ON STRONGLY ELLIPTIC SINGULARITIES

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1. Introduction. Let p be a normal singularity of the two-dimensional analytic space V. In [1], Artin introduced a definition for p to be rational. Rational singularities have also been studied by, for instance, DuVal [2], Tyurina [22], Laufer [12], and Lipmann [17]. In [23], Wagreich introduced a definition for p to be elliptic. Elliptic singularities have occurred naturally in papers by Grauert [3], Hirzebruch [7], Laufer [14], Orlik and Wagreich [18, 19], and Wagreich [24]. Karras [9, 10] and Saito [20] have studied some of these particular elliptic singularities. Choose V to be a Stein space with p as its only singularity. Let $\pi: M \to V$ be a resolution of V. It is known that $\dim H^1(M, \mathcal{O})$ is independent of resolution. Rational singularities are those singularities with $H^1(M, \mathcal{O}) = 0$.

Recently, Laufer [15] examined a class of elliptic singularities which satisfy a minimality condition. Let ${}_V\mathcal{O}_p$ be the germs at p of holomorphic functions on V. These minimally elliptic singularities are actually those singularities with $H^1(M, \mathcal{O}) = \mathbb{C}$ and ${}_V\mathcal{O}_p$ Gorenstein ring [15, Theorem 3.13]. Let $\pi: M \to V$ be the minimal resolution such that the irreducible components of $A = \pi^{-1}(p)$ are non-singular with normal crossings. Let Γ denote the associated dual graph (see e.g. [8] or [11]) including the genera of the irreducible components. In [13], a deformation theory preserving Γ was developed. This theory allows him to introduce the notion [15, Definition 4.1] of a property of the associated singularity holding generically for Γ. Let Z be the fundamental cycle. Let $\mathcal{O}(-Z)$ be the sheaf of germs of holomorphic functions on M whose divisors are at least Z. Let $\mathcal{O}_Z = \mathcal{O}/\mathcal{O}(-Z)$. Then $\chi(Z) = \dim H^0(M, \mathcal{O}_Z) - \dim H^1(M, \mathcal{O}_Z)$ may be computed from Γ via the Riemann-Roch theorem. Weak ellipticity is $\chi(Z) = 0$. If $\chi(Z) = 0$, then [15, Theorem 4.1] generically $H^1(M, \mathcal{O}) = \mathbb{C}$. So it is interesting to develop a theory for those singularities with $H^1(M, \mathcal{O}) = \mathbb{C}$.

In this paper, we complete the theory for those singularities with $H^1(M, \mathcal{O}) = \mathbb{C}$ and ${}_V\mathcal{O}_p$ non-Gorenstein. In Section 2, we first define Laufer sequence (Definition 2.1). We find that $\dim H^1(M, \mathcal{O}) \leqslant$ (length of Laufer sequence), i.e.,

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an analytic invariant is bounded by topological data. This Laufer sequence is completely determined by the topology of the singularity. In fact, the Laufer sequence may be computed from Γ explicitly via the intersection matrix. The reader should discover the difference between the Laufer sequence and the elliptic sequence which we introduced before [25]. In Sections 2 and 3, important invariants of singularities such as multiplicities and Hilbert functions are computed in terms of the Laufer sequence. Therefore, the analytic invariants are extracted from the topological information. In Section 4, we discuss the general lower bound for $m\mathcal{O}$, where m is the maximal ideal of ${}_V\mathcal{O}_p$.

In this paper, all the notation and terminology are standard; cf. [15], [25], and [26]. The basic knowledge necessary to read this paper is contained in [15] and [25].

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2. Calculation of Multiplicities. In this paper, Z will always denote the fundamental cycle and E will denote the minimally elliptic cycle [15, Definition 3.1].

Definition 2.1. Let A be the exceptional set of the minimal good resolution $\pi: M \to V$ of normal two-dimensional Stein space with p as its only weakly elliptic singularity. If $E \cdot Z < 0$, we say that the *Laufer sequence* is $\{Z\}$ and the *length of Laufer sequence* is equal to 1. Suppose $E \cdot Z = 0$. Let L_1 be the maximal connected subvariety of A such that $L_1 \supseteq |E|$ and $A_i \cdot Z = 0$ for all $A_i \subseteq L_1$. Since A is an exceptional set, L_1 is properly contained in A. Let Z_{L_1} be the fundamental cycle on L_1. Suppose $Z_{L_1} \cdot E = 0$. Let L_2 be the maximal connected subvariety of A such that $L_2 \supseteq |E|$ and $A_i \cdot (Z + Z_{L_1}) = 0$ for all $A_i \subseteq L_2$. Having defined L_{i-1}, let L_i be the maximal connected subvariety containing $|E|$ such that for all $A_j \subseteq L_i$, $A_j \cdot (Z_{L_0} + Z_{L_1} + \cdots + Z_{L_{i-1}}) \leqslant 0$ where $Z_{L_0} = Z$. Continuing this process, we finally obtain L_m with $(Z_{L_0} + \cdots + Z_{L_m}) \cdot E = Z_{L_m} \cdot E < 0$. (This will be justified in Proposition 2.2.) We call that $\{Z_{L_0}, Z_{L_1}, \ldots, Z_{L_m}\}$ is the *Laufer sequence* and the *length of Laufer sequence* is $m+1$.

PROPOSITION 2.2. *Use the notation of Definition* 2.1. *The Laufer sequence is well defined in the sense that the above process is stopped after a finite number of steps.*

Let $\{Z_{L_0}, Z_{L_1}, \dots, Z_{L_m}\}$ *be the Laufer sequence. Then* $\chi(\sum_{i=0}^h Z_{L_i})=0$, $0 \leqslant h \leqslant m$ *and* $(\sum_{i=0}^h Z_{L_i})\cdot A_j \leqslant 0$ *for all* $A_j \subseteq A$. *Moreover,* $\dim H^1(M, \mathcal{O}) \leqslant$ *(length of Laufer sequence)* $= m+1$.

Proof. Let Y be the minimal positive cycle $\geqslant Z$ such that $Y\cdot E<0$ and $Y\cdot A_i \leqslant 0$ for all $A_i \subseteq A$. Obviously, by the definition of Laufer sequence we have $(\sum_{i=0}^h Z_{L_i})\cdot A_j \leqslant 0$ for all $A_j \subseteq A$. If $\{Z_{L_0}, \dots, Z_{L_h}\}$ is a proper subset of the Laufer sequence, then $(\sum_{i=0}^h Z_{B_i})\cdot E=0$. Therefore $(\sum_{i=0}^h Z_{B_i}) \leqslant Y$ and the Laufer sequence is well defined. Moreover, it is easy to see that the summation of Laufer sequence is equal to Y:

$$\chi\left(\sum_{i=0}^{h} Z_{B_i}\right) = \chi\left(\sum_{i=0}^{h-1} Z_{B_i}\right) + \chi(Z_{L_k}) - \left(\sum_{i=0}^{h-1} Z_{L_i}\right)\cdot Z_{L_h}$$
$$= \chi\left(\sum_{i=0}^{h-1} Z_{L_i}\right)$$

By induction, we have $\chi(\sum_{i=0}^h Z_{L_i})=\chi(Z)=0$. The proof that $\dim H^1(M, \mathcal{O}) \leqslant m+1$ is quite similar to the proof of Theorem 3.9 of [25]. The details are left as an exercise to the reader. Q.E.D.

Definition 2.3. Let $\pi: M \to V$ be a resolution of a normal two-dimensional Stein space with p as its only singular point. Suppose $H^1(M, \mathcal{O}) = \mathbb{C}$. Then we call p a *strongly elliptic singularity*.

It is known that strong ellipticity implies weak ellipticity [15, Theorem 4.1].

THEOREM 2.4. *Let* $\pi: M \to V$ *be the minimal good resolution of a normal two-dimensional Stein space with p as its only strongly elliptic singularity. Let* $\{Z_{L_0}, Z_{L_1}, \dots, Z_{L_m}\}$ *be the Laufer sequence. Then* $m\mathcal{O} \subseteq \mathcal{O}(-\sum_{i=0}^m Z_{L_i})$. *If* $Z_{L_m}\cdot Z_E \leqslant -2$, *then* $m\mathcal{O} = \mathcal{O}(-\sum_{i=0}^m Z_{L_i})$ *provided that either one of the following holds:* (1) $Z_E = E$, *i.e.,* π *is the minimal resolution.* (2) $Z_{L_m}/|E| = Z_E$.

Proof. By [15, (2.6)] and $\chi(Z)=0$, we have $H^1(M, \mathcal{O}_Z) = \mathbb{C}$. The exact sequences

$$H^1\Big(M, \mathcal{O}_{\sum_{i=0}^{m} Z_{L_i}}\Big) \to H^1(M, \mathcal{O}_Z) = \mathbb{C} \to 0,$$
$$H^1(M, \mathcal{O}) = \mathbb{C} \to H^1\Big(M, \mathcal{O}_{\sum_{i=0}^{m} Z_{L_i}}\Big) \to 0$$

show that $H^1(M, \mathcal{O}_{\sum_{i=0}^{m} Z_{L_i}}) = \mathbb{C}$. As $\chi(\sum_{i=0}^m Z_{B_i})=0$, $H^0(M, \mathcal{O}_{\sum_{i=0}^{m} Z_{L_i}}) \cong \mathbb{C}$. Now look at the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0\to & H^0(M,\mathcal{O}(-Z)) & \to & H^0(M,\mathcal{O}) & \to & H^0(M,\mathcal{O}_Z)\simeq\mathbb{C} & \to 0\\ & \uparrow & & \uparrow & & \uparrow & \\ 0\to & H^0\left(M,\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)\right) & \to & H^0(M,\mathcal{O}) & \to & H^0\left(M,\mathcal{O}_{\sum_{i=0}^{m} Z_{L^i}}\right) & \to 0\end{array}$$

By the five lemma, $H^0(M,\mathcal{O}(-\sum_{i=0}^{m} Z_{L_i}))\to H^0(M,\mathcal{O}(-Z))$ is an isomorphism. Since $m\mathcal{O}\subseteq\mathcal{O}(-Z)$, it follows that $m\mathcal{O}\subseteq\mathcal{O}(-\Sigma_{i=0}^{m}Z_{L_i})$.

Suppose $Z_{L_m}\cdot Z_E\leqslant -2$, we want to prove $\mathcal{O}(-\Sigma_{i=0}^{m}Z_{L_i})=m\mathcal{O}$. There are two cases:

Case (i). *There exists* $A_i\subseteq|E|$ *such that* $E\cdot Z_{L_m}+1\leqslant A_i\cdot Z_{L_m}\leqslant -1$ *or* $E=A_i$ *is a non-singular elliptic curve.*

For any $A_1\subseteq|E|$, choose a computation sequence for Z with $A_{i_1}=A_1$, $E=Z_k$, and A_{i_k} such that $A_{i_k}\cdot Z_{L_m}<0$. Our hypothesis guarantees that such a computation sequence can be chosen. By Proposition 2.7 of [25], $H^1(M,\mathcal{O}(-\Sigma_{i=0}^{m}Z_{L_i}-A_1))=0$. So the map

$$H^0\left(M,\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}-A_1\right)\right)\to H^0\left[M,\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}\right]$$

is surjective. Given a point $a\in A_1$, let

$$\tilde{f}\in H^0\left[M,\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}-A_1\right)}\right]$$

be non-zero near a as a section of the line bundle $f\in H^0(M,\mathcal{O}(-\Sigma_{i=0}^{m}Z_{L_i}))$ projecting onto $\tilde{f}$ will generate $\mathcal{O}(-\Sigma_{i=0}^{m}Z_{L_i})$ near a, since it must vanish to the prescribed order on the A_1 near a and will have no other zeros near a.

For any $A_1\not\subseteq|E|$, choose a computation sequence of the following form: $Z_0=0$, $Z_1=A_{i_1}=A_1,\ldots,$ Z_r, $Z_{r+1},\ldots,$ $Z_{r+k}=Z_r+E,\ldots,$ $Z_{l}=Z$, where $|Z_r|\subseteq\overline{A\setminus|E|}$ and $Z_{r+1}-Z_r,\ldots,$ $Z_{r+k}-Z_r=E$ is part of a computation sequence for Z. Our hypothesis guarantees that the computation sequence for Z can be so chosen such that $A_{i_{r+k}}\cdot Z_{L_m}<0$. Consider the following exact sheaf sequence for $n \geqslant 0$:

$$0 \to \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_1\right) \to \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ\right) \to \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_1\right)} \to 0,$$

$$\vdots$$

$$0 \to \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_j\right) \to \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_{j-1}\right) \to \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_{j-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_j\right)} \to 0,$$

$$\vdots$$

$$0 \to \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_\eta\right) \to \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_{\eta-1}\right) \to \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_{\eta-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z\right)} \to 0. \tag{2.1}$$

By Proposition 2.2, $(\Sigma_{i=0}^{m} Z_{L_i}) \cdot A_j \leqslant 0$ for all $A_j \subseteq A$.

$$\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_{j-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_j\right)}$$

is the sheaf of germs of sections of a line bundle over A_{i_j} of Chern class $-A_{i_j} \cdot (\Sigma_{i=0}^{m} Z_{L_i} + nZ + Z_{j-1})$. If $\operatorname{supp} E$ has at least two irreducible components, we have $A_{i_{r+k}} \cdot (Z_{r+k-1}) = 2$ and $A_{i_j} \cdot Z_{j-1} = 1$ for $j \neq r+k$ by Proposition 2.5 of [25]. So $A_{i_j} \cdot (\Sigma_{i=0}^{m} Z_{L_i} + nZ + Z_{j-1}) \leqslant 1$ for all j and all n. Thus

$$H^1\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_{j-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_j\right)}\right] = 0,$$

and the maps $H^1(M, \mathcal{O}(-\Sigma_{i=0}^{m} Z_{L_i} - nZ - Z_j)) \to H^1(M, \mathcal{O}(-\Sigma_{i=0}^{m} Z_{L_i} - nZ - Z_{j-1}))$ in (2.1) are surjective. Comparing the maps, we see that

$$\rho : H^1\left(M, \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - nZ - Z_j\right)\right) \to H^1\left(M, \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_j\right)\right)$$

is surjective for all $n \geqslant 0$. For sufficiently large n, ρ is the 0 map by [3, §4, Satz

1, p. 355]. Hence $H^1(M, \mathcal{O}(-\Sigma_{i=0}^m Z_{L_i} - Z_j)) = 0$. If $\operatorname{supp} E = A_i$ is a non-singular elliptic curve, then $E \cdot Z_{L_m} \leqslant -2$. By Proposition 2.5 of [25], we have $A_{i_j} \cdot (\Sigma_{i=0}^m Z_{L_i} + nZ + Z_{j-1}) \leqslant 1$ for n and all $j \neq r+1$ and $A_{i_{r+1}} \cdot (\Sigma_{i=0}^m Z_{L_i} + nZ + Z_r) \leqslant -1$ for all n. It follows that

$$H^1\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - nZ - Z_{j-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - nZ - Z_j\right)}\right] = 0$$

for all j and n. A similar argument to the previous one will show that $H^1(M, \mathcal{O}(-\Sigma_{i=0}^m Z_{L_i} - Z_j)) = 0$. In particular, $H^1(M, \mathcal{O}(-\Sigma_{i=0}^m Z_{L_i} - A_1)) = 0$. We have proved

$$H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_1\right)}\right]$$

is surjective.

We remark that the argument in case (i) also handles the following case. There exists $A_i \subseteq |E|$ such that $A_i \cdot Z_{L_m} < 0$ and $e_i \geqslant 2$.

Case (ii). *$|E|$ has at least two irreducible components and there exists $A_i \subseteq |E|$ such that $e_i = 1$, $A_i \cdot Z_{L_m} < 0$ and $A_j \cdot Z_{L_m} = 0$ for all $A_j \subseteq |E|$ where $A_j \neq A_i$.*

Suppose $A_1 \subseteq |E|$. The proof of case (i) fails only because $A_1 = A_i$ and $e_1 = 1$. If π is the minimal resolution, then $Z_E = E$ and $A_1 \cdot Z_{L_m} = Z_E \cdot Z_{L_m} \leqslant -2$. Hence

$$\dim H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_1\right)}\right] \geqslant 3$$

and $\dim S \geqslant 2$, where S is the image of $H^0(M, \mathcal{O}(-\Sigma_{i=0}^m Z_{L_i}))$ in

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_1\right)}\right].$$

By Proposition 2.8 of [25], the elements of S have no common zeros as sections of the line bundle L on A_1 associated to

$$\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}-A_1\right)}.$$

If π is not the minimal resolution, then $Z_{L_m}/|E|=Z_E$. This implies that $A_1\cdot Z_E<0$. (In fact $A_j\cdot Z_{L_m}<0$ implies $A_j\cdot Z_E<0$ for all $A_j\subseteq|E|$). By Proposition 2.8 of [25] and case by case checking, the same argument as above still holds. (The crucial point is that the coefficient of A_1 in Z_E is equal to 1.)

Suppose $A_1\not\subseteq|E|$. The proof of case (i) fails only because $A_{i_{r+k}}\neq A_i$, i.e., $A_{i_{r+1}}\cdot Z_{L_m}<0$. Suppose first that $A_1\cap A_{i_{r+1}}=A_1\cap A_i\neq\varnothing$. Choose a computation sequence for Z with $E=Z_k$, $A_{i_k}=A_i$, $A_{i_{k+1}}=A_1$. By Proposition 2.7 of [25], $H^1(M,\mathcal{O}(-\Sigma_{i=0}^{m}Z_{L_i}-Z_j))=0$ for all j. Hence

$$H^0\left(M,\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)\right)\to H^0\left[M,\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}-Z_{k+1}\right)}\right]$$

is surjective. It follows that $H^0(M,\mathcal{O}(-\Sigma_{i=0}^{m}Z_{L_i}))$ and

$$H^0\left[M,\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\left(-\sum_{i=0}^{m} Z_{L_i}-Z_{k+1}\right)}\right]$$

have the same image R in

$$H^0\left[M,\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}-A_1\right)}\right],$$

and

$$\to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_k\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+1}\right)}\right] \to H^0\left(M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+1}\right)}\right)$$

$$\to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_k\right)}\right] \to 0$$

is an exact sequence. Thus the image of

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_k\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+1}\right)}\right]$$

which is injected into

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_1\right)}\right]$$

via the natural map is contained in R. If

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_k\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+1}\right)}\right] \neq 0,$$

then the elements of R have no common zeros on $A_1 - (A_1 \cap A_i)$ as sections of the line bundle L on A_1 associated to

$$\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_1\right)}.$$

If

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_k\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+1}\right)}\right] = 0,$$

then $A_1 \cdot (\sum_{i=0}^{m} Z_{L_i}) = 0$. So

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_1\right)}\right] \cong \mathbb{C}.$$

We claim that

$$H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_1\right)}\right]$$

is surjective. It suffices to prove that the map is not zero. But this is indeed the case, because the elements of image of

$$H^0\left(M, \mathcal{O}\left(-\sum_{j=0}^{m} Z_{L_j}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{j=0}^{m} Z_{L_j}\right)}{\mathcal{O}\left(-\sum_{j=0}^{m} Z_{L_j} - A_i\right)}\right]$$

have no common zeros on A_i as sections of the line bundle on A_i associated to

$$\frac{\mathcal{O}\left(-\sum_{j=0}^{m} Z_{L_j}\right)}{\mathcal{O}\left(-\sum_{j=0}^{m} Z_{L_j} - A_i\right)},$$

by Proposition 2.8 of [25]. To finish the proof of case (ii), it remains to consider those $A_1 \not\subseteq |E|$ such that $A_1 \cap A_i = \varnothing$ and the computation sequence for Z

starting from A_1 in order to reach $|E|$ must first reach A_i. Choose a computation sequence for Z with $E = Z_k$, $A_{i_k} = A_i$, $A_{i_{k+1}} \cap A_i \neq \varnothing$, $A_{i_{k+t}} = A_1$, $A_1 \not\subseteq \operatorname{supp}(Z_{k+t-1})$. Moreover A_{i_j} for $k+1 \leqslant j \leqslant k+t$ are distinct from each other and not contained in $|E|$. By Proposition 2.7 of [25], $H^1(M, \mathcal{O}(-\sum_{i=0}^m Z_{L_i} - Z_j)) = 0$ for all j. Hence

$$H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - Z_{k+t}\right)}\right]$$

is surjective. It follows that $H^0(M, \mathcal{O}(-\sum_{i=0}^m Z_{L_i}))$ and

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - Z_{k+t}\right)}\right]$$

have the same image R in

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_1\right)}\right],$$

and

$$0 \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - Z_{k+t-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - Z_{k+t}\right)}\right] \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - Z_{k+t}\right)}\right]$$

$$\to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - Z_{k+t-1}\right)}\right] \to 0.$$

is an exact sequence. Thus the image of

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+t-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+t}\right)}\right]$$

which is injected into

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_1\right)}\right]$$

via the natural map is contained in R. If

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+t-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+t}\right)}\right] \neq 0,$$

then the elements of R have no common zeros on $A_1 - (A_1 \cap A_{i_{k+t-1}})$ as sections of the line bundle L_1 on A_1 associated to

$$\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_1\right)}.$$

If

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+t-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+t}\right)}\right] = 0,$$

then $A_1 \cdot (\sum_{i=0}^m Z_{L_i}) = 0$. Hence

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_1\right)}\right] \cong \mathbb{C}.$$

But by induction, we know that the elements of the image of

$$H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_{i_{k+t-1}}\right)}\right]$$

have no common zeros on $A_{i_{k+t-1}}$ as sections of the line bundle $L_{i_{k+t-1}}$ on $A_{i_{k+t-1}}$ associated to

$$\frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_{i_{k+t-1}}\right)}.$$

It follows that

$$H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_1\right)}\right]$$

is surjective. This completes our proof of $\mathcal{O}(-\sum_{i=0}^m Z_{L_i}) \subseteq m\mathcal{O}$. Q.E.D.

3. Calculation of Hilbert Function.

THEOREM 3.1. *Let $\pi: M \to V$ be the minimal good resolution of normal two-dimensional Stein space with p as its only strongly elliptic singularity. Let $Z_{L_0}, Z_{L_1}, \ldots, Z_{L_m}$ be the Laufer sequence. Suppose $Z_{L_m} \cdot Z_E \leqslant -3$. Then $m^n \simeq H^0(A, \mathcal{O}(-n\sum_{i=0}^m Z_{L_i}))$ and $\dim m^n/m^{n+1} = -n(\sum_{i=0}^m Z_{L_i}^2)$ provided that either* *one of the following holds*: (1) $Z_E = E$, *i.e.*, π *is the minimal resolution*; (2) $Z_{L_m}/|E| = Z_E$.

Proof. It is true that $H^0(A, \mathcal{O}(-\sum_{i=0}^m Z_{L_i})) = \operatorname{dir\,lim} H^0(U, \mathcal{O}(-\sum_{i=0}^m Z_{L_i}))$, where U is a neighborhood of A. By the proof of Theorem 2.4, we have $H^0(A, \mathcal{O}(-\sum_{i=0}^m Z_{L_i})) = m$.

Since $H^1(M, \mathcal{O}(-n\sum_{i=0}^m Z_{L_i})) = 0$ for all n by Proposition 2.7 of [25],

$$H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^m Z_{L_i}\right)}\right] = 0$$

for all n. The long cohomology exact sequence corresponding to the sheaf exact sequence

$$0 \to \frac{\mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^m Z_{L_i}\right)} \to \frac{\mathcal{O}}{\mathcal{O}\left(-(n+1)\sum_{i=0}^m Z_{L_i}\right)} \to \frac{\mathcal{O}}{\mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i}\right)} \to 0$$

shows that

$$\begin{aligned}\dim H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^m Z_{L_i}\right)}\right] &= \chi\left((n+1)\sum_{i=0}^m Z_{L_i}\right) - \chi\left(n\sum_{i=0}^m Z_{L_i}\right) \\ &= -n\left(\sum_{i=0}^m Z_{L_i}\right)\cdot\left(\sum_{i=0}^m Z_{L_i}\right) \\ &= -n\sum_{i=0}^m Z_{L_i}^2.\end{aligned}$$

To show that $m^n = H^0(A, \mathcal{O}(-n\sum_{i=0}^m Z_{L_i}))$, we shall show that

$$H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)\right) \otimes_{\mathbf{C}} H^0\left(M, \mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i}\right)\right)$$
$$\to H^0\left(M, \mathcal{O}\left(-(n+1)\sum_{i=0}^m Z_{L_i}\right)\right)$$

is a surjective map. We claim that it suffices to prove

$$H^0\left[M,\frac{\mathcal{O}\left(-\sum_{i=0}^{m}Z_{L_i}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m}Z_{L_i}\right)}\right]\otimes_{\mathbf{C}}H^0\left[M,\frac{\mathcal{O}\left(-n\sum_{i=0}^{m}Z_{L_i}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m}Z_{L_i}\right)}\right]$$

$$\to H^0\left[M,\frac{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m}Z_{L_i}\right)}{\mathcal{O}\left(-(n+2)\sum_{i=0}^{m}Z_{L_i}\right)}\right] \tag{3.1}$$

is a surjective map for all n.

Let us first demonstrate this fact. We first show that the image of $H^0(M,\mathcal{O}(-\Sigma_{i=0}^{m}Z_{L_i}))\otimes_{\mathbf{C}}H^0(M,\mathcal{O}(-n\Sigma_{i=0}^{m}Z_{L_i}))$ contains $H^0(M,\mathcal{O}(-t\Sigma_{i=0}^{m}Z_{L_i}))$ for some t. Let $f_1,\ldots,f_s\in H^0(M,\mathcal{O}(-n\Sigma_{i=0}^{m}Z_{L_i}))$ generate $\mathcal{O}(-n\Sigma_{i=0}^{m}Z_{L_i})$ as an $\mathcal{O}$-module. The existence of such f_i follows from Theorem 2.4. The $\mathcal{O}$-module map

$$\rho:\oplus_s\mathcal{O}\left(-\sum_{i=0}^{m}Z_{L_i}\right)\to\mathcal{O}\left(-(n+1)\sum_{i=0}^{m}Z_{L_i}\right)$$

given by $(g_1,\ldots,g_s)\to\Sigma f_ig_i$ is then surjective. Let $K=\ker\rho$; then

$$0\to K\to\oplus_s\mathcal{O}\left(-\sum_{i=0}^{m}Z_{L_i}\right)\xrightarrow{\rho}\mathcal{O}\left(-(n+1)\sum_{i=0}^{m}Z_{L_i}\right)\to 0$$

is exact. Multiplying by $\mathcal{O}(-k\Sigma_{i=0}^{m}Z_{L_i})$, we get the following:

$$\begin{array}{ccccccccc}
0\to & K\mathcal{O}\left(-k\sum_{i=0}^{m}Z_{L_i}\right) & \to & \oplus_s\mathcal{O}\left(-(k+1)\sum_{i=0}^{m}Z_{L_i}\right) & \to & \mathcal{O}\left(-(n+k+1)\sum_{i=0}^{m}Z_{L_i}\right) & \to 0\\
 & \downarrow\sigma & & \downarrow & & \downarrow\lambda & \\
0\to & K & \to & \oplus_s\,\mathcal{O}\left(-\sum_{i=0}^{m}Z_{L_i}\right) & \to & \mathcal{O}\left(-(n+1)\sum_{i=0}^{m}Z_{L_i}\right) & \to 0
\end{array}$$

—a commutative diagram with the vertical maps the inclusion maps. The verification that the first row is exact is the same as the verification that (5.5) of [11] was exact.

$$\begin{array}{ccccc}
H^0\left(M,\oplus_s\mathcal{O}\left(-(k+1)\sum\limits_{i=0}^{m} Z_{L_i}\right)\right) & \to & H^0\left(M,\mathcal{O}\left(-(n+k+1)\sum\limits_{i=0}^{m} Z_{L_i}\right)\right) & \to & H^1\left(M,K\mathcal{O}\left(-k\sum\limits_{i=0}^{m} Z_{L_i}\right)\right) \\
\downarrow & & \downarrow\lambda_* & & \downarrow\sigma_* \\
H^0\left(M,\oplus_s\mathcal{O}\left(-\sum\limits_{i=0}^{m} Z_{L_i}\right)\right) & \xrightarrow{\rho_*} & H^0\left(M,\mathcal{O}\left(-(n+1)\sum\limits_{i=0}^{m} Z_{L_i}\right)\right) & \to & H^1(M,K)
\end{array}$$

is commutative with exact rows. By [3, §4, Satz 1, p. 355] σ_* is the zero map for sufficiently large k. Then given $h\in H^0(M,\mathcal{O}(-(n+k+1)\Sigma_{i=0}^m Z_{L_i}))$, $\lambda_*(h)=\rho_*(g)$ for some g, by exactness. Letting $t=n+k+1$, we have that the image of $H^0(M,\mathcal{O}(-\Sigma_{i=0}^m Z_{L_i}))\otimes_{\mathbf{C}} H^0(M,\mathcal{O}(-n\Sigma_{i=0}^m Z_{L_i}))$ contains $H^0(M,\mathcal{O}(-t\Sigma_{i=0}^m Z_{L_i}))$ as required.

If $t>n+1\geqslant 2$, we shall show that the image of $H^0(M,\mathcal{O}(-\Sigma_{i=0}^m Z_{L_i}))\otimes_{\mathbf{C}} H^0(M,\mathcal{O}(-n\Sigma_{i=0}^m Z_{L_i}))$ contains $H^0(M,\mathcal{O}(-(t-1)\Sigma_{i=0}^m Z_{L_i}))$. By decreasing induction, we will be done. Look at the following diagram:

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & H^0\left(M,\mathcal{O}\left(-t\sum\limits_{i=0}^{m} Z_{L_i}\right)\right) \\
 & & \downarrow \\
 & & H^0\left(M,\mathcal{O}(-t-1)\sum\limits_{i=0}^{m} Z_{L_i}\right) \\
 & & \downarrow \\
H^0\left[M,\dfrac{\mathcal{O}\left(-\sum\limits_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-2\sum\limits_{i=0}^{m} Z_{L_i}\right)}\right]\otimes_{\mathbf{C}} H^0\left[M,\dfrac{\mathcal{O}\left(-(t-2)\sum\limits_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-(t-1)\sum\limits_{i=0}^{m} Z_{L_i}\right)}\right] & \to & H^0\left[M,\dfrac{\mathcal{O}\left(-(t-1)\sum\limits_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-t\sum\limits_{i=0}^{m} Z_{L_i}\right)}\right] \\
 & & \downarrow \\
 & & 0
\end{array}$$

with the vertical sequence exact and the horizontal map surjective. It follows that the image of $H^0(M,\mathcal{O}(-\Sigma_{i=0}^m Z_{L_i})\otimes_{\mathbf{C}} H^0(M,\mathcal{O}(-n\Sigma_{i=0}^m Z_{L_i}))$ contains $H^0(M,\mathcal{O}(-(t-1)\Sigma_{i=0}^m Z_{L_i}))$.

It remains to prove that (3.1) is surjective for all n. The proof breaks up into three cases:

(i) There is an A_i (call it A_1) such that $Z_{L_m}\cdot Z_E+1\leqslant A_1\cdot Z_{L_m}\leqslant -2$.
(ii) There is an A_i (call it A_1) such that $A_1\cdot Z_{L_m}=Z_{L_m}\cdot Z_E$.
(iii) $A_i\cdot Z_{L_m}=-1$ or 0 all $A_i\subseteq|E|$. Take $A_1\cdot Z_{L_m}=-1$.

In case (i), all irreducible components are non-singular rational curves. Choose a computation sequence for Z_{L_j} as follows: $Z_0^j=0$, $Z_1^j,\ldots,$ $Z_k^j=E=Z_{k-1}^j+A_{i_k}^j,\ldots,$ $Z_{r_j}^j=Z_{L_j}$ where $A_{i_k}^j=A_1$. Consider

$$\tau_{L_0L_1,\ldots,L_h;s}:H^0\left[M,\frac{\mathcal{O}\left(-\sum_{i=0}^{m}Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m}Z_{L_i}-A_{i_s}^{h+1}\right)}\right]$$

$$\otimes_{\mathbf{C}}H^0\left[M,\frac{\mathcal{O}\left(-n\sum_{i=0}^{m}Z_{L_i}-\sum_{i=0}^{h}Z_{L_i}-Z_{s-1}^{h+1}\right)}{\mathcal{O}\left(-n\sum_{i=0}^{m}Z_{L_i}-\sum_{i=0}^{h}Z_{L_i}-Z_{s}^{h+1}\right)}\right]$$

$$\to H^0\left[M,\frac{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m}Z_{L_i}-\sum_{i=0}^{h}Z_{L_i}-Z_{s-1}^{h+1}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m}Z_{L_i}-\sum_{i=0}^{h}Z_{L_i}-Z_{s}^{h+1}\right)}\right] \tag{3.2}$$

for all $-1\leqslant h\leqslant m-1$ and $s\leqslant r_{h+1}$. To show that (3.1) is surjective, it will suffice to show that $\tau_{L_0,L_1,\ldots,L_h;s}$ is surjective for all $s\leqslant r_{h+1}$, $1\leqslant h\leqslant m-1$. Indeed, since $(\sum_{i=0}^{h}Z_{L_i})A_j\leqslant 0$ for all $A_j\subseteq A$, all of the first cohomology groups $H^1(M,\mathcal{O}(-n\sum_{i=0}^{m}Z_{L_i}-\sum_{i=0}^{h}Z_{L_i}-Z_s^{h-1}))=0$ by Proposition 2.7 of [25]. So

$$H^0\left[M,\frac{\mathcal{O}\left(-n\sum_{i=0}^{m}Z_{L_i}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m}Z_{L_i}\right)}\right]$$

can be written via successive quotients:

$$0 \to H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_j^{h+1}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m} Z_{L_i}\right)}\right]$$

$$\to H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{j-1}^{h+1}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m} Z_{L_i}\right)}\right]$$

$$\to H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{j-1}^{h+1}\right)}{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_j^{h+1}\right)}\right] \to 0,$$

$$-1 \leqslant h \leqslant m, \qquad 1 \leqslant j \leqslant r_{h+1},$$

where we denote $\Sigma_{i=0}^{-1} Z_{L_i} = 0$ and $Z_0^{h+1} = 0$ for all h. By the proof of Theorem 2.4 and Proposition 2.7 of [25], we have $H^1(M, \mathcal{O}(-\Sigma_{i=0}^{m} Z_{L_i} - A_{i_s}^{j})) = 0$ for all $0 \leqslant j \leqslant m$, $s \leqslant r_j$. Hence

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_s}\right)}\right]$$

is surjective. Look at the following commutative diagrams:

$$\begin{array}{ccc}
H^0\left(M, \frac{\mathcal{O}(-G)}{\mathcal{O}(-2G)}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nG - B_h - Z_{j+1}^{h+1})}{\mathcal{O}(-(n+1)G)}\right) & \to & H^0\left(M, \frac{\mathcal{O}(-(n+1)G - B_h - Z_{j+1}^{h+1})}{\mathcal{O}(-(n+2)G)}\right) \\
\downarrow \qquad\qquad\qquad \downarrow & & \downarrow \\
H^0\left(M, \frac{\mathcal{O}(-G)}{\mathcal{O}(-2G)}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nG - B_h - Z_{j}^{h+1})}{\mathcal{O}(-(n+1)G)}\right) & \to & H^0\left(M, \frac{\mathcal{O}(-(n+1)G - B_h - Z_{j}^{h+1})}{\mathcal{O}(-(n+2)G)}\right) \\
\downarrow \qquad\qquad\qquad \downarrow & & \downarrow \\
H^0\left(M, \frac{\mathcal{O}(-G)}{\mathcal{O}(-G - A_j^{h+1})}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nG - B_h - Z_{j}^{h+1})}{\mathcal{O}(-nG - B_h - Z_{j+1}^{h+1})}\right) & \to & H^0\left(M, \frac{\mathcal{O}(-(n+1)G - B_h - Z_{j}^{h+1})}{\mathcal{O}(-(n+1)G - B_h - Z_{j+1}^{h+1})}\right)
\end{array}$$

$$-1 \leqslant h \leqslant m, \qquad 0 \leqslant j \leqslant r_{h+1} - 1$$

where we denote $\Sigma_{i=0}^{-1} Z_{L_i} = 0$, $z_0^{h+1} = 0$, $G = \Sigma_{i=0}^{m} Z_{L_i}$ and $B_h = \Sigma_{i=0}^{h} Z_{L_i}$. Thus if (3.2) is surjective for all $s \leqslant r_{h+1}$, $-1 \leqslant h \leqslant m-1$, then (3.1) is also surjective.

Suppose that the target space in (3.2) is non-zero, i.e., $-A_{i_s}^{h+1} \cdot ((n+1)\Sigma_{i=0}^{m} Z_{L_i} + \Sigma_{i=0}^{h} Z_{L_i} + Z_{s-1}^{h+1}) \geqslant 0$. We need $-A_{i_s}^{h+1} \cdot (\Sigma_{i=0}^{m} Z_{L_i}) \geqslant 0$ and $-A_{i_s}^{h+1} \cdot (n\Sigma_{i=0}^{m} Z_{L_i} + \Sigma_{i=0}^{h} Z_{L_i} + Z_{s-1}^{h+1}) \geqslant 0$. For $s \neq k$, $A_{i_s}^{h+1} \cdot Z_{s-1}^{h+1} = 1$. If $-A_{i_s}^{h+1} \cdot (\Sigma_{i=0}^{m} Z_{L_i}) > 0$, then $-A_{i_s}^{h+1} \cdot (n\Sigma_{i=0}^{m} Z_{L_i} + \Sigma_{i=0}^{h} Z_{L_i} + Z_{s-1}^{h+1}) \geqslant 0$. If $-A_{i_s}^{h+1} \cdot (\Sigma_{i=0}^{m} Z_{L_i}) = 0$, then $-A_{i_s}^{h+1} \cdot (n\Sigma_{i=0}^{m} Z_{L_i} + \Sigma_{i=0}^{h} Z_{L_i} + Z_{s-1}^{h+1}) = -A_{i_s}^{h+1} \cdot ((n+1)\Sigma_{i=0}^{m} Z_{L_i} + \Sigma_{i=0}^{h} Z_{L_i} + Z_{s-1}^{h+1}) \geqslant 0$. For $s = k$, $A_{i_k}^{h+1} \cdot Z_{k-1}^{h+1} = 2$. But by construction $A_{i_k} \cdot Z_{L_m} \leqslant -2$ and so (3.2) is surjective for all $s \leqslant r_{h+1}$, $-1 \leqslant h \leqslant m-1$.

Let us now do case (ii). Suppose first that $\operatorname{supp} E$ has more than one irreducible component. The proof of case (i) fails only because the maps

$$H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_k}^{h+1}\right)}\right]$$

and the maps

$$H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{t+k}}^{h+1}\right)}\right]$$

need not be surjective, where $A_{i_{t+k}}^{h+1}$ has the following property. Any computation sequence for $Z_{B_{h+1}}$ starting from $A_{i_{t+k}}^{h+1}$ in order to reach $|E|$ must first reach A_1 and $A_1 \cdot Z_{L_m} < 0$. In (3.2),

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_k}^{h+1}\right)}\right] = H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_1\right)}\right]$$

must be replaced by the subspace S of Proposition 2.8 of [25], where $\dim S = -$

$A_1\cdot(\Sigma_{i=0}^m Z_{L_i}) = -A_1\cdot Z_{L_m} = -Z_E\cdot Z_{L_m} \geqslant 2$. Also

$$\dim H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_k^{h+1}\right)}\right]$$
$$= -A_1\cdot\left(n\sum_{i=0}^m Z_{L_i} + \sum_{i=0}^h Z_{L_i} + Z_{k-1}^{h+1}\right) = -nA_1\cdot Z_{L_m} - 1 \geqslant 2.$$

Under these conditions

$$\tau_{L_0,L_1,\ldots,L_h;k} S \otimes_{\mathbf{C}} H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-n\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_k^{h+1}\right)}\right]$$
$$\rightarrow H^0\left[M, \frac{\mathcal{O}\left(-(n+1)\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_k^{h+1}\right)}\right]$$

is still surjective for $-1 \leqslant h \leqslant m-1$. Namely, consider the subspace T of S of sections which vanish at some given point, say $a \in A_1$. T has codimension 1 in S. If all the elements of T have a common zero at some $b \neq a \in A_1$ or if all have a second order zero at a, then T, having codimension 2 in

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - A_{i_k}^{h+1}\right)}\right],$$

represents all sections of a suitable line bundle over A_1. Then $\tau_{L_0,\ldots,L_h;k}$ is readily seen to be surjective, as in the proof of [11, Lemma 7.9, pp. 144–146], but more easily. If the elements of T have no common zeros, then think of T as codimension-1 subspace of the sections of a line bundle, and replace S by T in the previous case. Eventually we see that $\tau_{L_0,\ldots,L_h;k}$ is surjective when $\dim T = 1$.

Also, in (3.2)

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{t+k}}^{h+1}\right)}\right].$$

must be replaced by the subspace S_{t+k} which is the image of

$$\varphi_{t+k}^{h+1} : H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{t+k}}^{h+1}\right)}\right]$$

if φ_{t+k}^{h+1} is not surjective. By the proof of Theorem 6.4, case (ii), we know that S_{t+k}^{h+1} has at most codimension 1 in

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{t+k}}^{h+1}\right)}\right].$$

Moreover, the elements of S_{t+k}^{h+1} have no common zeros as section of the line bundle on $A_{i_{t+k}}^{h+1}$ associated to

$$\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{t+k}}^{h+1}\right)}.$$

We are going to prove that

$$\dim H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{t+k-1}^{h+1}\right)}{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{t+k}^{h+1}\right)}\right] \geqslant 2.$$

Since we assume that

$$\varphi_{t+k}^{h+1}: H^0\left(M, \mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)\right) \to H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{t+k}}^{h+1}\right)}\right]$$

is not surjective, it follows from the proof of Theorem 2.4, case (ii), that $-A_{i_{t+k}}^{h+1} \cdot (\sum_{i=0}^{m} Z_{L_i}) \geqslant 1$. We claim that $-A_{i_{t+k}}^{h+1} \cdot (\sum_{i=0}^{m} Z_{L_i}) \neq 1$. Otherwise

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{k+t}}^{h+1}\right)}\right] \simeq \mathbb{C}^2.$$

Inductive argument will show that there exists

$$f \in H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{t+k}}^{h+1}\right)}\right]$$

such that f as section of the line bundle associated to

$$\frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{t+k}}^{h+1}\right)}$$

has no zero on $A_{i_{t+k}} \cap A_{i_{t+k-1}}$, and f is in S_{t+k}^{h+1}, the image of φ_{t+k}^{h+1}. Hence f cannot be in the image of

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+t-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{k+t}\right)}\right] \simeq \mathbb{C}$$

which is injected into

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_{i_{k+t}}\right)}\right]$$

via the natural map and which is contained in S_{t+k}^{h+1}. This implies that φ_{t+k}^{h+1} is surjective, which contradicts our assumption. We conclude that $-A_{i_{t+k}}^{h+1} \cdot (\sum_{i=0}^{m} Z_{L_i}) > 1$ and hence

$$\dim H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{t+k-1}^{h+1}\right)}{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{t+k}^{h+1}\right)}\right] \geqslant 2.$$

Now repeating the argument above, we get that

$$\tau_{L_0, L_1, \ldots, L_h; k}: S_{t+k}^{h+1} \otimes_{\mathbb{C}} H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{t-1}^{h+1}\right)}{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{t}^{h+1}\right)}\right]$$

$$\to H^0\left[M, \frac{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{t-1}^{h+1}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{t}^{h+1}\right)}\right]$$

is surjective for $-1 \leqslant h \leqslant m-1$. In case $E = Z_E = A_1$ is an elliptic curve, we know that

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - A_1\right)}\right] \otimes_{\mathbb{C}} H^0\left[M, \frac{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i}\right)}{\mathcal{O}\left(-n\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - A_1\right)}\right]$$

$$\to H^0\left[M, \frac{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i}\right)}{\mathcal{O}\left(-(n+1)\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - A_1\right)}\right]$$

is surjective. This is shown in [20]. The result follows from the proof above and the proof of case (i).

Let us now do case (iii). The proof of case (i) fails only because

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k}^{h+1}\right)}\right] = 0.$$

We can still get

$$H^0\left[M, \frac{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-3\sum_{i=0}^{m} Z_{L_i}\right)}\right]$$

as an image as follows. There are two subcases. First suppose that A_1 can be chosen so that $A_1 \cdot Z_{L_m} < 0$ and $e_1 > 1$ in $E = \Sigma e_i A_i$. In this subcase $Z_E = E$. Then choose a computation sequence for Z_E with $A_{i_1} \cdot Z_{L_m} < 0$, $E = Z_k = Z_E$, $A_1 = A_{i_k}$, and with a Z_q $(q < k)$ such that $A_{i_q} = A_1$, $A_1 \not\subseteq \operatorname{supp}(E - A_1 - Z_q)$ and $A_i \cdot Z_{q-1} \leqslant 0$, $i \neq 1$, $A_i \subseteq |E|$. Such a computation sequence can be formed by letting $A_{i_j} = A_1$ only when $A_{i_j} \subseteq |E|$ cannot be chosen otherwise. Then also $0, Z_q - Z_{q-1}, Z_{q+1} - Z_{q-1}, \ldots, Z_k - Z_{q-1}$ is a part of a computation sequence for $Z_E = E$ which, by Corollary 2.3 of [25] can be continued to terminate at A_{i_1}. Recall that $A_{i_1} \cdot Z_{L_m} < 0$ by construction. So by Proposition 2.7 of [25], we have $H^1(M, \mathcal{O}(-\Sigma_{i=0}^{m} Z_{L_i} - Z_q)) = 0$ and also $H^1(M, \mathcal{O}(-\Sigma_{i=0}^{m} Z_{L_i} - \Sigma_{i=0}^{h} Z_{L_i} - (Z_k - Z_{q-1})) = 0$, $H^1(M, \mathcal{O}(-\Sigma_{i=0}^{m} Z_{L_i} - \Sigma_{i=0}^{h} Z_{L_i} - Z_k^{h+1})) = 0$. In place of (3.2), we use

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{q-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_q\right)}\right] \otimes_{\mathbb{C}} H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - (Z_{k-1} - Z_{q-1})\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - (Z_k - Z_{q-1})\right)}\right]$$

$$\to H^0\left[M, \frac{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_k\right)}\right].$$

Look at the following commutative diagrams:

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] & \otimes_{\mathbf{C}} & H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_k^{h+1}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] & \to & H^0\left[M, \dfrac{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_k^{h+1}\right)}{\mathcal{O}\left(-3\sum_{i=0}^{m} Z_{L_i}\right)}\right] \\
\downarrow & & \downarrow & & \downarrow \\
H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] & \otimes_{\mathbf{C}} & H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] & \to & H^0\left[M, \dfrac{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-3\sum_{i=0}^{m} Z_{L_i}\right)}\right] \\
 & & & & \downarrow \\
H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{q-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{q-1} - A_1\right)}\right] & \otimes_{\mathbf{C}} & H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - \left(Z_{k-1}^{h+1} - Z_{q-1}\right)\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - \left(Z_{k}^{h+1} - Z_{q-1}\right)\right)}\right] & \to & H^0\left[M, \dfrac{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k}^{h+1}\right)}\right] \\
 & & & & \downarrow \\
 & & & & 0
\end{array}
$$

with the column on the right exact. Notice that the computation sequence for Z_{L_j} is so chosen that $Z_i^j = Z_i$ as above for $1 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant m$. The result follows easily.

In the other subcase, we claim that there must be A_1, A_2, and A_3 all distinct so that $A_i \cdot Z_{L_m} < 0$ and $e_i = 1$. For if $Z_E = E$, since $A_i \cdot Z_{L_m} = 0$ or -1 for all $A_i \subseteq |E|$, the claim follows easily. If $Z_{L_m}/|E| = Z_E$, then $A_i \cdot Z_E = 0$ implies that $A_i \cdot Z_{L_m} = 0$ for $A_i \subseteq |E|$. The claim follows from Proposition 2.4 of [25]. Choose a computation sequence for Z_E with $E = Z_k$ such that $A_{i_1} \cdot A_1 > 0$, $A_{i_k} = A_1$ and such that when Z_q with $q < k$, $A_{i_q} = A_2$ is reached, with $A_i \cdot Z_{q-1} \leqslant 0$ for $i \neq 1, 2$, $A_i \subseteq |E|$. We may suppose $A_3 \subseteq \operatorname{supp} Z_{q-1}$, for otherwise we reverse the roles of A_2 and A_3. Since $A_{i_1} \cdot A_1 > 0$ and $e_1 = 1$, $Z_{q-1} + A_1$ is part of a computation sequence for Z_E. $0, Z_q - Z_{q-1}, \ldots, Z_{k-1} - Z_{q-1}$ is also part of a computation sequence for Z_E. Hence $H^1(M, \mathcal{O}(-\sum_{i=0}^m Z_{L_i} - Z_{q-1} - A_1)) \simeq 0$, $H^1(M, \mathcal{O}(-\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - (Z_k^{h+1} - Z_{q-1})) \simeq 0$, and $H^1(M, \mathcal{O}(-2\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_k^h)) \simeq 0$. In place of (3.1), we use

$$H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - Z_{q-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - Z_{q-1} - A_1\right)}\right]$$

$$\otimes H^0\left[M, \frac{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - (Z_{k-1}^{h+1} - Z_{q-1})\right)}{\mathcal{O}\left(-\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - (Z_k^{h+1} - Z_{q-1})\right)}\right]$$

$$\rightarrow H^0\left[M, \frac{\mathcal{O}\left(-2\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-2\sum_{i=0}^m Z_{L_i} - \sum_{i=0}^h Z_{L_i} - Z_k^{h+1}\right)}\right]$$

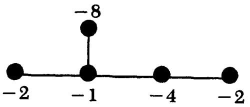
Look at the following commutative diagram:

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] & \otimes_{\mathbf{C}} & H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_k^{h+1}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] & \to & H^0\left[M, \dfrac{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_k^{h+1}\right)}{\mathcal{O}\left(-3\sum_{i=0}^{m} Z_{L_i}\right)}\right] \\
\downarrow & & \downarrow & & \downarrow \\
H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] & \otimes_{\mathbf{C}} & H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i}\right)}\right] & \to & H^0\left[M, \dfrac{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-3\sum_{i=0}^{m} Z_{L_i}\right)}\right] \\
 & & & & \downarrow \\
H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_{q-1}\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - Z_q\right)}\right] & \otimes_{\mathbf{C}} & H^0\left[M, \dfrac{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - \left(Z_{k-1}^{h+1} - Z_{q-1}\right)\right)}{\mathcal{O}\left(-\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - \left(Z_k^{h+1} - Z_{q-1}\right)\right)}\right] & \to & H^0\left[M, \dfrac{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_{k-1}^{h+1}\right)}{\mathcal{O}\left(-2\sum_{i=0}^{m} Z_{L_i} - \sum_{i=0}^{h} Z_{L_i} - Z_k^{h+1}\right)}\right] \\
 & & & & \downarrow \\
 & & & & 0
\end{array}
$$

with the vertical column on the right exact. Notice that the computation sequence of Z_{L_i} is so chosen that $Z_i^j = Z_i$ as above for $1 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant m$. The result follows easily. Q.E.D.

The following two examples show that our Theorem 3.1 is sharp.

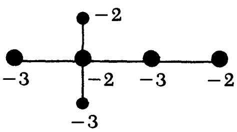
Example 1.



$$Z = 3\overset{1}{6}21 \qquad Z_E = 2\overset{1}{4}10$$

We have $\chi(Z)=0$, $Z_E \cdot Z = -3$, and $Z \cdot Z = -4$. The Laufer sequence consists of only one element, namely $\{Z\}$. (1) and (2) of Theorem 3.1 are not satisfied. We claim that $\dim m^n/m^{n+1} \neq -nZ \cdot Z$. Otherwise $\dim m/m^2 = -Z \cdot Z = 4$ and $\dim m^2/m^3 = -2Z \cdot Z = +8$. The ambient space has 4 variables, making 10 available dimensions of quadratic terms for m^2/m^3. Hence there are necessarily 2 defining equations which begin with quadratic terms. By the proof of Theorem 3.13 of [15] the variety is actually equal to the common zero set of these two equations. Since p has codimension 2, p is a complete intersection. In particular ${}_V\mathcal{O}_p$ is Gorenstein. However $Z_E \cdot Z < 0$ implies that p is a strongly elliptic singularity by Theorem 4.1 of [15]. This contradicts Theorem 3.10 of [15].

Example 2.



$$Z_E = 1\overset{1}{\underset{1}{2}}10 \qquad Z = 1\overset{1}{\underset{1}{2}}11$$

We have $\chi(Z)=0$, $Z \cdot Z_E = -2$, and $Z \cdot Z = -3$. Theorem 4.1 of [25] says that p is a strongly elliptic singularity. The hypothesis of Theorem 3.1 is not satisfied, because $Z \cdot Z_E > -3$. Notice that the Laufer sequence consists of only one element, namely $\{Z\}$. We claim that $\dim m^n/m^{n+1} \neq -nZ \cdot Z$. Otherwise $\dim m/m^2 = -Z \cdot Z = -3$. This implies that the singularity is a hypersurface singularity. In particular, ${}_V\mathcal{O}_p$ is Gorenstein. But this contradicts Theorem 3.10 of [15].

4. General Lower Bound for $m\mathcal{O}$. In this section we introduce the definition of universal cycle. We then give a lower bound for $m\mathcal{O}$ in terms of universal cycle. The universal cycle is defined purely topologically, and it can be computed readily by a weighted dual graph.

Definition 4.1. Let A be an exceptional set in the resolution $\pi: M \to V$ of a normal 2-dimensional singularity p. Suppose that $\{A_i\}$, $1 \leqslant i \leqslant n$, the irreducible components of A, are non-singular. The *universal cycle* $L = \Sigma a_i A_i$ is the minimal positive cycle such that $A_i \cdot L < 0$ for all $A_i \subseteq A$.

PROPOSITION 4.2. *There is a unique universal cycle L on the exceptional set A.*

Proof. By Lemma 4.10 of [11] there always exists a positive cycle $L_1 = \Sigma a_i^1 A_i$ with the property that $L_1 \cdot A_i < 0$ for all $A_i \subseteq A$. If $L_2 = \Sigma a_i^2 A_i$ is another positive cycle with this property, then so is $L^1 = \Sigma a_i A_i$, where $a_i = \inf(a_i^1, a_i^2)$ and moreover L^1 is positive. In fact, if (say) $a_j^1 \leqslant a_j^2$, we have

$$L^1 \cdot A_j = a_j^1(A_j \cdot A_j) + \sum_{i \neq j} a_i(A_i \cdot A_j)$$
$$\leqslant a_j^1(A_j \cdot A_j) + \sum_{i \neq j} a_i^1(A_1 \cdot A_j) = L_1 \cdot A_j < 0. \qquad \text{Q.E.D.}$$

The following proposition tells us how to compute the universal cycle L in terms of weighted dual graph.

PROPOSITION 4.3. *L may be computed as follows. Let $L_1 = A_{i_0}$, any A_{i_0}. Having defined $L_j = \Sigma a_{ji} A_i$, if there exists an A_{i_j} such that $A_{i_j} \cdot L_j \geqslant 0$, let $L_{j+1} = L_j + A_{i_j}$. If $A_i \cdot Z_j < 0$ for all A_i, then $L = L_j$.*

Proof. We prove by induction that $L_j \leqslant L$. This is true if $j = 1$. If $L_j < L$, since L is minimal, there exists A_{i_j} such that $A_{i_j} \cdot L \geqslant 0$. However, $a_{ji_j} = a_{i_j}$ is impossible for $A_{i_j} \cdot L < 0$. Thus $a_{ji_j} = a_{i_j}$ would imply that $A_{i_j} \cdot L_j < 0$, since $a_{ji} \leqslant a_i$ all i and $A_k \cdot A_l \geqslant 0$ if $k \neq l$. Hence $a_{ji_j} < a_{i_j}$ if $L_j < L$, so that $L_{j+1} \leqslant L$.

THEOREM 4.4. *Let $\pi: M \to V$ be the minimal good resolution of normal 2-dimensional Stein space with p as its only weakly elliptic singularity. Then $m\mathcal{O} \supseteq \mathcal{O}(-2L)$. If $\operatorname{supp} E$ has at least two irreducible components or $E = A_1$ is an elliptic curve and $A_1 \cdot L \leqslant -2$, then $m\mathcal{O} \supseteq \mathcal{O}(-L)$.*

Proof. By the proof of Proposition 2.7 of [25] and the proof of Theorem 2.4, we have $H^1(M, \mathcal{O}(-2L - A_i)) = 0$ for all $A_i \subseteq A$. Hence

$$H^0(M, \mathcal{O}(-2L)) \to H^0\left(M, \frac{\mathcal{O}(-2L)}{\mathcal{O}(-2L - A_i)}\right)$$

is surjective.

If we assume that $\operatorname{supp} E$ has at least two irreducible components or $E = A_1$ is an elliptic curve and $A_1 \cdot L \leqslant -2$, then similar reasoning to the above also shows that $H^1(M, \mathcal{O}(-L-A_i)) = 0$ for all $A_i \subseteq A$. Hence

$$H^0(M, \mathcal{O}(-L)) \to H^0\left(M, \frac{\mathcal{O}(-L)}{\mathcal{O}(-L-A_i)}\right)$$

is surjective. Q.E.D.

THEOREM 4.5. *Let $\pi: M \to V$ be the minimal good resolution of normal two-dimensional Stein space with p as its only singular point. Let $g = p(Z) = 1 - \chi(Z)$ be the arithmetic genus of the fundamental cycle. Then $m\mathcal{O} \supseteq \mathcal{O}(-3gL)$.*

Proof. By (2.6) of [15], $g = \dim H^1(M, \mathcal{O}_Z)$. By (2.7) of [15] and the Riemann-Roch theorem, $A_{i_j} \cdot Z_{j-1} \leqslant g+1$ for all j where $Z_0 = 0$, $Z_1 = A_{i_1}, \ldots, Z_j = Z_{j-1} + A_{i_j}, \ldots, Z$ is a computation sequence for the fundamental cycle. Similar reasoning as Theorem 4.4 will show that $m\mathcal{O} \supseteq \mathcal{O}(-3gL)$. Q.E.D.

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