{r_0} dr \int_0^{\pi/2} G(\theta_{n-1}, r, (x_1, \ldots, x_{n-1}, 0)) d\theta_{n-1}
\]
\[
= J_h(x', 0).
\]
Thus, we see that \( J_h \) is continuous on \( \{x_n \leq 0\} \) and has boundary value smooth over \( \{x_n = 0\} \). The proof of the Fact is complete. The proof of Lemma 3.4 and thus the proof of Proposition 3.2 are complete, too. This then finally completes the proof of Theorem 3.1.\[\text{q.e.d.}\]

Completion of the Proof of the Main Theorem. With Theorem 3.1 at our disposal, the proof of the remaining statements in the Main Theorem can be easily achieved: Let \( f \) be the smooth CR embedding of \( M_0 \) into \( \C^m \) as in Main Theorem (IV). By Theorem 3.1, we can find a holomorphic extension \( F \) of \( f \) to \( \widehat{X}_{\epsilon'} \) (\( |\epsilon'| \ll 1 \)) with \( F \) smooth up to \( X_{\epsilon'} \). Then the map \( T = (F, \pi) \) embeds both \( X_{\epsilon} \) and \( \widehat{X}_{\epsilon} \) into \( \C^m \times \Delta_\epsilon \) for \( |\epsilon| \ll \epsilon' \). Notice that \( T \) must be proper from \( \widehat{X}_{\epsilon} \) into \( \C^m \times \Delta_\epsilon \setminus T(X_{\epsilon'}) \), by the assumption. Write \( \widehat{X}_{\epsilon} = T(X_{\epsilon'}) \). We conclude easily that \( X_{\epsilon} \) must be a Stein space with the properties stated in (IV1)–(IV3). The proof of the Main Theorem is complete.\[\text{q.e.d.}\]

Remark 3.5. Fix a distance function \( \text{dist} \) over \( \overline{X}_{\epsilon} \). Fix certain \( C^k \)-norms \( \| \cdot \|_k \) over \( M_0 \) (\( k = 1, 2, \ldots \)). Let \( \Phi \) be an extension of \( \phi \) as constructed in the proof of Theorem 3.1. From our proof of Theorem 3.1, \( \Phi \) can be written as \( \Phi_1 + t\Phi_2 \), (see (3.1)), with \( \Phi_1 \) a certain smooth extension of \( \phi \) to \( \overline{X}_{\epsilon} \) and \( \Phi_2 \) a certain correction function from solving the \( \overline{\partial} \)-equation. We can make use of the estimates in [6] to handle \( \Phi_1 \) and those in [10] to handle \( \Phi_2 \) over \( \Omega_{\epsilon} \). Meanwhile, we can apply Siu’s version of Cartan’s Theorem A and B with bounds to handle the bounds for solutions from solving the Cousin problem [30, §9]. One can then conclude the following statement: For any \( p_1 \in M_{t_1}, p_2 \in M_{t_2}, \delta > 0,\)
4. Extension of holomorphic functions and simultaneous embeddings

Let $(X, \pi, \Delta)$ be the strongly pseudoconvex CR family and let $(\hat{X}, \hat{\pi}, \Delta)$ be the Siu-Ling completion as in the Main Theorem. Still write $M_t := \pi^{-1}(t)$ for the connected strongly pseudoconvex manifold of dimension at least 5. In this section, we discuss the question when a smooth CR function defined over $M_0$ can be extended holomorphically to $\hat{M}_0 := \hat{\pi}^{-1}(0)$. This can then be applied with the Main Theorem to study the simultaneous embedding and blowing-down problems. We first briefly recall the definition of the Kohn-Rossi cohomology group.

Let $M$ be a strongly pseudoconvex CR manifold of real dimension $2n - 1$ with contact form $\theta$ and holomorphic complex tangent bundle $T^{(1,0)}M$. Assign the Hermitian metric in $M$ be the Siu-Ling completion as in the Main Theorem. Still write $C_T$ be the Levi-form defined there (see §2). Let $T$ be the Reeb vector field associated with $\theta$ in the sense that $\langle \theta, T \rangle = 1$ and the contraction of $d\theta$ along $T$ is zero. For each $p \in M$, let $\{L_j(p)\}_{j=1}^{n-1}$ be an orthonormal basis of $T_p^{(1,0)}M$ with dual frame $\{\omega_j(p)\}$. $(\langle \omega_j, T \rangle = 0$ and $\omega_j$ annihilates any vector of type $(0,1))$. Then $\{\omega_1, \ldots, \omega_{n-1}, \overline{\omega_1}, \ldots, \overline{\omega}_{n-1}, \theta\}$ forms a basis of $CT^*M$ and any $k$-form $\alpha$ at $p$ has a unique representation:

$$\alpha = \sum a_{j_1, \ldots, j_l, i_1, \ldots, i_q} \overline{\omega_{j_1}}(p) \wedge \cdots \wedge \overline{\omega_{j_l}}(p) \wedge \omega_{i_1}(p) \wedge \cdots \wedge \omega_{i_q}(p) + \sum b_{j_1, \ldots, j_l, i_1, \ldots, i_q} \theta \wedge \omega_{j_1}(p) \wedge \cdots \wedge \omega_{j_l}(p) \wedge \overline{\omega_{i_1}}(p) \wedge \cdots \wedge \overline{\omega_{i_q}}(p),$$

where in the first summation, $j_1 < \cdots < j_l, i_1 < \cdots < i_q, l + q = k$ and in the second summation, $j_1 < \cdots < j_l, i_1 < \cdots < i_q, l + q + 1 = k$.

A form at $p$ is called of type $(0,k)$ if it can be expressed as

$$\sum_{i_1 < \cdots < i_q} a_{00, i_1 \ldots i_q} \omega_{i_1}(p) \wedge \cdots \wedge \omega_{i_q}(p).$$

Namely, in the above representation,

$$b_{j_1, \ldots, j_l, i_1, \ldots, i_q}(p) = 0,$$

$$a_{j_1, \ldots, j_l, i_1, \ldots, i_q}(p) = a_{j_1, \ldots, j_l, i_1, \ldots, i_q}(p).$$

Let $\pi_{(0,q)}$ be the projection from the space of $q$-forms to the space of $(0,q)$-forms $\Lambda^{(0,q)}$ over $M$. Then we define $\overline{\partial}_b = \pi_{(0,q)} \circ d_{q-1}$, where $d_{q-1}$ is the regular De Rham differential operator at the degree $(q - 1)$. The Kohn-Rossi cohomology group $H_{KR}^{(0,q)}(M)$ of order $(0,q)$ is defined

$C^* > 0$, there exist an $\varepsilon'(\delta, C^*) > 0$, depending only on $\delta$ and $C^*$, and a certain fixed positive integer $k_0$, such that for any $|t_1|, |t_2| < \frac{1}{2} \varepsilon$, $\|\phi\|_{k_0} < C^*$ with $\phi$ as described in Theorem 3.1, when $\operatorname{dist}(p_1, p_2) < \varepsilon'(\delta, C^*)$, one has $|\Phi(p_1) - \Phi(p_2)| < \delta$ for a certain holomorphic extension $\Phi$ of $\phi$ with properties described in Theorem 3.1.

4. Extension of holomorphic functions and simultaneous embeddings

Let $(X, \pi, \Delta)$ be the strongly pseudoconvex CR family and let $(\hat{X}, \hat{\pi}, \Delta)$ be the Siu-Ling completion as in the Main Theorem. Still write $M_t := \pi^{-1}(t)$ for the connected strongly pseudoconvex manifold of dimension at least 5. In this section, we discuss the question when a smooth CR function defined over $M_0$ can be extended holomorphically to $\hat{M}_0 := \hat{\pi}^{-1}(0)$. This can then be applied with the Main Theorem to study the simultaneous embedding and blowing-down problems. We first briefly recall the definition of the Kohn-Rossi cohomology group.

Let $M$ be a strongly pseudoconvex CR manifold of real dimension $2n - 1$ with contact form $\theta$ and holomorphic complex tangent bundle $T^{(1,0)}M$. Assign the Hermitian metric in $T_p^{(1,0)}M$ for each $p \in M$ to be the Levi-form defined there (see §2). Let $T$ be the Reeb vector field associated with $\theta$ in the sense that $\langle \theta, T \rangle = 1$ and the contraction of $d\theta$ along $T$ is zero. For each $p \in M$, let $\{L_j(p)\}_{j=1}^{n-1}$ be an orthonormal basis of $T_p^{(1,0)}M$ with dual frame $\{\omega_j(p)\}$. $(\langle \omega_j, T \rangle = 0$ and $\omega_j$ annihilates any vector of type $(0,1))$. Then $\{\omega_1, \ldots, \omega_{n-1}, \overline{\omega_1}, \ldots, \overline{\omega}_{n-1}, \theta\}$ forms a basis of $CT^*M$ and any $k$-form $\alpha$ at $p$ has a unique representation:

$$\alpha = \sum a_{j_1, \ldots, j_l, i_1, \ldots, i_q} \overline{\omega_{j_1}}(p) \wedge \cdots \wedge \overline{\omega_{j_l}}(p) \wedge \omega_{i_1}(p) \wedge \cdots \wedge \omega_{i_q}(p) + \sum b_{j_1, \ldots, j_l, i_1, \ldots, i_q} \theta \wedge \omega_{j_1}(p) \wedge \cdots \wedge \omega_{j_l}(p) \wedge \overline{\omega_{i_1}}(p) \wedge \cdots \wedge \overline{\omega_{i_q}}(p),$$

where in the first summation, $j_1 < \cdots < j_l, i_1 < \cdots < i_q, l + q = k$ and in the second summation, $j_1 < \cdots < j_l, i_1 < \cdots < i_q, l + q + 1 = k$.

A form at $p$ is called of type $(0,k)$ if it can be expressed as

$$\sum_{i_1 < \cdots < i_q} a_{00, i_1 \ldots i_q} \omega_{i_1}(p) \wedge \cdots \wedge \omega_{i_q}(p).$$

Namely, in the above representation,

$$b_{j_1, \ldots, j_l, i_1, \ldots, i_q}(p) = 0,$$

$$a_{j_1, \ldots, j_l, i_1, \ldots, i_q}(p) = a_{j_1, \ldots, j_l, i_1, \ldots, i_q}(p).$$

Let $\pi_{(0,q)}$ be the projection from the space of $q$-forms to the space of $(0,q)$-forms $\Lambda^{(0,q)}$ over $M$. Then we define $\overline{\partial}_b = \pi_{(0,q)} \circ d_{q-1}$, where $d_{q-1}$ is the regular De Rham differential operator at the degree $(q - 1)$. The Kohn-Rossi cohomology group $H_{KR}^{(0,q)}(M)$ of order $(0,q)$ is defined
as the quotient of the space of $\bar{\partial}_b$-closed $(0,q)$-forms with the space of all $\bar{\partial}_b$-exact $(0,q)$-forms. Our definition of the Kohn-Rossi cohomology group $H^{(0,q)}_{KR}(M)$ is taken from Tanaka [35] and is isomorphic to the intrinsic definition given by [13] in the strongly pseudoconvex case. It is well known that $H^{(0,q)}_{KR}(M)$ is a pure CR invariant, independent of the choice of the contact form $\theta$. (See [35], [13] or the following Theorem 4.1 (4.II) and [38].)

Now, suppose that $M$ is a connected compact strongly pseudoconvex CR manifold of real dimension at least 5. Suppose $M$ bounds a complex space $\hat{M}$, that has $M$ as its smooth boundary. Let $\rho_0$ be a smooth function defined over $\hat{M} \cup M$ such that $\rho_0 < 0$ in $\hat{M}$, $\rho_0 = 0$ along $M$, and $d\rho_0 \neq 0$ along $M$. Moreover, we assume that $\rho_0$ is strongly plurisubharmonic in a small neighborhood of $M$ in $\hat{M} \cup M$. Let $\text{codh}_x(\hat{M}) := \text{codh}_{\mathcal{O}_x(\hat{M})}(\mathcal{O}_x(\hat{M}))$ be the homological codimension of $\hat{M}$ at $x \in \hat{M}$. Then the following statements are well-known.

**Theorem 4.1.** With the above notation, we have

(4.I) $\text{codh}_x(\hat{M}) \geq 3$ for any $x \in \hat{M}$, if and only if $\hat{M}$ is normal and $H^{(0,1)}_{KR}(M) = 0$.

(4.II) Let $\hat{M}_\epsilon := \{x \in \hat{M}, 0 > \rho_0(x) > -\epsilon\}$. Assume that $0 < \epsilon \ll 1$. Then $H^{(0,1)}_{KR}(M)$ is isomorphic to $H^1(\hat{M}_\epsilon, \mathcal{O})$.

(4.III) ([13, Corollary 3.3.5]) Any smooth CR function defined over $M$ extends holomorphically to $\hat{M}$ if either $\hat{M}$ is smooth or only has isolated normal singularities.

**Proof.** (4.I) and (4.II) follow from the arguments in [38] as follows.

Let $\{y_1, \ldots, y_m\}$ be the set of singular points of $\hat{M}$. It is well known that $\text{codh}_x(\hat{M}) \geq 3$ for any $x \in \hat{M}$ is equivalent to $H^k_{\{y_i\}}(\hat{M}, \mathcal{O}) = 0$ for $0 \leq k < 3$ and $0 \leq i \leq m$ (cf. Theorem 3.3 of [33]). In particular, $y_1, \ldots, y_m$ are normal singularities. On the other hand, $\dim H^{(0,1)}_{KR}(M) = \sum_{i=1}^m \dim H^2_{\{y_i\}}(\hat{M}, \mathcal{O})$ by Theorem B of [38]. Hence (4.I) is proved.

Let $\hat{M}_{\text{res}}$ be a resolution of singularity of $\hat{M}$. By definition, $H^i_{\infty}(\hat{M}, \mathcal{O})$ is the $i$-th cohomology of the quotient complex

$$C^\infty(\hat{M}_{\text{res}}, \Lambda^{0,*})/C^\infty_c(\hat{M}_{\text{res}}, \Lambda^{0,*}).$$

Here $C^\infty(\hat{M}_{\text{res}}, \Lambda^{0,*})$ is the $C^\infty$-Dolbeault complex, and $C^\infty_c(\hat{M}_{\text{res}}, \Lambda^{0,*})$ is the subcomplex of smooth compactly supported $(0,*)$-forms. Then by Laufer [22], $\lim_{\epsilon \to 0} H^i(\hat{M}_\epsilon, \mathcal{O}) \cong H^i_{\infty}(\hat{M}, \mathcal{O})$. On the other hand, by Andreotti and Grauert (Théorème 15 of [1]), $H^i(\hat{M} - \{y_1, \ldots, y_m\}, \mathcal{O}) \cong$
Proof of Theorem 4.2. Let \((X, \pi, \Delta)\) be the strongly pseudoconvex CR family as in the Main Theorem. Let \((\tilde{X}_\epsilon, \tilde{\pi}, \Delta_\epsilon)\) be the Siu-Ling completion of \((X_\epsilon, \pi, \Delta_\epsilon)\) with \(\epsilon_0 < 1\). Assume that \(\text{dim}_R M_0 \geq 5\) and \(t\) is not a zero divisor of the germ of the first direct image sheaf \(\mathcal{G}_1\) at \(t = 0\). Let \(\phi\) be a smooth CR function over \(M_0 = \pi^{-1}(0)\). Then it admits an extension that is holomorphic over \(\tilde{X}_\epsilon\) and smooth up to the strongly pseudoconvex manifold \(X_\epsilon\), where \(0 < \epsilon < \epsilon_0\).

Proof of Theorem 4.2. By Theorem 3.1, to prove Theorem 4.2, it suffices to explain the \(\phi\) defined above admits a holomorphic extension to \(\tilde{M}_0\).

Let \(0 < \eta_1 < \epsilon_0\) be such that \(H^1(\Omega_{\eta_1} \cap \tilde{\pi}^{-1}(\Delta_{\eta_1}), \mathcal{O})\) has a finite set of generators \(\{\xi_j\}\), whose restrictions to \((\mathcal{G}_1)_x\) also generate \((\mathcal{G}_1)_x\) for any \(x \in \Delta_{\eta_1}\). Pick an \(\eta_2 < \eta_1\) and a Stein open covering \(\{V_j\}\) of \(\tilde{\pi}^{-1}(\Delta_{\eta_2}) \cap \Omega_{\eta_2}\) such that \(\phi\) has a holomorphic extension \(\phi_j\) to each \(V_j\). Define \(\phi_{jl} = \frac{\phi_j - \phi_l}{t}\) over \(V_j \cap V_l\). Then \(E := \{\phi_{jl}\}\) is a closed 1-cochain and thus defines an element in \(H^1(\Omega_{\eta_2} \cap \tilde{\pi}^{-1}(\Delta_{\eta_2}), \mathcal{O})\). By our choices and after shrinking \(\eta_2\) if necessary, we have holomorphic functions \(a_j\) over \(\Delta_{\eta_2}\) such that \(E = \sum_j a_j(t) \xi_j\) in \(H^1(\Omega_{\eta_2} \cap \tilde{\pi}^{-1}(\Delta_{\eta_2}), \mathcal{O})\). (If we need to shrink \(\eta_2\), the new \(E\) is taken as the naturally restricted element and \(V_j's\) will be naturally restricted too. For simplicity, we do not use new notation).

Hence there is a holomorphic function \(\psi_j\) over \(V_j\) for each \(j\) such that 
\[
(E - \sum_j a_j(t) \xi_j) (V_j \cap V_l) = \psi_j - \psi_l.
\]
It thus follows that 
\[
\left(\sum_j t a_j(t) \xi_j\right) (V_j \cap V_l) = (-t \psi_j + \phi_j) - (-t \psi_l + \phi_l).
\]
Hence, $\sum_j t a_j(t) \xi_j = 0$. Since under the assumption of Theorem 4.2, $t$ is not a zero divisor of $(G_1)_0$, it follows that at the very beginning, we can already choose $a_j's$ to be $0$. Hence, we have that $(t \psi_j - \phi_j) = (t \psi_l - \phi_l)$ on $V_j \cap V_l$. Hence $\Phi := \phi_j - t \psi_j$ over $V_j$ for each $j$, well defines a holomorphic extension of $\phi$ to $\tilde{\pi}^{-1}(\Delta_{\eta_0}) \cap \Omega_{\eta_0}$. Now by the way $\hat{X}_{\eta_2}$ was constructed, one sees that $\Phi$ extends holomorphically to $\hat{X}_{\eta_2}$. (See [24, Proposition 4.3.4].) Applying Theorem 3.1, we see the proof of Theorem 4.2.

**Remark 4.3.** By a result of Siu, if $H^1(\Omega_{\epsilon_0} \cap \tilde{\pi}^{-1}(t), \mathcal{O})$ has a fixed dimension and $N = \dim \Omega_{\epsilon_0} \geq 5$ for each $t$, then $G_1$ is a locally free coherent sheaf. In particular, $t$ is not a zero divisor of $G_1$ (31, Theorem 2)). Combining this fact with the above mentioned Theorem 4.1 (4.II), we see that the hypothesis in Theorem 4.2 holds if $H^{[0,1]}_{KR}(M_t)$ has a constant dimension for $|t| < 1$ and $\dim_{\mathbb{R}}(M_t) \geq 7$.

As an immediate application of the Main Theorem and Theorem 4.2, we obtain:

**Corollary 4.4.** Let $(X := \cup_{|t| < 1} M_t, \pi, \Delta)$ be a CR family of compact strongly pseudoconvex CR manifolds. Assume that $\dim_{\mathbb{R}} M_0 \geq 5$ and $t$ is not a zero divisor of the germ of the first direct image sheaf $G_1$ defined above at $t = 0$. Suppose that $f_0$ is a smooth CR embedding from $M_0$ into $\mathbb{C}^m$. Then there is a smooth CR diffeomorphism $F = (\tilde{f}, \pi)$ from $X_\epsilon := \cup_{|t| < \epsilon} M_t$ into $\mathbb{C}^m \times \Delta_\epsilon$ with $\epsilon < 1$ such that $f|_{M_0} = \tilde{f}$.

**Remark 4.5.** Corollary 4.4, in particular, implies Corollary 1.6 when $\dim_{\mathbb{R}}(M_t) \geq 7$ by the above mentioned theorem of Siu.

Next, by applying Theorem 3.1 (Remark 3.5), Theorem 4.2 and Tanaka’s theorem, we will complete the proof of Corollary 1.6 when $\dim_{\mathbb{R}}(M_t) \geq 5$ as follows:

**Proof of Corollary 1.6.** Let $(X, \pi, \Delta)$ be as in Corollary 1.6 with $(\tilde{X}, \tilde{\pi}, \Delta)$ as its Siu-Ling completion. Fix $\epsilon_0 < 1$ and define $G_1 = \hat{\pi}^{-1}(\mathcal{O}(\Omega_{\epsilon_0}))$ as before. Since $N = \dim \hat{X} \geq 4$, by Ling’s theorem, $G_1$ is a coherent sheaf over $\Delta_{\epsilon_0}$. Without loss of generality, by what we did above, we can assume that $t - t_0$ is not a zero divisor of $(G_1)_{t = t_0}$ except at $t_0 = 0$ (see [34, 3.10]). Let $f_0$ be a CR diffeomorphism from $M_0$ into $\mathbb{C}^m$. Then, by the Tanaka theorem [35], there is a certain small $0 < \epsilon'_0 < \epsilon_0$ such that $f_0$ extends to a smooth family $\{f_t\}$ with $f_t$ a CR embedding from $M_t$ into $\mathbb{C}^m$ for $|t| < \epsilon'_0$. Now, by the assumption and making use of Theorem 4.2, $f_t$ extends holomorphically to $(\tilde{X}_{\epsilon_1}, \tilde{\pi}, \Delta_{\epsilon_1})$ for a certain fixed $\epsilon_1$ with $\epsilon_1 < 1$ for all $t$ with $0 < |t| < \epsilon_1$. Notice that $f_t$ is a smooth function over $X_{\epsilon_1}$. Write the holomorphic extension $f_t|_{M_t}$ to $(\tilde{X}_{\epsilon_1}, \tilde{\pi}, \Delta_{\epsilon_1})$ as $F^t$. By Remark 3.5, we can choose $F^t$ such
that \( \| F^t|_{M_0} - f_0 \|_{C^0(M_0)} \to 0 \) as \( t \to 0 \). By the maximum principle in complex spaces, \( \{ F^t|_{\tilde{M}_0} \} \) converges uniformly over \( \tilde{M}_0 \). By the normal family for holomorphic functions over complex spaces (see \cite[Theorem 8, p. 171]{15}), we conclude that \( F^t \to F^0 \), that is holomorphic over \( \tilde{M}_0 \) and has boundary value \( f_0 \) over \( M_0 \). Namely, we proved that \( f_0 \) extends holomorphically to \( \tilde{M}_0 \). The rest of the argument now follows easily from Theorem 3.1.

We call \( \tilde{M} \) a smooth strongly pseudoconvex complex manifold if \( \tilde{M} \) is a complex manifold with smooth boundary \( \partial \tilde{M} \), that is strongly pseudoconvex with respect to \( \tilde{M} \). Let \( \tilde{X} \) be a complex manifold with \( X \) as part of its strongly pseudoconvex boundary. We call \((\tilde{X}, \bar{\pi}, \Delta)\) a family of smooth strongly pseudoconvex complex manifolds if (I): \( \bar{\pi} \) is a surjective holomorphic map from \( \tilde{X} \) to \( \Delta \), which extends smoothly to \( X = \bigcup \partial \bar{\pi}^{-1}(t) = \bigcup \partial \tilde{X}_t \), where \( \tilde{X}_t = \bar{\pi}^{-1}(t) \); (II) \((X, \pi, \Delta)\) is a CR family of strongly pseudoconvex manifolds. Now, let \( f \) be a holomorphic map from \( X_0 := \bar{\pi}^{-1}(0) \) into \( \mathbb{C}^m \), that is biholomorphic near \( \partial \tilde{X}_0 \) and extends to a smooth CR diffeomorphism from the boundary to its image. Also, assume that \( f(\partial \tilde{X}_0) \) bounds a complex space, denoted by \( Y \), with at most isolated singularities and has \( f(\partial \tilde{X}_0) \) as its smooth boundary. We say \( \tilde{X}_0 \) resolves the singularities of \( Y \) through \( f \) when \( Y \) does have isolated singularities and \( f \) is proper from \( \tilde{X} \) to \( Y \). Notice that \( f \) then must be biholomorphic from \( \tilde{X}_0 \setminus E \) into \( Y \setminus \text{Sing}(Y) \), where \( \text{Sing}(Y) \) is the singular set of \( Y \) and \( E = f^{-1}(\text{Sing}(Y)) \) is the exceptional set of \( \tilde{X} \). In this setting, we also call \( f \) a blowing-down map from \( \tilde{X}_0 \) to its image. There have been many papers in the past on when a family of strongly pseudoconvex complex manifolds with exceptional sets can be simultaneously blown-down. (See the papers \cite{26}, \cite{27}, \cite{28}, \cite{22} and the references therein.) As an immediate application of Corollaries 1.6, 4.4, we have the following result, a certain local version of which was already proved by Riemenschneider by different methods in \cite{26}–\cite{27}.

**Corollary 4.6.** Let \((\tilde{X}, \bar{\pi}, \Delta)\) be a smooth holomorphic family of strongly pseudoconvex complex manifolds. Assume that \( X = \bigcup_{t \in \Delta} \partial \tilde{X}_t \) can be CR embedded into a complex manifold. Suppose that

\[
\dim H^{(0,1)}_{KR}(\partial \tilde{X}_t) = \text{constant}
\]

and \( \tilde{X}_0 \) is at least of complex dimension 3. Suppose that \( f_0 \) is a blowing-down map from \( \tilde{X}_0 \) to \( \mathbb{C}^m \). Then there is a map \( F = (\bar{f}, \bar{\pi}) \) from \( \tilde{X}_\epsilon := \bar{\pi}^{-1}(\Delta_\epsilon) \) to \( \mathbb{C}^m \times \mathbb{C} \), which extends smoothly over \( \bigcup_{|t| < \epsilon < 1} \partial \tilde{X}_t \) such that \( \bar{f}|_{\tilde{X}_t} \) is a (holomorphic) blowing-down map from \( \tilde{X}_t \) to \( \mathbb{C}^m \) with \( \bar{f}|_{\tilde{X}_0} = f_0 \).
Proof of Corollary 4.6. Applying Corollary 1.6 to $f_0$, we conclude that $f_0$ extends to a smooth CR diffeomorphism to a neighborhood of $\partial(\tilde{X}_0)$ in $\cup_{|t|<1} \partial \tilde{X}_t$. Then applying the Kohn-Rossi extension theorem (4.III), the CR diffeomorphism extends to the required map as in the corollary. q.e.d.

Finally, let $(X, \pi, \Delta)$ be as in Corollary 1.5 with $(\tilde{X}, \tilde{\pi}, \Delta)$ as its Siu-Ling completion. On the other hand, by the work of Rossi, Andreotti-Siu [3], for each $t \in \Delta$, the fiber $X_t$ itself admits a normal Stein filling $\tilde{X}_{t,\text{nor}}$ with $\text{codh}((\tilde{X}_{t,\text{nor}}) \geq 2$. Assume that $\text{codh}(\tilde{X}_{t,\text{nor}}) \geq 3$, which is equivalent to $H_{KR}^{(0,1)}(M_t) = 0$ by Theorem 4.1 (4.I). Fujiki then showed in [14] that $(\cup_{t \in \Delta} \tilde{X}_{t,\text{nor}}, \pi, \Delta)$ carries a normal Stein complex structure. In view of our Main Theorem, we see that the completion by Fujiki must then be isomorphic to the Siu-Ling completion of $(X, \pi, \Delta)$ by the uniqueness of both fillings.

Proof of Corollary 1.5. Indeed, we need only to show that $\tilde{M}_0$ is normal under the assumption of Corollary 1.5. By the theorems of Boutet de Monvel and Harvey-Lawson [8], [16], there is a CR diffeomorphism $f$ from $M_0$ to $M'_0 = f(M_0)$, that bounds a Stein space $V_0$ with only isolated normal singularities. Moreover $V_0$ is smooth near $M'_0$. From the proof in Corollary 1.6, we see that $f$ extends holomorphically to $\tilde{M}_0$. Let $g$ be the inverse of $f$ near $M_0$ in $\tilde{M}_0$. Since $\tilde{M}_0$ is also Stein, we see easily that $g$ extends to a holomorphic map, still denoted by $g$, from $V_0$ into $\tilde{M}_0$. Clearly, from the uniqueness property of holomorphic functions, $f$ and $g$ are the holomorphic inverse of each other. Hence, $\tilde{M}_0$ is biholomorphically equivalent to $V_0$ and thus must be normal, too. q.e.d.

Remark 4.7. From the proof of Corollary 1.6, it is clear that in Corollary 1.6, one can weaken the assumption: $\dim(H_{KR}^{(0,1)}(M_t)) = \text{const}$ for each $t$ by that of $\dim(H_{KR}^{(0,1)}(M_{\gamma(t)})) = \text{const}$ for a certain smooth function $\gamma(t)$ with $\gamma(0) = 0$ and $d(\gamma)(0) \neq 0$. Also, if the last condition holds, then $\tilde{M}_0$ in the Siu-Ling completion must be normal. However, this observation only makes sense when $M_t$ has real dimension 5, for otherwise, by a result of Siu [31, Theorem 1], the set $\{t \in \Delta : d_1(t) = \dim(H_{KR}^{(0,1)}(M_t)) = d_1(0)\}$ is either the whole $\Delta$ or has 0 as an isolated point. Also, one can similarly define the notion of the CR family of compact strongly pseudoconvex CR manifolds with the parameter space being polydisks in any complex space and establish the similar results in this setting.
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