On Number Theoretic Conjecture of Positive Integral Points in 5-Dimension Tetrahedron and a Sharp Estimate of Dickman-De Bruijn Function

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Dedicated to Professor Ronald Graham on the occasion of his 75th birthday

Abstract

It is well known that getting the estimate of integral points in right-angled simplices is equivalent to getting the estimate of Dickman-De Bruijn function $\psi(x, y)$ which is the number of positive integers $\leq x$ and free of prime factors $> y$. Motivating from the Yau Geometry Conjecture, the third author formulated the Number Theoretic Conjecture which gives a sharp polynomial upper estimate that counts the number of positive integral points in n-dimensional ($n \geq 3$) real right-angled simplices. In this paper, we prove this Number Theoretic Conjecture for $n = 5$. As an application, we give a sharp estimate of Dickman-De Bruijn function $\psi(x, y)$ for $5 \leq y < 13$.

1 Introduction

Let $\Delta(a_1, a_2, ..., a_n)$ be an n-dimensional simplex described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1, \ x_1, x_2, ..., x_n \geq 0 \tag{1.1}$$

where $a_1 \geq a_2 \geq ... \geq a_n \geq 1$ are positive real numbers. Let $P_n = P(a_1, a_2, ..., a_n)$ and $Q_n = Q(a_1, a_2, ..., a_n)$ be defined as the number of positive and nonnegative integral solutions of (1.1) respectively. They are related by the following formula

$$Q(a_1, a_2, ..., a_n) = P(a_1(1 + a), a_2(1 + a), ..., a_n(1 + a)), \tag{1.2}$$

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where $a = \frac{1}{a_1} + \frac{1}{a_2} + .. + \frac{1}{a_n}$. The estimate of integral points has many applications in number theory, complex geometry, toric variety and tropical geometry.

One of the central topics in computational number theory is the estimate of $\psi(x, y)$, the Dickman-De Bruijn function (see [4],[5],[6],[10]). Let $S(x, y)$ be the set of positive integers $\leq x$, composed only of prime factors $\leq y$. The Dickman-De Bruijn function $\psi(x, y)$ is the cardinality of this set. It turns out that the computation of $\psi(x, y)$ is equivalent to compute the number of integral points in an n-dimensional tetrahedron $\triangle(a_1, a_2, \cdots, a_n)$ with real vertices $(a_1, 0, \cdots, 0), \cdots, (0, \cdots, 0, a_n)$. Let $p_1 < p_2 < \cdots < p_n$ denotes the primes up to y. It is clear that $p_1^{l_1} p_2^{l_2} \cdots p_n^{l_n} \leq x$ if and only if $l_1 \log p_1 + l_2 \log p_2 + \cdots + l_n \log p_n \leq \log x$. Therefore, $\psi(x, y)$ is precisely the number Q_n of (integer) lattice points inside the n-dimensional tetrahedron (1.1) with $a_i = \frac{\log x}{\log p_i}$, $1 \leq i \leq n$.

The general problem of counting the number Q_n has been a challenging problem for many years. Tremendous researches have been putting to develop an exact formula when $a_1, \cdots, a_n$ are positive integers (see [2],[1],[7],[14]). Mordell gave a formula for Q_3, expressed in terms of three Dedekind sums, in the case that a_1, a_2, and a_3 are pairwise relatively prime [21]; Pommersheim extended the formula for Q_3 to arbitrary a_1, a_2, and a_3 using toric varieties [22] and so forth. Meanwhile, the problem of counting the number of integral points in an n-dimensional tetrahedron with real vertices is a classical subject which has attracted a lot of famous mathematicians. Also from the view of estimating the Dickman-De Bruijn function, a_i, $1 \leq i \leq n$, are not always integers. Hardy and Littlewood wrote several papers that have been applied on Diophantine approximation ([11], [12], [13]). A more general approximation of Q_n was obtained by D. C. Spencer [23], [24] via complex function-theoretic methods.

According to Granville [9], an upper polynomial estimate of P_n is a key topic in number theory. Such an estimate could be applied to finding large gaps between primes, to Waring's problem, to primality testing and factoring algorithms, and to bounds for the least prime k-th power residues and non-residues (mod n). Granville ([9]) obtained the following estimate

$$P_n \leq \frac{1}{n!} a_1 a_2 ... a_n \tag{1.3}$$

This estimate of $P_{(a_1, a_2, ..., a_n)}$ given by (1.3) is interesting, but not strong enough to be useful, particularly when many of the a_i's are small [9]. In geometry and singularity theory, estimating P_n for real right-angled simplices is related to the Durfee Conjecture [27]. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : f(z_1, \cdots, z_n) = 0\}$. The Milnor number of the singularity $(V, 0)$ is defined as

$$\mu = \dim \mathbb{C}\{z_1, \cdots, z_n\}/(f_{z_1}, \cdots, f_{z_n})$$

the geometric genus p_g of $(V, 0)$ is defined as

$$p_g = \dim H^{n-2}(M, \Omega^{n-1})$$

where M is a resolution of V and Ω^{n-1} is the sheaf of germs of holomorphic $n-1$ forms on M. In 1978, Durfee [8] made the following conjecture:

Durfee Conjecture. $n!p_g \le \mu$ with equality only when $\mu = 0$.

If $f(z_1, \cdots, z_n)$ is a weighted homogeneous polynomial of type $(a_1, a_2, \cdots, a_n)$ with an isolated singularity at the origin, Milnor and Orlik [20] proved that $\mu = (a_1 - 1)(a_2 - 1)\cdots(a_n - 1)$. On the other hand, Merle and Teissier [19] showed that $p_g = P_n$. Finding a sharp estimate of P_n will lead to a resolution of the Durfee Conjecture.

Starting from early 1990's, the authors of [16], [26] and [28] tried to get sharp upper estimates of P_n where a_i are positive real numbers. They were successful for $n = 3,\ 4$, and 5:

$$
\begin{aligned}
3!P_3 &\le f_3 = a_1a_2a_3 - (a_1a_2 + a_1a_3 + a_2a_3) + a_1 + a_2 \\
4!P_4 &\le f_4 = a_1a_2a_3a_4 - \frac{3}{2}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) \\
&\qquad + \frac{11}{3}(a_1a_2 + a_1a_3 + a_2a_3) - 2(a_1 + a_2 + a_3) \\
5!P_5 &\le f_5 = a_1a_2a_3a_4a_5 - 2(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5) \\
&\qquad + \frac{35}{4}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) \\
&\qquad - \frac{50}{6}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_2a_5) + 6(a_1 + a_2 + a_3 + a_4).
\end{aligned}
$$

They then proposed a general conjecture:

Conjecture 1.1 (Granville-Lin-Yau (GLY) Conjecture) Let P_n = number of element of set
$\left\{(x_1, x_2, ...x_n) \in \mathbf{Z}_+^n; \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \le 1\right\}$. Let $n \ge 3$,
(1) Sharp Estimate: if $a_1 \ge a_2 \ge ... \ge a_n \ge n-1$, then

$$
n!P_n \le f_n := A_0^n + \frac{s(n, n-1)}{n}A_1^n + \sum_{l=1}^{n-2} \frac{s(n, n-1-l)}{\binom{n-1}{l}} A_l^{n-1}, \tag{1.4}
$$

where $s(n,k)$ is the Stirling number of the first kind defined by generating function:

$$
x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k,
$$

and A_k^n is defined as

$$
A_k^n = (\prod_{i=1}^n a_i)(\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}}),
$$

for $k = 1, 2, \cdots, n-1$. Equality holds if and only if $a_1 = a_2 = ... = a_n$ = integer.
(2) Weak Estimate: If $a_1 \ge a_2 \ge ... \ge a_n > 1$

$$
n!P_n < q_n := \prod_{i=1}^n (a_i - 1). \tag{1.5}
$$

These estimates are all polynomials of a_i. They are sharp because the equality holds true if and only if all a_i take the same integer. In ([15],[16],[26],[28]) the authors showed that (1.5) holds for $3 \leq n \leq 5$. The sharp estimate conjecture was first formulated in [17]. In private communication to the third author, Granville formulated this sharp estimated conjecture independently after reading [15]. Again, the sharp GLY Conjecture has been proven individually for $n = 3, 4, 5$ by [27], [28] and [16] respectively. It has also been proven generally for $n \leq 6$ in [25]. However, for $n = 7$, a counterexample to the conjecture has been given.

Counterexample to GLY Conjecture Take $n = 7$. Let $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 2000$ and $a_7 = 6.09$. Consider the following 7-dimensional tetrahedron: $x_i > 0, 1 \leq i \leq 7$,

$$\frac{x_1}{2000} + \frac{x_2}{2000} + \frac{x_3}{2000} + \frac{x_4}{2000} + \frac{x_5}{2000} + \frac{x_6}{2000} + \frac{x_7}{6.09} \leq 1.$$

P_7 has been computed to be $3.9656226290532420 \times 10^{16}$. Meanwhile, $f_7 = 1.99840413 \times 10^{20}$ when $a_1 = a_2 = ... = a_6 = 2000$, $a_7 = 6.09$. Thus,

$$f_7 - 7!P_7 = -2.69675 \times 10^{16}.$$

This implies that the sharp estimate of GLY Conjecture fails in the case $n = 7$.

The breakthrough in the subject is the following theorem by Yau and Zhang [29] which states that the weak GLY conjecture holds for all $n \geq 3$.

Theorem 1.1 (Yau-Zhang [29]) *For $n \geq 3$, let $a_1 \geq a_2 \geq \cdots \geq a_n > 1$ be real numbers. Let P_n be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1$, i.e.*

$$P_n = \sharp\{(x_1, x_2, \cdots, x_n) \in \mathbb{Z}_+^n : \ \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1\},$$

where $\mathbb{Z}_+$ is the set of positive integers. Then

$$n!P_n \leq (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)$$

and the equality holds if and only if $a_n = 1$.

Theorem 1.1 above implies Durfee Conjecture for weighted homogeneous singularities. However, the Yau-Zhang estimate is not sharp. It is not good enough to characterize the homogeneous polynomial with isolated singularity. In order to do that, the third author made the following conjecture in 1995.

Conjecture 1.2 (Yau Geometric Conjecture) Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a weighted homogeneous polynomial with isolated critical points at the origin. Let μ, P_g and ν be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z : f(z) = 0\}$. Then

$$\mu - h(\nu) \geq (n + 1)!P_g, \tag{1.6}$$

where $h(\nu) = (\nu - 1)^{n+1} - \nu(\nu - 1) \cdots (\nu - n)$, and equality holds if and only if f is a homogeneous polynomial.

The Yau Geometric Conjecture was answered affirmatively for $n = 3, 4, 5$ by [27], [16] and [3] respectively.

In order to overcome the difficulty that the GLY sharp estimate conjecture is only true if a_n is larger than $y(n)$, a positive integer depending on n, the third author proposes to prove a new sharp polynomial estimate conjecture which is motivated from the Yau Geometric Conjecture. The importance of this conjecture is that we only need $a_n > 1$ and hence the conjecture will give a sharp upper estimate of the Dickman-De Bruijn function $\psi(x, y)$.

Conjecture 1.3 *Assume that $a_1 \geq a_2 \geq \cdots \geq a_n > 1$, $n \geq 3$ and let P_n = number of element of set $\{(x_1, x_2, ...x_n) \in \boldsymbol{Z}_+^n; \frac{x_1}{a_1} + \frac{x_2}{a_2} + ... + \frac{x_n}{a_n} \leq 1\}$. If $P_n > 0$, then*

$$n!P_n \leq (a_1 - 1)(a_2 - 1) \cdots (a_n - 1) - (a_n - 1)^n + a_n(a_n - 1) \cdots (a_n - (n - 1)) \tag{1.7}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_n$ = integer.

Obviously, there is an intimate relation between the Yau Geometric Conjecture (1.6) and the number theoretic conjecture (1.7). Recall that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a weighted homogeneous polynomial with isolated singularity at the origin, then the multiplicity ν of f at the origin is given by $\inf\{n \in \mathbb{Z}_+ : n \geq \inf\{w_1, \cdots, w_n\}\}$, where w_i is the weight of x_i. Notice that in general, w_i is only a rational number. In case the minimal weight is an integer, then the Yau Geometric Conjecture (1.6) and the number theoretic conjecture (1.7) are the same. In general, these two conjectures do not imply each other, although they are intimately related.

The number theoretic conjecture (1.7) is much sharper than the weak GLY conjecture (1.5). The estimate in (1.7) is optimal in the sense that the equality occurs precisely when $a_1 = a_2 = \cdots = a_n$ = integer. Moreover, the sharp GLY conjecture (1.4) does not hold for $n = 7$ as the counterexample shows. However, the number theoretic conjecture (1.7) does hold for this example.

By the previous works of Xu and Yau [26], [28], it was shown that the number theoretic conjecture is true for $n = 3$. $n = 4$ has been shown in our previous work [18]. The purpose of this paper is to prove that the number theoretic conjecture is true for $n = 5$. The basic strategy of proving $n = 4$ and $n = 5$ are the same. But the feasibility of the strategy has been challenged, even if the dimension has only been increased by 1. As we will see in our proof, the number of subcases has been increased from 4 (when $n = 4$) to 11 (when $n = 5$). Showing subcases one by one will absolutely cause tremendous involved computations. And it is tedious to our readers. In this paper, we, based on the intrinsic observation, simplify 11 subcases into 5 major classes ($k = 1, 2, 3, 4$ and $a_5 \geq 5$), and modify the former 4 classes with delicate analysis of A_i's domain, where $A_i = a_i(1 - \frac{k}{a_5})$, $i = 1, 2, 3, 4$ to deal with the subcases one by one. Furthermore, we give an explicit formula for the estimate of Dickman-De Bruijn function $\psi(x, y)$, when $5 \leq y < 13$. Mathematica 4.0 is adopted to do some involved computations. The following are our main theorems.

Theorem 1.2 (Number theoretic conjecture for $n = 5$) *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 > 1$ be real numbers. Let P_5 be the number of positive integral solutions of $\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3}+\frac{x_4}{a_4}+\frac{x_5}{a_5} \leq 1$, i.e.*

$$P_5 = \sharp \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}_+^5 : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} \leq 1 \right\},$$

where $\mathbb{Z}_+$ is the set of positive integers. If $P_5 > 0$, then

$$\begin{aligned} 120P_5 \leq & (a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1) - (a_5-1)^5 \\ & + a_5(a_5-1)(a_5-2)(a_5-3)(a_5-4) \end{aligned}$$

and the equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5$=integer. This can also be expressed as

$$\begin{aligned} 120P_5 \leq & a_1a_2a_3a_4a_5 - (a_1a_2a_3a_4 + a_1a_2a_4a_5 + a_2a_3a_4a_5 + a_1a_3a_4a_5) - 5a_5^4 \\ & + (a_1a_2a_3 + a_1a_2a_4 + a_1a_2a_5 + a_1a_3a_4 + a_1a_3a_5 + a_1a_4a_5 \\ & \quad + a_2a_3a_4 + a_2a_3a_5 + a_2a_4a_5 + a_3a_4a_5) + 25a_5^3 \\ & + (a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5 + a_2a_3 + a_2a_4 + a_2a_5 + a_3a_4 + a_3a_5 + a_4a_5) - 40a_5^2 \\ & - (a_1 + a_2 + a_3 + a_4) + 20a_5. \end{aligned} \tag{1.8}$$

Theorem 1.3 (Estimate of Dickman-De Bruijn function) *Let $\psi(x, y)$ be the Dickman-De Bruijn function. We have the following upper estimate for $5 \leq y < 13$:*

(i) when $5 \leq y < 7$ and $x > 5$, we have

$$\begin{aligned} \psi(x,y) \leq & \frac{1}{6}\{\frac{1}{\log 2 \log 3 \log 5}(\log x + \log 15)(\log x + \log 10)(\log x + \log 6) \\ & - \frac{1}{\log^3 5}[(\log x + \log 6)^3 \\ & - (\log x + \log 6 + \log 5)(\log x + \log 6)(\log x + \log 6 - \log 5)]\}; \end{aligned}$$

(ii) when $7 \leq y < 11$ and $x > 11$, we have

$$\begin{aligned} \psi(x,y) \leq & \frac{1}{24}\{\frac{1}{\log 2 \log 3 \log 5 \log 7}(\log x + \log 105)(\log x + \log 70) \\ & \cdot (\log x + \log 42)(\log x + \log 30) \\ & - \frac{1}{\log^4 7}[(\log x + \log 30)^4 \\ & - (\log x + \log 7 + \log 30)(\log x + \log 30) \\ & \cdot (\log x + \log 30 - \log 7)(\log x + \log 30 - 2\log 7)]\}; \end{aligned}$$

(iii) when $11 \le y < 13$ *and* $x > 13$, *we have*

$$\psi(x,y) \le \frac{1}{120}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11}(\log x + \log 1155)(\log x + \log 770)(\log x + \log 462)$$
$$\cdot (\log x + \log 330)(\log x + \log 210)$$
$$- \frac{1}{\log^5 11}[(\log x + \log 210)^5$$
$$- (\log x + \log 11 + \log 210)(\log x + \log 210)(\log x + \log 210 - \log 11)$$
$$\cdot (\log x + \log 210 - 2\log 11)(\log x + \log 210 - 3\log 11)]\}.$$

2 Proof of Theorems

2.1 Proof of Theorem 1.2

Our strategy is to divide our proof of the main theorem into five cases:
(1) $a_5 \ge 5$;
(2) $5 > a_5 > 4$;
(3) $4 \ge a_5 > 3$;
(4) $3 \ge a_5 > 2$;
(5) $2 \ge a_5 > 1$.

To prove case (1), we only need to notice the main theorem in [16].

Theorem 2.4 ([16]) Let $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5 \ge 4$ be real numbers and P_5 be the number of positive integral points satisfying

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} \le 1.$$

Then

$$\begin{aligned} 120P_5 \le & a_1a_2a_3a_4a_5 - 2(a_1a_2a_3a_4 + a_1a_2a_4a_5 + a_2a_3a_4a_5 + a_1a_3a_4a_5 + a_1a_2a_3a_5) \\ & + \frac{35}{4}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) \\ & - \frac{50}{6}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4) + 6(a_1 + a_2 + a_3 + a_4), \end{aligned} \tag{2.9}$$

and the equality is attained if and only if $a_1 = a_2 = a_3 = a_4 = a_5 =$ integer.

Case (1) is solved by showing that our sharp upper bound is larger than or equal to theirs, and the equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5$.

Lemma 2.1 *When $a_5 \geq 5$, R.H.S. of (1.8) $\geq$ R.H.S. of (2.9).*

Proof. Let $A_i = \frac{a_i}{a_5}$, $i = 1, 2, 3, 4$. From condition $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 > 1$, we have that $A_i \geq 1$, $i = 1, 2, 3, 4$. Now, subtract R.H.S. of (1.8) by R.H.S. of (2.9), and substitute a_i by A_i, $i = 1, 2, 3, 4$:

$$\begin{aligned}\triangle_1 &\triangleq \text{R.H.S. of (1.8)} - \text{R.H.S. of (2.9)} \\ &= A_1A_2A_3A_4a_5^4 + (A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4)(a_5^4 - \frac{31}{4}a_5^3) \\ &\quad + (A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4)(a_5^3 + \frac{22}{3}a_5^2) \\ &\quad + (A_1 + A_2 + A_3 + A_4)(-a_5^2 - 5a_5) + (-5a_5^4 + 25a_5^3 - 40a_5^2 + 20a_5) \qquad (2.10)\end{aligned}$$

The idea is to show that for all $a_5 \geq 5$, the minimum of $\triangle_1$ in $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$ and $\triangle_1 |_{A_1=A_2=A_3=A_4=1} = 0$, for all $a_5 \geq 5$. Note that $\triangle_1$ is symmetric with respect with A_1, A_2, A_3 and A_4.

$$\frac{\partial^4 \triangle_1}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} = a_5^4 > 0,$$

for $a_5 > 1$. It follows that $\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 for $a_5 > 1$, $A_4 \geq 1$. Hence the minimum of $\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_3}$ occurs at $A_4 = 1$,

$$\left.\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=1} = \left.\left[A_4a_5^4 + (a_5^4 - \frac{31}{4}a_5^3)\right]\right|_{A_4=1} = a_5^3(2a_5 - \frac{31}{4}) > 0,$$

for $a_5 > \frac{31}{8}$. It follows that $\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_3} > 0$ for $A_4 \geq 1$, $a_5 > \frac{31}{8}$. Note that $\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_4} > 0$, for $A_3 \geq 1$, $a_5 > \frac{31}{8}$. Moreover, we have $\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $a_5 > \frac{31}{8}$. The minimum of $\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2}$ occurs at $A_3 = A_4 = 1$,

$$\begin{aligned}\left.\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2}\right|_{A_3=A_4=1} &= \left.\left[A_3A_4a_5^4 + (A_3 + A_4)(a_5^4 - \frac{31}{4}a_5^3) + (a_5^3 + \frac{22}{3}a_5^2)\right]\right|_{A_3=A_4=1} \\ &= 3a_5^4 - \frac{29}{2}a_5^3 + \frac{22}{3}a_5^2 = a_5^2(3a_5^2 - \frac{29}{2}a_5 + \frac{22}{3}) > 0,\end{aligned}$$

for $a_5 \geq 5$, since the largest solution to $3a_5^2 - \frac{29}{2}a_5 + \frac{22}{3} = 0$ is around 4.26. It follows that $\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2} > 0$, for $A_3 \geq 1$, $A_4 \geq 1$ and $a_5 \geq 5$. From the property that $\frac{\partial \triangle_1}{\partial A_1}$ is symmetric with respect to A_2, A_3 and A_4, we also get $\frac{\partial \triangle_1}{\partial A_1 \partial A_3} > 0$, for $A_2 \geq 1$, $A_4 \geq 1$ and $a_5 \geq 5$ and $\frac{\partial \triangle_1}{\partial A_1 \partial A_4} > 0$, for $A_2 \geq 1$, $A_3 \geq 1$ and $a_5 \geq 5$. Therefore, we have $\frac{\partial \triangle_1}{\partial A_1}$ is an increasing function of A_2, A_3 and A_4 for $A_2 \geq 1$, $A_3 \geq 1$, $A_4 \geq 1$ and $a_5 \geq 5$. Hence the minimum of $\frac{\partial \triangle_1}{\partial A_1}$ occurs at $A_2 = A_3 = A_4 = 1$,

$$\begin{aligned}\frac{\partial \triangle_1}{\partial A_1}\bigg|_{A_2=A_3=A_4=1} &= \Big[A_2A_3A_4a_5^4 + (A_2A_3 + A_2A_4 + A_3A_4)(a_5^4 - \frac{31}{4}a_5^3) \\ &\quad +(A_2 + A_3 + A_4)(a_5^3 + \frac{22}{3}a_5^2) + (-a_5^2 - 5a_5)\Big]\bigg|_{A_2=A_3=A_4=1} \\ &= 4a_5^4 - \frac{81}{4}a_5^3 + 21a_5^2 - 5a_5 = a_5(4a_5^3 - \frac{79}{4}a_5^2 + 21a_5 - 5) > 0,\end{aligned}$$

for $a_5 \geq 5$, since $4a_5^3 - \frac{81}{4}a_5^2 + 21a_5 - 5 \geq a_5(4a_5^2 - \frac{81}{4}a_5 + 20) = 4a_5[(a_5 - \frac{81}{32})^2 - \frac{1441}{1024}]$, and let $f(a_5) = (a_5 - \frac{81}{32})^2 - \frac{1441}{1024} > f(5) = \frac{75}{16} > 0$, for $a_5 \geq 5$. It follows that $\frac{\partial \triangle_1}{\partial A_1} > 0$, for $A_2 \geq 1$, $A_3 \geq 1$, $A_4 \geq 1$ and $a_5 \geq 5$. By the property that $\triangle_1$ is symmetric with respect to A_1, A_2, A_3 and A_4. We have the minimum of $\triangle_1$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$,

$$\triangle_1|_{A_1=A_2=A_3=A_4=1} = 0,$$

for $a_5 \geq 5$. Therefore, we have $\triangle_1 \geq 0$ when $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq 5$ and $\triangle_1 = 0$ if and only if $a_1 = a_2 = a_3 = a_4 = a_5$. The equality of (2.9) holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 =$ integer, so does the equality of (1.8). □

For case (2) to (5), we adopt the similar strategy: basically, we partition the 5-dimension tetrahedron into 4-dimension tetrahedra [25]. We have:

$$\begin{aligned}&\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{k}{a_5} \leq 1 \\ &\frac{x_1}{a_1(1 - \frac{k}{a_5})} + \frac{x_2}{a_2(1 - \frac{k}{a_5})} + \frac{x_3}{a_3(1 - \frac{k}{a_5})} + \frac{x_4}{a_4(1 - \frac{k}{a_5})} \leq 1,\end{aligned} \tag{2.11}$$

for $k = 1, \cdots, [a_5]$, where $[a_5]$ is the largest integer less than or equal to a_5. Let $P_4(k)$ be the number of positive integral solutions of (2.11). Then

$$P_5 = \sum_{k=1}^{[a_5]} P_4(k). \tag{2.12}$$

According to Theorem 1.1 in [18], if $P_4(k) > 0$, then we have

$$\begin{aligned}5!P_4(k) \leq &5[(a_1(1 - \frac{k}{a_5}) - 1)(a_2(1 - \frac{k}{a_5}) - 1)(a_3(1 - \frac{k}{a_5}) - 1)(a_4(1 - \frac{k}{a_5}) - 1) \\ &- (a_4(1 - \frac{k}{a_5}) - 1)^4 \\ &+ a_4(1 - \frac{k}{a_5})(a_4(1 - \frac{k}{a_5}) - 1)(a_4(1 - \frac{k}{a_5}) - 2)(a_4(1 - \frac{k}{a_5}) - 3)].\end{aligned}$$

Suppose there exists some k_0, $1 \le k_0 \le [a_5]$, which is the largest integer such that $P_4(k_0) > 0$ and $P_4(k) = 0$, for all $k_0 < k \le [a_5]$. In fact, the integer k_0 does exist due to the condition $P_5 > 0$. By (2.12), we have

$$\begin{aligned}
5!P_5 =&5!\sum_{k=1}^{k_0} P_4(k)\\
\le&5\sum_{k=1}^{k_0}[(a_1(1-\frac{k}{a_5})-1)(a_2(1-\frac{k}{a_5})-1)(a_3(1-\frac{k}{a_5})-1)(a_4(1-\frac{k}{a_5})-1)\\
&-(a_4(1-\frac{k}{a_5})-1)^4\\
&+a_4(1-\frac{k}{a_5})(a_4(1-\frac{k}{a_5})-1)(a_4(1-\frac{k}{a_5})-2)(a_4(1-\frac{k}{a_5})-3)]. \qquad (2.13)
\end{aligned}$$

In order to prove (1.8), it is sufficient to show that R.H.S. of (1.8) $\ge$ R.H.S. of (2.13). For case (2) to (5), the equality in (1.8) can't be attained by any chance. On the one hand, $P_5 > 0$ won't be satisfied if $a_1 = a_2 = a_3 = a_4 = a_5 < 5$. On the other hand, we could show that R.H.S. of (1.8) is strictly larger than R.H.S. of (2.13) in these cases. Therefore, no such $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5$ and $a_5 \in (1, 5)$ could make the equality in (1.8) happen.

Now, for case (5), there are two levels $k = 1$ and $k = 2$. It is easy to see that $P_4(2) = 0$. From the condition $P_5 > 0$, we know that the level $k = 1$ can't have no positive integral solution, i.e. $P_4(1) = P_5 > 0$. It is also implied that the smallest positive integral solution $(1, 1, 1, 1, 1)$ must be its solution, which gives that $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4} \le 1-\frac{1}{a_5} \triangleq \alpha$, $\alpha \in (0, \frac{1}{2}]$, since $a_5 \in (1, 2]$. And let $A_i = a_i\alpha$, $i = 1, 2, 3, 4$. Also notice that

$$A_1 \ge 4,\ A_2 \ge 3,\ A_3 \ge 2 \text{ and } A_4 \ge 1, \qquad (2.14)$$

since $\frac{1}{A_4} \le 1$, $\frac{2}{A_3} \le \frac{1}{A_3}+\frac{1}{A_4} \le 1$, $\frac{3}{A_2} \le \frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4} \le 1$ and $\frac{4}{A_1} \le \frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4} \le 1$. Here, (2.13) can be rewritten as

$$\begin{aligned}
5!P_5 = 5!P_4(1) \le&5[(A_1-1)(A_2-1)(A_3-1)(A_4-1)-(A_4-1)^4\\
&+A_4(A_4-1)(A_4-2)(A_4-3)]. \qquad (2.15)
\end{aligned}$$

To prove (1.8) in this case, it is sufficient to show that R.H.S. of (1.8) > R.H.S. of (2.15).

Lemma 2.2 *When $1 < a_5 \le 2$, R.H.S. of (1.8) > R.H.S. of (2.15).*

Proof. Substitute $a_i = \frac{A_i}{\alpha}$, $i = 1, 2, 3, 4$ and $a_5 = \frac{1}{1-\alpha}$ to R.H.S. of (1.8), subtract that by

R.H.S. of (2.15), and multiply $(1-\alpha)^4$, we get

$$\begin{aligned}\triangle_2 \triangleq & A_1A_2A_3A_4(\frac{1}{\alpha^3}-\frac{3}{\alpha^2}+\frac{3}{\alpha}-6+20\alpha-30\alpha^2+20\alpha^3-5\alpha^4)\\ &+(A_1A_2A_3+A_1A_2A_4+A_1A_3A_4+A_2A_3A_4)\\ &\qquad\cdot(-\frac{1}{\alpha^2}+\frac{3}{\alpha}+2-19\alpha+30\alpha^2-20\alpha^3+5\alpha^4)\\ &+(A_1A_2+A_1A_3+A_1A_4+A_2A_3+A_2A_4+A_3A_4)(\frac{1}{\alpha}-8+23\alpha-31\alpha^2+20\alpha^3-5\alpha^4)\\ &+(A_1+A_2+A_3)(4-17\alpha+27\alpha^2-19\alpha^3+5\alpha^4)\\ &+A_4^3(10-40\alpha+60\alpha^2-40\alpha^3+10\alpha^4)+A_4^2(-25+100\alpha-150\alpha^2+100\alpha^3-25\alpha^4)\\ &+A_4(14-57\alpha+87\alpha^2-59\alpha^3+15\alpha^4)+(-5\alpha+20\alpha^2-20\alpha^3)\end{aligned}$$

The idea is to show that for all $\alpha\in(0,\frac{1}{2}]$, the minimum of $\triangle_2$ in $A_1\geq 4$, $A_2\geq 3$, $A_3\geq 2$ and $A_4\geq 1$ occurs at $A_1=4$, $A_2=3$, $A_3=2$ and $A_4=1$ and $\triangle_2\mid_{A_1=4,A_2=3,A_3=2,A_4=1}>0$, for all $\alpha\in(0,\frac{1}{2}]$.

$$\begin{aligned}\frac{\partial^4\triangle_2}{\partial A_1\partial A_2\partial A_3\partial A_4} =&\frac{1}{\alpha^3}-\frac{3}{\alpha^2}+\frac{3}{\alpha}-6+20\alpha-30\alpha^2+20\alpha^3-5\alpha^4\\ =&\frac{1}{\alpha^3}(1-\alpha)^3(1-5\alpha^3+5\alpha^4)>0,\end{aligned}$$

for $\alpha\in(0,1)$. In fact, let $f(\alpha)\triangleq 1-5\alpha^3+5\alpha^4$. And $f'(\alpha)=20\alpha^3-15\alpha^2=5\alpha^2(4\alpha-3)$, which implies that $f'(\alpha)\leq 0$, for $\alpha\in(0,\frac{3}{4}]$, while $f'(\alpha)>0$, for $\alpha\in(\frac{3}{4},1)$. Thus, $\min_{\alpha\in(0,1)} f(\alpha)=f(\frac{3}{4})=\frac{121}{256}>0$. Therefore, $f(\alpha)>0$, for $\alpha\in(0,1)$. It follows that $\frac{\partial^3\triangle_2}{\partial A_1\partial A_2\partial A_3}$ is an increasing function of A_4 for $\alpha\in(0,1)$, $A_4\geq 1$. Hence the minimum of $\frac{\partial^3\triangle_2}{\partial A_1\partial A_2\partial A_3}$ occurs at $A_4=1$,

$$\begin{aligned}\left.\frac{\partial^3\triangle_2}{\partial A_1\partial A_2\partial A_3}\right|_{A_4=1} =&\Bigg[A_4(\frac{1}{\alpha^3}-\frac{3}{\alpha^2}+\frac{3}{\alpha}-6+20\alpha-30\alpha^2+20\alpha^3-5\alpha^4)\\ &+(-\frac{1}{\alpha^2}+\frac{3}{\alpha}+2-19\alpha+30\alpha^2-20\alpha^3+5\alpha^4)\Bigg]\Bigg|_{A_4=1}\\ =&\frac{1}{\alpha^3}(\alpha-1)^4>0,\end{aligned}$$

for $\alpha\in(0,1)$. It follows that $\frac{\partial^3\triangle_2}{\partial A_1\partial A_2\partial A_3}>0$ for $A_4\geq 1$, $\alpha\in(0,1)$. Note that $\frac{\partial^2\triangle_2}{\partial A_1\partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3\triangle_1}{\partial A_1\partial A_2\partial A_4}>0$, for $A_3\geq 1$, $\alpha\in(0,1)$. Moreover, we have $\frac{\partial^2\triangle_1}{\partial A_1\partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3\geq A_4\geq 1$ and $\alpha\in(0,1)$. The minimum of $\frac{\partial^2\triangle_2}{\partial A_1\partial A_2}$ occurs at $A_3=A_4=1$,

$$\begin{aligned}
\frac{\partial^2\triangle_2}{\partial A_1\partial A_2}\Big|_{A_3=A_4=1} &= \Big[A_3A_4(\frac{1}{\alpha^3}-\frac{3}{\alpha^2}+\frac{3}{\alpha}-6+20\alpha-30\alpha^2+20\alpha^3-5\alpha^4) \\
&\quad + (A_3+A_4)(-\frac{1}{\alpha^2}+\frac{3}{\alpha}+2-19\alpha+30\alpha^2-20\alpha^3+5\alpha^4) \\
&\quad +(\frac{1}{\alpha}-8+23\alpha-31\alpha^2+20\alpha^3-5\alpha^4)\Big]\Big|_{A_3=A_4=1} \\
&= -\frac{1}{\alpha^3}(-1+\alpha)^5>0,
\end{aligned}$$

for $\alpha\in(0,1)$. It follows that $\frac{\partial^2\triangle_2}{\partial A_1\partial A_2}>0$, for $A_3\geq A_4\geq 1$ and $\alpha\in(0,1)$. From the property that $\frac{\partial\triangle_2}{\partial A_1}$ is symmetric with respect to A_2, A_3 and A_4, we also get $\frac{\partial^2\triangle_2}{\partial A_1\partial A_3}>0$, for $A_2\geq A_4\geq 1$ and $\alpha\in(0,1)$ and $\frac{\partial^2\triangle_2}{\partial A_1\partial A_4}>0$, for $A_2\geq A_3\geq 1$ and $\alpha\in(0,1)$. Therefore, we have $\frac{\partial\triangle_2}{\partial A_1}$ is an increasing function of A_2, A_3 and A_4 for $A_2\geq A_3\geq A_4\geq 1$ and $\alpha\in(0,1)$. Hence the minimum of $\frac{\partial\triangle_2}{\partial A_1}$ occurs at $A_2=A_3=A_4=1$,

$$\begin{aligned}
\frac{\partial\triangle_2}{\partial A_1}\Big|_{A_2=A_3=A_4=1} &= \Big[A_2A_3A_4(\frac{1}{\alpha^3}-\frac{3}{\alpha^2}+\frac{3}{\alpha}-6+20\alpha-30\alpha^2+20\alpha^3-5\alpha^4) \\
&\quad + (A_2A_3+A_2A_4+A_3A_4)(-\frac{1}{\alpha^2}+\frac{3}{\alpha}+2-19\alpha+30\alpha^2-20\alpha^3+5\alpha^4) \\
&\quad + (A_2+A_3+A_4)(\frac{1}{\alpha}-8+23\alpha-31\alpha^2+20\alpha^3-5\alpha^4) \\
&\quad +(4-17\alpha+27\alpha^2-19\alpha^3+5\alpha^4)\Big]\Big|_{A_2=A_3=A_4=1} \\
&= \frac{1}{\alpha^3}(-1+\alpha)^6>0,
\end{aligned}$$

for $\alpha\in(0,1)$. It follows that $\frac{\partial\triangle_2}{\partial A_1}>0$, for $A_2\geq A_3\geq A_4\geq 1$ and $\alpha\in(0,1)$. By the property that $\triangle_2$ is symmetric with respect to A_1, A_2 and A_3. We also have $\frac{\partial\triangle_2}{\partial A_2}>0$, for $A_1\geq A_3\geq A_4\geq 1$ and $\frac{\partial\triangle_2}{\partial A_3}>0$, for $A_1\geq A_2\geq A_4\geq 1$. Meanwhile,

$$\frac{\partial^3\triangle_2}{\partial A_4^3}=10(-1+\alpha)^4>0,$$

for $\alpha\in(0,1)$. It follows that $\frac{\partial^2\triangle_2}{\partial A_4^2}$ is an increasing function of A_4, for $A_4\geq 1$ and $\alpha\in(0,1)$. Thus, the minimum of $\frac{\partial^2\triangle_2}{\partial A_4^2}$ occurs at $A_4=1$,

$$\begin{aligned}
\frac{\partial^2\triangle_2}{\partial A_4^2}\Big|_{A_4=1} &= \big[6A_4(10-40\alpha+60\alpha^2-40\alpha^3+10\alpha^4) \\
&\quad +2(-25+100\alpha-150\alpha^2+100\alpha^3-25\alpha^4)\big]\Big|_{A_4=1} \\
&= 10(-1+\alpha)^4>0,
\end{aligned}$$

for $\alpha \in (0,1)$. It follows that $\frac{\partial^2 \triangle_2}{\partial A_4^2} > 0$, for $A_4 \geq 1$ and $\alpha \in (0,1)$. Thus, $\frac{\partial \triangle_2}{\partial A_4}$ is an increasing function of A_4, for $A_4 \geq 1$ and $\alpha \in (0,1)$. Moreover, it's an increasing function with respect to A_1, A_2, A_3 and A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$, $\alpha \in (0,1)$, since $\frac{\partial \triangle_2}{\partial A_4}$ is symmetric with respect to A_1, A_2 and A_3. Take condition (2.14) into consideration, the minimum of $\frac{\partial \triangle_2}{\partial A_4}$ occurs at $A_1 = 4,\ A_2 = 3,\ A_3 = 2,\ A_4 = 1$,

$$\begin{aligned}
\frac{\partial \triangle_2}{\partial A_4}\Big|_{A_1=4,A_2=3,A_3=2,A_4=1} &= \Big[A_1A_2A_3(\frac{1}{\alpha^3} - \frac{3}{\alpha^2} + \frac{3}{\alpha} - 6 + 20\alpha - 30\alpha^2 + 20\alpha^3 - 5\alpha^4) \\
&\quad + (A_1A_2 + A_1A_3 + A_2A_3) \\
&\qquad \cdot (-\frac{1}{\alpha^2} + \frac{3}{\alpha} + 2 - 19\alpha + 30\alpha^2 - 20\alpha^3 + 5\alpha^4) \\
&\quad + (A_1 + A_2 + A_3)(\frac{1}{\alpha} - 8 + 23\alpha - 31\alpha^2 + 20\alpha^3 - 5\alpha^4) \\
&\quad + (14 - 57\alpha + 87\alpha^2 - 59\alpha^3 + 15\alpha^4) \\
&\quad + 3A_4^2(10 - 40\alpha + 60\alpha^2 - 40\alpha^3 + 10\alpha^4) \\
&\quad + 2A_4(-25 + 100\alpha - 150\alpha^2 + 100\alpha^3 - 25\alpha^4)\Big]\Big|_{A_1=4,A_2=3,A_3=2,A_4=1} \\
&= -\frac{1}{\alpha^3}(-1+\alpha)^3(24 - 26\alpha + 9\alpha^2 - 41\alpha^3 + 40\alpha^4) > 0,
\end{aligned}$$

for $\alpha \in (0, \frac{4}{5})$. In fact, let $g(\alpha) \triangleq 24 - 26\alpha + 9\alpha^2 - 41\alpha^3 + 40\alpha^4$. And $g'(\alpha) = -26 + 18\alpha - 123\alpha^2 + 160\alpha^3 < -8\alpha - 123\alpha^2 + 160\alpha = \alpha(-8 - 123\alpha + 160\alpha^2)$. Let $h(\alpha) = -8 - 123\alpha + 160\alpha^2 = 160(\alpha - \frac{123}{320})^2 - \frac{20249}{640} < 0$, for $\alpha \in (0, \frac{4}{5})$, since $h(0) = -8$ and $h(\frac{4}{5}) = -4$, $max_{\alpha \in (0,\frac{4}{5})} h(\alpha) = -4 < 0$. Thus, $g'(\alpha) < 0$ for $\alpha \in (0, \frac{4}{5})$. It follows that $g(\alpha)$ is a decreasing function in $\alpha \in (0, \frac{4}{5})$. Moreover, $g(\alpha) \geq g(\frac{4}{5}) = \frac{544}{125} > 0$, for $\alpha \in (0, \frac{4}{5})$. It follows that $\frac{\partial \triangle_2}{\partial A_4} > 0$, for $A_1 \geq 4,\ A_2 \geq 3,\ A_3 \geq 2,\ A_4 \geq 1$ and $\alpha \in (0, \frac{4}{5})$. Therefore, $\triangle_2$ is an increasing function of A_1, A_2, A_3 and A_4, for $A_1 \geq 4,\ A_2 \geq 3,\ A_3 \geq 2,\ A_4 \geq 1$ and $\alpha \in (0, \frac{4}{5})$. Thus, the minimum of $\triangle_2$ occurs at $A_1 = 4,\ A_2 = 3,\ A_3 = 2,\ A_4 = 1$,

$$\triangle_2|_{A_1=4,A_2=3,A_3=2,A_4=1} = -\frac{1}{\alpha^3}(-24 + 122\alpha - 257\alpha^2 + 289\alpha^3 - 180\alpha^4 + 45\alpha^5 + 10\alpha^6) > 0,$$

for $\alpha \in (0, \frac{1}{2}]$. Indeed, let $f(\alpha) = -24 + 122\alpha - 257\alpha^2 + 289\alpha^3 - 180\alpha^4 + 45\alpha^5 + 10\alpha^6$. And $f^{(3)}(\alpha) = 1734 - 4320\alpha + 2700\alpha^2 + 1200\alpha^3 > 1734 - 4320\alpha + 2700\alpha^2 = 2700(\alpha - \frac{4}{5})^2 + 6 > 0$, for $\alpha \in (0, \frac{1}{2}]$. Thus, $f''(\alpha)$ is increasing in $\alpha \in (0, \frac{1}{2}]$ and $f''(\alpha) < f''(\frac{1}{2}) = -\frac{223}{4} < 0$. So $f'(\alpha)$ is decreasing in $\alpha \in (0, \frac{1}{2}]$ and $f'(\alpha) > f'(\frac{1}{2}) = \frac{123}{16} > 0$. It implies that $f(\alpha)$ is increasing in $\alpha \in (0, \frac{1}{2}]$ and $f(\alpha) < f(\frac{1}{2}) = -\frac{13}{16} < 0$. Therefore, $f(\alpha) < 0$, for $\alpha \in (0, \frac{1}{2}]$. It follows that $\triangle_2 > 0$ for $A_1 \geq 4,\ A_2 \geq 3,\ A_3 \geq 2,\ A_4 \geq 1$ and $\alpha \in (0, \frac{1}{2}]$. □

For case (4), there are three levels $k = 1$, $k = 2$ and $k = 3$. Also it is easy to see that $P_4(3) = 0$. The condition $P_5 > 0$ guarantees that $P_4(1) > 0$, but the positivity of $P_4(2)$ is unknown. Therefore, we split this case into following two subcases:

(4a) $P_4(2) = 0$ (i.e. $k_0 = 1$ in (2.13));

(4b) $P_4(2) > 0$ (i.e. $k_0 = 2$ in (2.13)).

For subcase (4a), the proof is actually the same as case (5). As we know, in this case $P_5 = P_4(1) > 0$, thus $(1,1,1,1,1)$ is the smallest positive integral solution, i.e. $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \leq 1 - \frac{1}{a_5} \triangleq \alpha$, $\alpha \in (\frac{1}{2}, \frac{2}{3}]$, since $a_5 \in (2,3]$. The new range of α helps us to improve the condition (2.14) to the following one:

$$A_1 \geq 4, \ A_2 \geq 3, \ A_3 \geq A_4 \geq \frac{\alpha}{1-\alpha}, \tag{2.16}$$

since $A_i = a_i\alpha \geq a_5\alpha = \frac{\alpha}{1-\alpha}$. With $\alpha \in (\frac{1}{2}, \frac{2}{3}]$, it is easy to check that $1 < \frac{\alpha}{1-\alpha} \leq 2$. Therefore, it is sufficient to show that $\triangle_2 > 0$, for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq \frac{\alpha}{1-\alpha}$ and $\alpha \in (\frac{1}{2}, \frac{2}{3}]$. Notice that in the proof of Lemma 2.2 all the partial derivatives of A_i are positive for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\alpha \in (0, \frac{4}{5})$ until the last step to compute $\triangle_2|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ only for $\alpha \in (0, \frac{1}{2}]$. Thus, we need to take condition (2.16) instead of the rough estimate (2.14) of A_i, $i = 1,2,3,4$.

$$\triangle_2|_{A_1=4,A_2=3,A_3=2,A_4=\frac{\alpha}{1-\alpha}} = \frac{1}{\alpha}(24 - 39\alpha - 82\alpha^2 + 223\alpha^3 - 152\alpha^4 + 20\alpha^5) > 0,$$

for $\alpha \in (\frac{1}{2}, \frac{2}{3}]$. In fact, let $f(\alpha) \triangleq 24 - 39\alpha - 82\alpha^2 + 223\alpha^3 - 152\alpha^4 + 20\alpha^5$, then $f''(\alpha) = -162 + 1338\alpha - 1824\alpha^2 + 400\alpha^3 > -162 + 1338\alpha - 1624\alpha^2 = -1624(\alpha - \frac{669}{1624})^2 + \frac{184473}{1624} \triangleq g(\alpha)$, and $g(\alpha) > g(\frac{2}{3}) = \frac{74}{9} > 0$. So $f''(\alpha) > 0$, for $\alpha \in (\frac{1}{2}, \frac{2}{3}]$. Thus, $f'(\alpha)$ is increasing in $\alpha \in (\frac{1}{2}, \frac{2}{3}]$, i.e. $f'(\alpha) < f'(\frac{2}{3}) = -\frac{815}{81} < 0$. So $f'(\alpha) < 0$ for $\alpha \in (\frac{1}{2}, \frac{2}{3}]$. It follows that $f(\alpha)$ is decreasing in $\alpha \in (\frac{1}{2}, \frac{2}{3}]$, and $f(\alpha) > f(\frac{2}{3}) = \frac{166}{243} > 0$. Therefore, $f(\alpha) > 0$, for $\alpha \in (\frac{1}{2}, \frac{2}{3}]$.

For subcase (4b), $P_4(2) > 0$ which implies that $(1,1,1,1,2)$ is the smallest positive integer solution to the level $k = 2$. So we have $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \leq 1 - \frac{2}{a_5} \triangleq \beta$, $\beta \in (0, \frac{1}{3}]$, since $a_5 \in (2,3]$. Let $A_i = a_i\beta$, $i = 1,2,3,4$. Also notice that condition (2.14) still holds here. (2.13) can be written as

$$\begin{aligned}
5!P_5 =& 5!(P_4(1) + P_4(2)) \\
\leq& 5[(A_1\frac{1+\beta}{2\beta} - 1)(A_2\frac{1+\beta}{2\beta} - 1)(A_3\frac{1+\beta}{2\beta} - 1)(A_4\frac{1+\beta}{2\beta} - 1) \\
&- (A_4\frac{1+\beta}{2\beta} - 1)^4 + A_4\frac{1+\beta}{2\beta}(A_4\frac{1+\beta}{2\beta} - 1)(A_4\frac{1+\beta}{2\beta} - 2)(A_4\frac{1+\beta}{2\beta} - 3) \\
&+ (A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1) - (A_4 - 1)^4 + A_4(A_4 - 1)(A_4 - 2)(A_4 - 3)].
\end{aligned} \tag{2.17}$$

It is sufficient to show that R.H.S. of (1.8) is strictly larger than R.H.S. of (2.17).

Lemma 2.3 *When $2 < a_5 \leq 3$, R.H.S. of (1.8) $>$ R.H.S. of (2.17).*

Proof. Substitute $a_i = \frac{A_i}{\beta}$, $i = 1,2,3,4$ and $a_5 = \frac{2}{1-\beta}$ to R.H.S. of (1.8), subtract that by

R.H.S. of (2.17), and multiply $(1-\beta)^5\beta^4$, we get

$$\begin{aligned}
\triangle_3 \triangleq & A_1A_2A_3A_4 \\
& \quad \cdot (\frac{11}{16} - \frac{43}{16}\beta + \frac{13}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{79}{8}\beta^4 + \frac{223}{8}\beta^5 - \frac{195}{4}\beta^6 + \frac{195}{4}\beta^7 - \frac{405}{16}\beta^8 + \frac{85}{16}\beta^9) \\
& + (A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4) \\
& \quad \cdot (-\frac{3}{8}\beta + \frac{7}{4}\beta^2 - \frac{13}{4}\beta^3 + \frac{27}{4}\beta^4 - 22\beta^5 + \frac{181}{4}\beta^6 - \frac{195}{4}\beta^7 + \frac{105}{4}\beta^8 - \frac{45}{8}\beta^9) \\
& + (A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4) \\
& \quad \cdot (-\frac{1}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{17}{4}\beta^4 + \frac{83}{4}\beta^5 - \frac{187}{4}\beta^6 + \frac{209}{4}\beta^7 - \frac{115}{4}\beta^8 + \frac{25}{4}\beta^9) \\
& + (A_1 + A_2 + A_3)(\frac{3}{2}\beta^3 - 2\beta^4 - \frac{29}{2}\beta^5 + 48\beta^6 - \frac{119}{2}\beta^7 + 34\beta^8 - \frac{15}{2}\beta^9) \\
& + A_4^3(\frac{5}{4}\beta - \frac{5}{2}\beta^2 - \frac{5}{2}\beta^3 + \frac{35}{2}\beta^4 - 50\beta^5 + \frac{185}{2}\beta^6 - \frac{195}{2}\beta^7 + \frac{105}{2}\beta^8 - \frac{45}{4}\beta^9) \\
& + A_4^2(-\frac{25}{4}\beta^2 + \frac{75}{4}\beta^3 - \frac{125}{4}\beta^4 + \frac{375}{4}\beta^5 - \frac{875}{4}\beta^6 + \frac{1025}{4}\beta^7 - \frac{575}{4}\beta^8 + \frac{125}{4}\beta^9) \\
& + A_4(\frac{13}{2}\beta^3 - 12\beta^4 - \frac{79}{2}\beta^5 + 148\beta^6 - \frac{369}{2}\beta^7 + 104\beta^8 - \frac{45}{2}\beta^9) \\
& + (-40\beta^6 + 40\beta^8)
\end{aligned}$$

The idea is to show that for all $\beta \in (0, \frac{1}{3}]$, the minimum of $\triangle_3$ in $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$ and $A_4 \geq 1$ occurs at $A_1 = 4$, $A_2 = 3$, $A_3 = 2$ and $A_4 = 1$ and $\triangle_3 \mid_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$, for all $\beta \in (0, \frac{1}{3}]$.

$$\begin{aligned}
& \frac{\partial^4 \triangle_3}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \\
= & \frac{11}{16} - \frac{43}{16}\beta + \frac{13}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{79}{8}\beta^4 + \frac{223}{8}\beta^5 - \frac{195}{4}\beta^6 + \frac{195}{4}\beta^7 - \frac{405}{16}\beta^8 + \frac{85}{16}\beta^9 \\
= & \frac{1}{16}(-1+\beta)^4(11 + \beta - 10\beta^2 + 10\beta^3 - 65\beta^4 + 85\beta^5) > 0,
\end{aligned}$$

for $\beta \in (0, \frac{3}{5})$. In fact, $11 + \beta - 10\beta^2 + 10\beta^3 - 65\beta^4 + 85\beta^5 > 11 - 9\beta - 55\beta^3 + 85\beta^5 \triangleq f(\beta)$, for $\beta \in (0, \frac{3}{5})$. Then $f'(\beta) = -9 - 165\beta^2 + 425\beta^4 = 425(\beta^2 - \frac{33}{170})^2 - \frac{1701}{68}$. Thus, $f'(\beta) < f'(\frac{3}{5}) = -\frac{333}{25} < 0$, for $\beta \in (0, \frac{3}{5})$, which implies $f(\beta)$ is an decreasing function, i.e. $f(\beta) > f(\frac{3}{5}) = \frac{206}{625} > 0$, for $\beta \in (0, \frac{3}{5})$. It follows that $\frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of

A_4 for $\beta \in (0, \frac{3}{5})$, $A_4 \geq 1$. Hence the minimum of $\frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_3}$ occurs at $A_4 = 1$,

$$\begin{aligned}
&\left.\frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=1} \\
&= \left[A_4\left(\frac{11}{16} - \frac{43}{16}\beta + \frac{13}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{79}{8}\beta^4 + \frac{223}{8}\beta^5 - \frac{195}{4}\beta^6 + \frac{195}{4}\beta^7 - \frac{405}{16}\beta^8 + \frac{85}{16}\beta^9\right)\right. \\
&\quad \left.\left. +\left(-\frac{3}{8}\beta + \frac{7}{4}\beta^2 - \frac{13}{4}\beta^3 + \frac{27}{4}\beta^4 - 22\beta^5 + \frac{181}{4}\beta^6 - \frac{195}{4}\beta^7 + \frac{105}{4}\beta^8 - \frac{45}{8}\beta^9\right)\right]\right|_{A_4=1} \\
&= -\frac{1}{16}(-1+\beta)^5(1+\beta)(11 - 5\beta + 5\beta^2 + 5\beta^3) > 0,
\end{aligned}$$

for $\beta \in (0, \frac{3}{5})$, since $11 - 5\beta + 5\beta^2 + 5\beta^3 > 8 + 5\beta^2 + 5\beta^3 > 0$, for $\beta \in (0, \frac{3}{5})$. It follows that $\frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_3} > 0$ for $A_4 \geq 1$, $\beta \in (0, \frac{3}{5})$. Note that $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_4} > 0$, for $A_3 \geq 1$, $\beta \in (0, \frac{3}{5})$. Moreover, we have $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$. The minimum of $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2}$ occurs at $A_3 = A_4 = 1$,

$$\begin{aligned}
&\left.\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2}\right|_{A_3=A_4=1} \\
&= \left[A_3 A_4\left(\frac{11}{16} - \frac{43}{16}\beta + \frac{13}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{79}{8}\beta^4 + \frac{223}{8}\beta^5 - \frac{195}{4}\beta^6 + \frac{195}{4}\beta^7 - \frac{405}{16}\beta^8 + \frac{85}{16}\beta^9\right)\right. \\
&\quad + (A_3 + A_4)\left(-\frac{3}{8}\beta + \frac{7}{4}\beta^2 - \frac{13}{4}\beta^3 + \frac{27}{4}\beta^4 - 22\beta^5 + \frac{181}{4}\beta^6 - \frac{195}{4}\beta^7 + \frac{105}{4}\beta^8 - \frac{45}{8}\beta^9\right) \\
&\quad \left.\left. +\left(-\frac{1}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{17}{4}\beta^4 + \frac{83}{4}\beta^5 - \frac{187}{4}\beta^6 + \frac{209}{4}\beta^7 - \frac{115}{4}\beta^8 + \frac{25}{4}\beta^9\right)\right]\right|_{A_3=A_4=1} \\
&= \frac{1}{16}(-1+\beta)^6(1+\beta)(11 + 5\beta^2) > 0,
\end{aligned}$$

for $\beta \in (0, \frac{3}{5})$. It follows that $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2} > 0$, for $A_3 \geq A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$. From the property that $\frac{\partial \triangle_3}{\partial A_1}$ is symmetric with respect to A_2, A_3 and A_4, we also get $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_3} > 0$, for $A_2 \geq A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$ and $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_4} > 0$, for $A_2 \geq A_3 \geq 1$ and $\beta \in (0, \frac{3}{5})$. Therefore, we have $\frac{\partial \triangle_3}{\partial A_1}$ is an increasing function of A_2, A_3 and A_4 for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$.

Hence the minimum of $\frac{\partial \triangle_3}{\partial A_1}$ occurs at $A_2 = A_3 = A_4 = 1$,

$$\begin{aligned}
&\frac{\partial \triangle_3}{\partial A_1}\bigg|_{A_2=A_3=A_4=1} \\
&= \Big[A_2A_3A_4(\frac{11}{16} - \frac{43}{16}\beta + \frac{13}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{79}{8}\beta^4 + \frac{223}{8}\beta^5 - \frac{195}{4}\beta^6 + \frac{195}{4}\beta^7 - \frac{405}{16}\beta^8 + \frac{85}{16}\beta^9) \\
&\quad + (A_2A_3 + A_2A_4 + A_3A_4) \\
&\qquad \cdot (-\frac{3}{8}\beta + \frac{7}{4}\beta^2 - \frac{13}{4}\beta^3 + \frac{27}{4}\beta^4 - 22\beta^5 + \frac{181}{4}\beta^6 - \frac{195}{4}\beta^7 + \frac{105}{4}\beta^8 - \frac{45}{8}\beta^9) \\
&\quad + (A_2 + A_3 + A_4)(-\frac{1}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{17}{4}\beta^4 + \frac{83}{4}\beta^5 - \frac{187}{4}\beta^6 + \frac{209}{4}\beta^7 - \frac{115}{4}\beta^8 + \frac{25}{4}\beta^9) \\
&\quad +(\frac{3}{2}\beta^3 - 2\beta^4 - \frac{29}{2}\beta^5 + 48\beta^6 - \frac{119}{2}\beta^7 + 34\beta^8 - \frac{15}{2}\beta^9)\Big]\bigg|_{A_2=A_3=A_4=1} \\
&= \frac{1}{16}(-1+\beta)^6(1+\beta)(11+5\beta^2) > 0,
\end{aligned}$$

for $\beta \in (0, \frac{3}{5})$. It follows that $\frac{\partial \triangle_3}{\partial A_1} > 0$, for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$. By the property that $\triangle_3$ is symmetric with respect to A_1, A_2 and A_3. We also have $\frac{\partial \triangle_3}{\partial A_2} > 0$, for $A_1 \geq A_3 \geq A_4 \geq 1$ and $\frac{\partial \triangle_3}{\partial A_3} > 0$, for $A_1 \geq A_2 \geq A_4 \geq 1$. Meanwhile,

$$\begin{aligned}
\frac{\partial^3 \triangle_3}{\partial A_4^3} =& 6(\frac{5}{4}\beta - \frac{5}{2}\beta^2 - \frac{5}{2}\beta^3 + \frac{35}{2}\beta^4 - 50\beta^5 + \frac{185}{2}\beta^6 - \frac{195}{2}\beta^7 + \frac{105}{2}\beta^8 - \frac{45}{4}\beta^9) \\
=& -\frac{15}{2}(-1+\beta)^5(1+3\beta)(1+3\beta^2)\beta > 0,
\end{aligned}$$

for $\beta \in (0, \frac{3}{5})$. It follows that $\frac{\partial^2 \triangle_3}{\partial A_4^2}$ is an increasing function of A_4, for $A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$. Thus, the minimum of $\frac{\partial^2 \triangle_3}{\partial A_4^2}$ occurs at $A_4 = 1$,

$$\begin{aligned}
&\frac{\partial^2 \triangle_3}{\partial A_4^2}\bigg|_{A_4=1} \\
&= \Big[6A_4(\frac{5}{4}\beta - \frac{5}{2}\beta^2 - \frac{5}{2}\beta^3 + \frac{35}{2}\beta^4 - 50\beta^5 + \frac{185}{2}\beta^6 - \frac{195}{2}\beta^7 + \frac{105}{2}\beta^8 - \frac{45}{4}\beta^9) \\
&\quad +2(-\frac{25}{4}\beta^2 + \frac{75}{4}\beta^3 - \frac{125}{4}\beta^4 + \frac{375}{4}\beta^5 - \frac{875}{4}\beta^6 + \frac{1025}{4}\beta^7 - \frac{575}{4}\beta^8 + \frac{125}{4}\beta^9)\Big]\bigg|_{A_4=1} \\
&= -\frac{5}{2}(-1+\beta)^5\beta(3+4\beta-\beta^2+2\beta^3) > 0,
\end{aligned}$$

for $\beta \in (0, \frac{3}{5})$, since $3 + 4\beta - \beta^2 + 2\beta^3 > 3 + 4\beta - \frac{3}{5}\beta = 3 + \frac{17}{5}\beta > 0$, for $\beta \in (0, \frac{3}{5})$. It follows that $\frac{\partial^2 \triangle_3}{\partial A_4^2} > 0$, for $A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$. Thus, $\frac{\partial \triangle_3}{\partial A_4}$ is an increasing function of A_4, for $A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$. Moreover, it's an increasing function with respect to A_1, A_2, A_3 and

A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$, $\beta \in (0, \frac{3}{5})$, since $\frac{\partial \triangle_3}{\partial A_4}$ is symmetric with respect to A_1, A_2 and A_3. The minimum of $\frac{\partial \triangle_3}{\partial A_4}$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$,

$$\begin{aligned}
&\frac{\partial \triangle_3}{\partial A_4}\bigg|_{A_1=A_2=A_3=A_4=1} \\
&= [A_1A_2A_3 \\
&\qquad \cdot (\frac{11}{16} - \frac{43}{16}\beta + \frac{13}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{79}{8}\beta^4 + \frac{223}{8}\beta^5 - \frac{195}{4}\beta^6 + \frac{195}{4}\beta^7 - \frac{405}{16}\beta^8 + \frac{85}{16}\beta^9) \\
&\quad + (A_1A_2 + A_1A_3 + A_2A_3) \\
&\qquad \cdot (-\frac{3}{8}\beta + \frac{7}{4}\beta^2 - \frac{13}{4}\beta^3 + \frac{27}{4}\beta^4 - 22\beta^5 + \frac{181}{4}\beta^6 - \frac{195}{4}\beta^7 + \frac{105}{4}\beta^8 - \frac{45}{8}\beta^9) \\
&\quad + (A_1 + A_2 + A_3)(-\frac{1}{4}\beta^2 + \frac{3}{4}\beta^3 - \frac{17}{4}\beta^4 + \frac{83}{4}\beta^5 - \frac{187}{4}\beta^6 + \frac{209}{4}\beta^7 - \frac{115}{4}\beta^8 + \frac{25}{4}\beta^9) \\
&\quad + (\frac{13}{2}\beta^3 - 12\beta^4 - \frac{79}{2}\beta^5 + 148\beta^6 - \frac{369}{2}\beta^7 + 104\beta^8 - \frac{45}{2}\beta^9) \\
&\quad + 3A_4^2(\frac{5}{4}\beta - \frac{5}{2}\beta^2 - \frac{5}{2}\beta^3 + \frac{35}{2}\beta^4 - 50\beta^5 + \frac{185}{2}\beta^6 - \frac{195}{2}\beta^7 + \frac{105}{2}\beta^8 - \frac{45}{4}\beta^9) \\
&\quad + 2A_4(-\frac{25}{4}\beta^2 + \frac{75}{4}\beta^3 - \frac{125}{4}\beta^4 + \frac{375}{4}\beta^5 \\
&\qquad\qquad -\frac{875}{4}\beta^6 + \frac{1025}{4}\beta^7 - \frac{575}{4}\beta^8 + \frac{125}{4}\beta^9)\bigg]\bigg|_{A_1=A_2=A_3=A_4=1} \\
&= -\frac{1}{16}(-1+\beta)^5(11 + 54\beta + 64\beta^2 + 66\beta^3 + 285\beta^4) > 0,
\end{aligned}$$

for $\beta \in (0, \frac{3}{5})$. It follows that $\frac{\partial \triangle_3}{\partial A_4} > 0$, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$. Therefore, $\triangle_3$ is an increasing function of A_1, A_2, A_3 and A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$. Thus, the minimum of $\triangle_3$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$ and take condition (2.14) into consideration,

$$\begin{aligned}
&\triangle_3|_{A_1=4, A_2=3, A_3=2, A_4=1} > \triangle_3|_{A_1=A_2=A_3=A_4=1} \\
&= -\frac{1}{16}(-1+\beta)(1+\beta)(-1+3\beta)(-11 + 14\beta + 31\beta^2 - 172\beta^3 + 115\beta^4 - 322\beta^5 + 25\beta^6) > 0,
\end{aligned}$$

for $\beta \in (0, \frac{1}{3}]$, since $-11 + 14\beta + 31\beta^2 - 172\beta^3 + 115\beta^4 - 322\beta^5 + 25\beta^6 < -11 + \frac{14}{3} + \frac{31}{9} - 172\beta^3 + \frac{115}{3}\beta^3 - 322\beta^5 + \frac{25}{3}\beta^5 = -\frac{26}{9} - \frac{401}{3}\beta^3 - \frac{941}{3}\beta^5 < 0$, for $\beta \in (0, \frac{1}{3}]$. It follows that $\triangle_3 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\beta \in (0, \frac{1}{3}]$. □

For case (3), there are four levels $k = 1$, $k = 2$, $k = 3$ and $k = 4$. It's easy to see that $P_4(4) = 0$. From the condition $P_5 > 0$, we know that $P_4(1) > 0$, but the positivity of $P_4(2)$ and $P_4(3)$ are unknown. Therefore, we split this case into three subcases:

(3a) $P_4(2) = P_4(3) = 0$ (i.e. $k_0 = 1$ in (2.13));
(3b) $P_4(2) > 0$, $P_4(3) = 0$ (i.e. $k_0 = 2$ in (2.13));
(3c) $P_4(2) > 0$, $P_4(3) > 0$ (i.e. $k_0 = 3$ in (2.13)).

For subcase (3a), the proof is nearly the same as case (5). As we know, in this case $P_5 = P_4(1) > 0$, thus $(1,1,1,1,1)$ is the smallest positive integral solution, i.e. $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4} \leq 1-\frac{1}{a_5} \triangleq \alpha$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$, since $a_5 \in (3,4]$. The new range of α helps us to improve the condition (2.14) to the following one:

$$A_1 \geq 4,\ A_2 \geq 3,\ A_3 \geq A_4 \geq \frac{\alpha}{1-\alpha}, \tag{2.18}$$

since $A_i = a_i\alpha \geq a_5\alpha = \frac{\alpha}{1-\alpha}$. With $\alpha \in (\frac{2}{3}, \frac{3}{4}]$, it is easy to check that $2 < \frac{\alpha}{1-\alpha} \leq 3$. Therefore, it is sufficient to show that $\triangle_2 > 0$, for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq A_4 \geq \frac{\alpha}{1-\alpha}$ and $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. Notice that in the proof of Lemma 2.2 all the partial derivatives of A_i are positive for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\alpha \in (0, \frac{4}{5})$ until the last step to compute $\triangle_2|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ only for $\alpha \in (0, \frac{1}{2}]$. Thus, we need to take condition (2.18) instead of the rough estimate (2.14) of A_i, $i = 1,2,3,4$.

$$\triangle_2|_{A_1=4,A_2=3,A_3=A_4=\frac{\alpha}{1-\alpha}} = -25+176\alpha-411\alpha^2+415\alpha^3-160\alpha^4 > 0,$$

for $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. In fact, let $f(\alpha) \triangleq -25+176\alpha-411\alpha^2+415\alpha^3-160\alpha^4$, then $f''(\alpha) = -822+2490\alpha-1920\alpha^2 = -1920(\alpha-\frac{83}{128})^2-\frac{1881}{128} < 0$, for $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. Thus, $f'(\alpha)$ is decreasing in α, and $f'(\alpha) \leq f'(\frac{2}{3}) = -\frac{224}{27} < 0$. Therefore, $f(\alpha)$ is decreasing in $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ and $f(\alpha) \geq f(\frac{3}{4}) = \frac{17}{64} > 0$, for $\alpha \in (\frac{2}{3}, \frac{3}{4}]$.

For case (3b), the proof is nearly the same as subcase (2b). As we know, in this case $P_5 = P_4(1)+P_4(2) > 0$, thus $(1,1,1,1,2)$ is the smallest positive integral solution for the level $k=2$, i.e. $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4} \leq 1-\frac{2}{a_5} \triangleq \beta$, $\beta \in (\frac{1}{3}, \frac{1}{2}]$, since $a_5 \in (3,4]$. Also, let $A_i = a_i\beta$, $i = 1,2,3,4$. The new range of β helps us to improve the condition (2.14) to the following one:

$$A_1 \geq 4,\ A_2 \geq 3,\ A_3 \geq 2, A_4 \geq \frac{2\beta}{1-\beta}, \tag{2.19}$$

since $A_i = a_i\beta \geq a_5\beta = \frac{2\beta}{1-\beta}$. With $\beta \in (\frac{1}{3}, \frac{1}{2}]$, it is easy to check that $1 < \frac{2\beta}{1-\beta} \leq 2$. Therefore, it is sufficient to show that $\triangle_3 > 0$, for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq \frac{2\beta}{1-\beta}$ and $\beta \in (\frac{1}{3}, \frac{1}{2}]$. Notice that in the proof of Lemma 2.3 all the partial derivatives of A_i are positive for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\beta \in (0, \frac{3}{5})$ until the last step to compute $\triangle_3|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ only for $\beta \in (0, \frac{1}{3}]$. Thus, we need to take condition (2.19) instead of the rough estimate (2.14) of A_i, $i = 1,2,3,4$.

$$\begin{aligned}&\triangle_3|_{A_1=4,A_2=3,A_3=2,A_4=\frac{2\beta}{1-\beta}} \\ =&\beta(-1+\beta)(-24+56\beta-26\beta^2-65\beta^3+222\beta^4-256\beta^5+28\beta^6+145\beta^7) > 0,\end{aligned}$$

for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Indeed, $-24+56\beta-26\beta^2-65\beta^3+222\beta^4-256\beta^5+28\beta^6+145\beta^7 \leq -24+56\beta-26\beta^2-65\beta^3+222\beta^4-\frac{823}{4}\beta^5 < -24+56\beta-26\beta^2-65\beta^3+\frac{1841}{12}\beta^4 \leq -24+56\beta-26\beta^2+\frac{281}{24}\beta^3 \leq -24+56\beta-\frac{967}{48}\beta^2 \triangleq f(\beta)$, and $f(\beta) = -\frac{967}{48}(\beta-\frac{1344}{967})^2+\frac{14424}{967}$. Thus, $f(\beta) < f(\frac{1}{2}) = -\frac{199}{192} < 0$, for $\beta \in (\frac{1}{3}, \frac{1}{2}]$.

For subcase (3c), $P_4(3) > 0$ which implies that $(1,1,1,1,3)$ is the smallest positive integer solution to the level $k = 3$. So we have $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \le 1 - \frac{3}{a_5} \triangleq \gamma$, $\gamma \in (0, \frac{1}{4}]$, since $a_5 \in (3, 4]$. Let $A_i = a_i\gamma$, $i = 1, 2, 3, 4$. Also notice that condition (2.14) still holds here. (2.13) can be written as

$$\begin{aligned}
5!P_5 =& 5!(P_4(1) + P_4(2) + P_4(3)) \\
\le& 5[(A_1\frac{2+\gamma}{3\gamma} - 1)(A_2\frac{2+\gamma}{3\gamma} - 1)(A_3\frac{2+\gamma}{3\gamma} - 1)(A_4\frac{2+\gamma}{3\gamma} - 1) \\
&- (A_4\frac{2+\gamma}{3\gamma} - 1)^4 + A_4\frac{2+\gamma}{3\gamma}(A_4\frac{2+\gamma}{3\gamma} - 1)(A_4\frac{2+\gamma}{3\gamma} - 2)(A_4\frac{2+\gamma}{3\gamma} - 3) \\
&+ (A_1\frac{1+2\gamma}{3\gamma} - 1)(A_2\frac{1+2\gamma}{3\gamma} - 1)(A_3\frac{1+2\gamma}{3\gamma} - 1)(A_4\frac{1+2\gamma}{3\gamma} - 1) \\
&- (A_4\frac{1+2\gamma}{3\gamma} - 1)^4 + A_4\frac{1+2\gamma}{3\gamma}(A_4\frac{1+2\gamma}{3\gamma} - 1)(A_4\frac{1+2\gamma}{3\gamma} - 2)(A_4\frac{1+2\gamma}{3\gamma} - 3) \\
&+ (A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1) - (A_4 - 1)^4 + A_4(A_4 - 1)(A_4 - 2)(A_4 - 3)].
\end{aligned} \tag{2.20}$$

It is sufficient to show that R.H.S. of (1.8) is strictly larger than R.H.S. of (2.20).

Lemma 2.4 *When $3 < a_5 \le 4$, R.H.S. of (1.8) $>$ R.H.S. of (2.20).*

Proof. Substitute $a_i = \frac{A_i}{\gamma}$, $i = 1, 2, 3, 4$ and $a_5 = \frac{3}{1-\gamma}$ to R.H.S. of (1.8), subtract that by R.H.S. of (2.20), and multiply $(1-\gamma)^5\gamma^4$, we get

$$\begin{aligned}
\triangle_4 \triangleq& A_1A_2A_3A_4 \\
&\cdot(\frac{77}{81} - \frac{38}{9}\gamma + \frac{62}{9}\gamma^2 - \frac{104}{27}\gamma^3 - \frac{53}{9}\gamma^4 + \frac{224}{9}\gamma^5 - \frac{1300}{27}\gamma^6 + \frac{460}{9}\gamma^7 - \frac{250}{9}\gamma^8 + \frac{490}{81}\gamma^9) \\
&+ (A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4) \\
&\quad\cdot(-\frac{1}{3}\gamma + 2\gamma^2 - \frac{14}{3}\gamma^3 + \frac{26}{3}\gamma^4 - 23\gamma^5 + \frac{142}{3}\gamma^6 - \frac{160}{3}\gamma^7 + 30\gamma^8 - \frac{20}{3}\gamma^9) \\
&+ (A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4) \\
&\quad\cdot(-\frac{7}{9}\gamma^2 + \frac{22}{9}\gamma^3 - \frac{16}{3}\gamma^4 + \frac{182}{9}\gamma^5 - \frac{443}{9}\gamma^6 + \frac{178}{3}\gamma^7 - \frac{310}{9}\gamma^8 + \frac{70}{9}\gamma^9) \\
&+ (A_1 + A_2 + A_3)(3\gamma^3 - 8\gamma^4 - 8\gamma^5 + 52\gamma^6 - 73\gamma^7 + 44\gamma^8 - 10\gamma^9) \\
&+ A_4^3(\frac{10}{3}\gamma - 10\gamma^2 + \frac{20}{3}\gamma^3 + \frac{40}{3}\gamma^4 - 50\gamma^5 + \frac{290}{3}\gamma^6 - \frac{320}{3}\gamma^7 + 60\gamma^8 - \frac{40}{3}\gamma^9) \\
&+ A_4^2(-\frac{125}{9}\gamma^2 + \frac{425}{9}\gamma^3 - \frac{200}{3}\gamma^4 + \frac{1000}{9}\gamma^5 - \frac{2125}{9}\gamma^6 + \frac{875}{3}\gamma^7 - \frac{1550}{9}\gamma^8 + \frac{350}{9}\gamma^9) \\
&+ A_4(13\gamma^3 - 38\gamma^4 - 8\gamma^5 + 152\gamma^6 - 223\gamma^7 + 134\gamma^8 - 30\gamma^9) \\
&+ (-30\gamma^4 - 105\gamma^5 - 45\gamma^6 + 120\gamma^7 + 60\gamma^8)
\end{aligned}$$

The idea is to show that for all $\gamma \in (0, \frac{1}{4}]$, the minimum of $\triangle_4$ in $A_1 \ge 4$, $A_2 \ge 3$, $A_3 \ge 2$ and $A_4 \ge 1$ occurs at $A_1 = 4$, $A_2 = 3$, $A_3 = 2$ and $A_4 = 1$ and $\triangle_4 |_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$, for all $\gamma \in (0, \frac{1}{4}]$.

$$\frac{\partial^4 \triangle_4}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} = \frac{1}{81}(-1+\gamma)^4(77 - 34\gamma - 40\gamma^2 + 40\gamma^3 - 290\gamma^4 + 490\gamma^5) > 0,$$

for $\gamma \in (0, \frac{2}{5})$, since $77 - 34\gamma - 40\gamma^2 + 40\gamma^3 - 290\gamma^4 + 490\gamma^5 > 77 - 34 \times \frac{2}{5} - 40 \times (\frac{2}{5})^2 - 290 \times (\frac{2}{5})^4 + 40\gamma^3 + 490\gamma^5 = \frac{6197}{125} + 40\gamma^3 + 490\gamma^5 > 0$. It follows that $\frac{\partial^3 \triangle_4}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 for $\gamma \in (0, \frac{2}{5})$, $A_4 \geq 1$. Hence the minimum of $\frac{\partial^3 \triangle_4}{\partial A_1 \partial A_2 \partial A_3}$ occurs at $A_4 = 1$,

$$\begin{aligned}
&\frac{\partial^3 \triangle_4}{\partial A_1 \partial A_2 \partial A_3}\bigg|_{A_4=1} \\
&= \bigg[A_4(\frac{77}{81} - \frac{38}{9}\gamma + \frac{62}{9}\gamma^2 - \frac{104}{27}\gamma^3 - \frac{53}{9}\gamma^4 + \frac{224}{9}\gamma^5 - \frac{1300}{27}\gamma^6 + \frac{460}{9}\gamma^7 - \frac{250}{9}\gamma^8 + \frac{490}{81}\gamma^9) \\
&\quad + (-\frac{1}{3}\gamma + 2\gamma^2 - \frac{14}{3}\gamma^3 + \frac{26}{3}\gamma^4 - 23\gamma^5 + \frac{142}{3}\gamma^6 - \frac{160}{3}\gamma^7 + 30\gamma^8 - \frac{20}{3}\gamma^9)\bigg]\bigg|_{A_4=1} \\
&= -\frac{1}{81}(-1+\gamma)^5(77 + 16\gamma + 30\gamma^2 + 70\gamma^3 + 50\gamma^4) > 0,
\end{aligned}$$

for $\gamma \in (0, \frac{2}{5})$. It follows that $\frac{\partial^3 \triangle_4}{\partial A_1 \partial A_2 \partial A_3} > 0$ for $A_4 \geq 1$, $\gamma \in (0, \frac{2}{5})$. Note that $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3 \triangle_4}{\partial A_1 \partial A_2 \partial A_4} > 0$, for $A_3 \geq 1$, $\beta \in (0, \frac{3}{5})$. Moreover, we have $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. The minimum of $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_2}$ occurs at $A_3 = A_4 = 1$,

$$\begin{aligned}
&\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_2}\bigg|_{A_3=A_4=1} \\
&= \bigg[A_3 A_4(\frac{77}{81} - \frac{38}{9}\gamma + \frac{62}{9}\gamma^2 - \frac{104}{27}\gamma^3 - \frac{53}{9}\gamma^4 + \frac{224}{9}\gamma^5 - \frac{1300}{27}\gamma^6 + \frac{460}{9}\gamma^7 - \frac{250}{9}\gamma^8 + \frac{490}{81}\gamma^9) \\
&\quad + (A_3 + A_4)(-\frac{1}{3}\gamma + 2\gamma^2 - \frac{14}{3}\gamma^3 + \frac{26}{3}\gamma^4 - 23\gamma^5 + \frac{142}{3}\gamma^6 - \frac{160}{3}\gamma^7 + 30\gamma^8 - \frac{20}{3}\gamma^9) \\
&\quad + (-\frac{7}{9}\gamma^2 + \frac{22}{9}\gamma^3 - \frac{16}{3}\gamma^4 + \frac{182}{9}\gamma^5 - \frac{443}{9}\gamma^6 + \frac{178}{3}\gamma^7 - \frac{310}{9}\gamma^8 + \frac{70}{9}\gamma^9)\bigg]\bigg|_{A_3=A_4=1} \\
&= \frac{1}{81}(-1+\gamma)^6(77 + 66\gamma + 60\gamma^2 + 40\gamma^3) > 0,
\end{aligned}$$

for $\gamma \in (0, \frac{2}{5})$. It follows that $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_2} > 0$, for $A_3 \geq A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. From the property that $\frac{\partial \triangle_4}{\partial A_1}$ is symmetric with respect to A_2, A_3 and A_4, we also get $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_3} > 0$, for $A_2 \geq A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$ and $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_4} > 0$, for $A_2 \geq A_3 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. Therefore, we have $\frac{\partial \triangle_4}{\partial A_1}$ is an increasing function of A_2, A_3 and A_4 for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$.

Hence the minimum of $\frac{\partial \triangle_4}{\partial A_1}$ occurs at $A_2 = A_3 = A_4 = 1$,

$$
\begin{aligned}
&\frac{\partial \triangle_4}{\partial A_1}\bigg|_{A_2=A_3=A_4=1} \\
=&[A_2A_3A_4 \\
&\qquad \cdot (\frac{77}{81} - \frac{38}{9}\gamma + \frac{62}{9}\gamma^2 - \frac{104}{27}\gamma^3 - \frac{53}{9}\gamma^4 + \frac{224}{9}\gamma^5 - \frac{1300}{27}\gamma^6 + \frac{460}{9}\gamma^7 - \frac{250}{9}\gamma^8 + \frac{490}{81}\gamma^9) \\
&+ (A_2A_3 + A_2A_4 + A_3A_4) \\
&\qquad \cdot (-\frac{1}{3}\gamma + 2\gamma^2 - \frac{14}{3}\gamma^3 + \frac{26}{3}\gamma^4 - 23\gamma^5 + \frac{142}{3}\gamma^6 - \frac{160}{3}\gamma^7 + 30\gamma^8 - \frac{20}{3}\gamma^9) \\
&+ (A_2 + A_3 + A_4)(-\frac{7}{9}\gamma^2 + \frac{22}{9}\gamma^3 - \frac{16}{3}\gamma^4 + \frac{182}{9}\gamma^5 - \frac{443}{9}\gamma^6 + \frac{178}{3}\gamma^7 - \frac{310}{9}\gamma^8 + \frac{70}{9}\gamma^9) \\
&+(3\gamma^3 - 8\gamma^4 - 8\gamma^5 + 52\gamma^6 - 73\gamma^7 + 44\gamma^8 - 10\gamma^9)]\big|_{A_2=A_3=A_4=1} \\
=&-\frac{1}{81}(-1+\gamma)^7(77 + 116\gamma + 50\gamma^2) > 0,
\end{aligned}
$$

for $\gamma \in (0, \frac{2}{5})$. It follows that $\frac{\partial \triangle_4}{\partial A_1} > 0$, for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. By the property that $\triangle_4$ is symmetric with respect to A_1, A_2 and A_3. We also have $\frac{\partial \triangle_4}{\partial A_2} > 0$, for $A_1 \geq A_3 \geq A_4 \geq 1$ and $\frac{\partial \triangle_4}{\partial A_3} > 0$, for $A_1 \geq A_2 \geq A_4 \geq 1$. Meanwhile,

$$\frac{\partial^3 \triangle_4}{\partial A_4^3} = -20(-1+\gamma)^5\gamma(1+2\gamma)(1+2\gamma^2) > 0,$$

for $\gamma \in (0, \frac{2}{5})$. It follows that $\frac{\partial^2 \triangle_4}{\partial A_4^2}$ is an increasing function of A_4, for $A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. Thus, the minimum of $\frac{\partial^2 \triangle_4}{\partial A_4^2}$ occurs at $A_4 = 1$,

$$
\begin{aligned}
&\frac{\partial^2 \triangle_4}{\partial A_4^2}\bigg|_{A_4=1} \\
=&\Big[6A_4(\frac{10}{3}\gamma - 10\gamma^2 + \frac{20}{3}\gamma^3 + \frac{40}{3}\gamma^4 - 50\gamma^5 + \frac{290}{3}\gamma^6 - \frac{320}{3}\gamma^7 + 60\gamma^8 - \frac{40}{3}\gamma^9) \\
&+2(-\frac{125}{9}\gamma^2 + \frac{425}{9}\gamma^3 - \frac{200}{3}\gamma^4 + \frac{1000}{9}\gamma^5 - \frac{2125}{9}\gamma^6 + \frac{875}{3}\gamma^7 - \frac{1550}{9}\gamma^8 + \frac{350}{9}\gamma^9)\Big]\Big|_{A_4=1} \\
=&-\frac{10}{9}(-1+\gamma)^5\gamma(18 + 11\gamma - 4\gamma^2 + 2\gamma^3) > 0,
\end{aligned}
$$

for $\gamma \in (0, \frac{2}{5})$, since $18 + 11\gamma - 4\gamma^2 + 2\gamma^3 > 18 + 11\gamma - \frac{8}{5}\gamma + 2\gamma^3 = 18 + \frac{47}{5}\gamma + 2\gamma^3 > 0$, for $\gamma \in (0, \frac{2}{5})$. It follows that $\frac{\partial^2 \triangle_4}{\partial A_4^2} > 0$, for $A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. Thus, $\frac{\partial \triangle_4}{\partial A_4}$ is an increasing function of A_4, for $A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. Moreover, it's an increasing function with respect to A_1, A_2, A_3 and A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$, $\gamma \in (0, \frac{2}{5})$, since $\frac{\partial \triangle_4}{\partial A_4}$ is symmetric with respect to A_1, A_2 and A_3. The minimum of $\frac{\partial \triangle_4}{\partial A_4}$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$,

$$
\begin{aligned}
&\frac{\partial \triangle_4}{\partial A_4}\bigg|_{A_1=A_2=A_3=A_4=1}\\
=&[A_1A_2A_3\\
&\quad\cdot(\frac{77}{81}-\frac{38}{9}\gamma+\frac{62}{9}\gamma^2-\frac{104}{27}\gamma^3-\frac{53}{9}\gamma^4+\frac{224}{9}\gamma^5-\frac{1300}{27}\gamma^6+\frac{460}{9}\gamma^7-\frac{250}{9}\gamma^8+\frac{490}{81}\gamma^9)\\
&+(A_1A_2+A_1A_3+A_2A_3)\\
&\quad\cdot(-\frac{1}{3}\gamma+2\gamma^2-\frac{14}{3}\gamma^3+\frac{26}{3}\gamma^4-23\gamma^5+\frac{142}{3}\gamma^6-\frac{160}{3}\gamma^7+30\gamma^8-\frac{20}{3}\gamma^9)\\
&+(A_1+A_2+A_3)(-\frac{7}{9}\gamma^2+\frac{22}{9}\gamma^3-\frac{16}{3}\gamma^4+\frac{182}{9}\gamma^5-\frac{443}{9}\gamma^6+\frac{178}{3}\gamma^7-\frac{310}{9}\gamma^8+\frac{70}{9}\gamma^9)\\
&+(13\gamma^3-38\gamma^4-8\gamma^5+152\gamma^6-223\gamma^7+134\gamma^8-30\gamma^9)\\
&+3A_4^2(\frac{10}{3}\gamma-10\gamma^2+\frac{20}{3}\gamma^3+\frac{40}{3}\gamma^4-50\gamma^5+\frac{290}{3}\gamma^6-\frac{320}{3}\gamma^7+60\gamma^8-\frac{40}{3}\gamma^9)\\
&+2A_4(-\frac{125}{9}\gamma^2+\frac{425}{9}\gamma^3-\frac{200}{3}\gamma^4+\frac{1000}{9}\gamma^5\\
&\qquad-\frac{2125}{9}\gamma^6+\frac{875}{3}\gamma^7-\frac{1550}{9}\gamma^8+\frac{350}{9}\gamma^9)\Big]\bigg|_{A_1=A_2=A_3=A_4=1}\\
=&\frac{1}{81}(-1+\gamma)^5(-77-772\gamma+735\gamma^2+1154\gamma^3+1390\gamma^4)>0,
\end{aligned}
$$

for $\gamma \in (0, \frac{2}{5})$, since $-77-772\gamma+735\gamma^2+1154\gamma^3+1390\gamma^4 < -77-773\gamma+735\times\frac{2}{5}\gamma+1154\times(\frac{2}{5})^2\gamma+1390\times(\frac{2}{5})^4 = -\frac{5177}{125}-\frac{7334}{25}\gamma<0$, for $\gamma\in(0,\frac{2}{5})$. It follows that $\frac{\partial \triangle_4}{\partial A_4}>0$, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. Therefore, $\triangle_4$ is an increasing function of A_1, A_2, A_3 and A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$. Thus, the minimum of $\triangle_4$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$ and take condition (2.14) into consideration,

$$
\begin{aligned}
&\triangle_4|_{A_1=4,A_2=3,A_3=2,A_4=1}\\
=&-\frac{1}{27}(-1+\gamma)(616-2480\gamma+3304\gamma^2-647\gamma^3-3023\gamma^4-2180\gamma^5\\
&\qquad-4235\gamma^6-2570\gamma^7+280\gamma^8)>0,
\end{aligned}
$$

for $\gamma\in(0,\frac{1}{4}]$. In fact, $616-2480\gamma+3304\gamma^2-647\gamma^3-3023\gamma^4-2180\gamma^5-4235\gamma^6-2570\gamma^7+280\gamma^8 > 616-2480\gamma+(3304-647\times\frac{1}{4}-3023\times(\frac{1}{4})^2-2180\times(\frac{1}{4})^3-4235\times(\frac{1}{4})^4-2570\times(\frac{1}{4})^5)\gamma^2 = 616-2480\gamma+\frac{1484901}{8192}\gamma^2 \triangleq f(\gamma)$. And $f(\gamma)=\frac{1484901}{8192}(\gamma-\frac{10158080}{1484901})^2-\frac{11681320184}{1484901}$, so $f(\gamma)>f(\frac{1}{4})=\frac{960613}{131072}>0$, for $\gamma\in(0,\frac{1}{4}]$. It follows that $\triangle_4>0$ for $A_1\geq 4$, $A_2\geq 3$, $A_3\geq 2$, $A_4\geq 1$ and $\gamma\in(0,\frac{1}{4}]$. □

For case (2), there are four levels $k=1$, $k=2$, $k=3$ and $k=4$. From the condition $P_5>0$, we know that $P_4(1)>0$, but the positivity of $P_4(2)$, $P_4(3)$ and $P_4(4)$ are unknown. Therefore, we split this case into four subcases:

(2a) $P_4(2) = P_4(3) = P_4(4) = 0$ (i.e. $k_0 = 1$ in (2.13));
(2b) $P_4(2) > 0$, $P_4(3) = P_4(4) = 0$ (i.e. $k_0 = 2$ in (2.13));
(2c) $P_4(2) > 0$, $P_4(3) > 0$, $P_4(4) = 0$ (i.e. $k_0 = 3$ in (2.13));
(2d) $P_4(2) > 0$, $P_4(3) > 0$, $P_4(4) > 0$ (i.e. $k_0 = 4$ in (2.13)).

For subcase (2a), the proof is nearly the same as case (5). As we know, in this case $P_5 = P_4(1) > 0$, thus $(1,1,1,1,1)$ is the smallest positive integral solution, i.e. $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \le 1 - \frac{1}{a_5} \triangleq \alpha$, $\alpha \in (\frac{3}{4}, \frac{4}{5})$, since $a_5 \in (4,5)$. The new range of α helps us to improve the condition (2.14) to the following one:

$$A_1 \ge 4,\ A_2 \ge A_3 \ge A_4 \ge \frac{\alpha}{1-\alpha}, \tag{2.21}$$

since $A_i = a_i\alpha \ge a_5\alpha = \frac{\alpha}{1-\alpha}$. With $\alpha \in (\frac{3}{4}, \frac{4}{5})$, it is easy to check that $3 < \frac{\alpha}{1-\alpha} < 4$. Therefore, it is sufficient to show that $\triangle_2 > 0$, for $A_1 \ge 4$, $A_2 \ge A_3 \ge A_4 \ge \frac{\alpha}{1-\alpha}$ and $\alpha \in (\frac{3}{4}, \frac{4}{5})$. Notice that in the proof of Lemma 2.2 all the partial derivatives of A_i are positive for $A_1 \ge 4$, $A_2 \ge 3$, $A_3 \ge 2$, $A_4 \ge 1$ and $\alpha \in (0, \frac{4}{5})$ until the last step to compute $\triangle_2|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ only for $\alpha \in (0, \frac{1}{2}]$. Here, we take condition (2.21) into consideration,

$$\triangle_2|_{A_1=4,A_2=A_3=A_4=\frac{\alpha}{1-\alpha}} = (-4+5\alpha)(-5+24\alpha-37\alpha^2+16\alpha^3) > 0,$$

for $\alpha \in (\frac{3}{4}, \frac{4}{5})$. In fact, $-5+24\alpha-37\alpha^2+16\alpha^3 < -5+24\alpha-37\alpha^2+16\times\frac{4}{5}\alpha^2 = -5+24\alpha-\frac{121}{5}\alpha^2 \triangleq f(\alpha)$, for $\alpha \in (\frac{3}{4}, \frac{4}{5})$, and $f(\alpha) = -\frac{121}{5}(\alpha-\frac{60}{121})^2 + \frac{115}{121}$. So $f(\alpha)$ is decreasing in $\alpha \in (\frac{3}{4}, \frac{4}{5})$. Thus, $f(\alpha) < f(\frac{3}{4}) = -\frac{49}{80}$, for $\alpha \in (\frac{3}{4}, \frac{4}{5})$.

For case (2b), the proof is nearly the same as subcase (3b). As we know, in this case $P_5 = P_4(1) + P_4(2) > 0$, thus $(1,1,1,1,2)$ is the smallest positive integral solution for the level $k=2$, i.e. $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \le 1 - \frac{2}{a_5} \triangleq \beta$, $\beta \in (\frac{1}{2}, \frac{3}{5})$, since $a_5 \in (4,5)$. Also, let $A_i = a_i\beta$, $i = 1,2,3,4$. The new range of β helps us to improve the condition (2.14) to the following one:

$$A_1 \ge 4,\ A_2 \ge 3,\ A_3 \ge A_4 \ge \frac{2\beta}{1-\beta}, \tag{2.22}$$

since $A_i = a_i\beta \ge a_5\beta = \frac{2\beta}{1-\beta}$. With $\beta \in (\frac{1}{2}, \frac{3}{5})$, it is easy to check that $2 < \frac{2\beta}{1-\beta} < 3$. Therefore, it is sufficient to show that $\triangle_3 > 0$, for $A_1 \ge 4$, $A_2 \ge 3$, $A_3 \ge A_4 \ge \frac{2\beta}{1-\beta}$ and $\beta \in (\frac{1}{2}, \frac{3}{5})$. Notice that in the proof of Lemma 2.3 all the partial derivatives of A_i are positive for $A_1 \ge 4$, $A_2 \ge 3$, $A_3 \ge 2$, $A_4 \ge 1$ and $\beta \in (0, \frac{3}{5})$ until the last step to compute $\triangle_3|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ only for $\beta \in (0, \frac{1}{3}]$. Thus, we need to take condition (2.22) instead of the rough estimate (2.14) of A_i, $i = 1,2,3,4$.

$$\begin{aligned}&\triangle_3|_{A_1=4,A_2=3,A_3=A_4=\frac{2\beta}{1-\beta}}\\ =&-\beta^2(-12-17\beta+99\beta^2-270\beta^3+718\beta^4-953\beta^5+515\beta^6) > 0,\end{aligned}$$

for $\beta \in (\frac{1}{2}, \frac{3}{5})$. Indeed, let $f(\beta) \triangleq -12-17\beta+99\beta^2-270\beta^3+718\beta^4-953\beta^5+515\beta^6$. Then $f^{(3)}(\beta) = 12(-135+1436\beta-4765\beta^2+5150\beta^3) > 12(-270\beta+1436\beta-4765\beta^2+5150\beta^3) = 12\beta(1166-4765\beta+5150\beta^2) = 61800\beta(\beta-\frac{953}{2060})^2+\frac{52575}{824} > 0$, for $\beta \in (\frac{1}{2}, \frac{3}{5})$. It implies that $f''(\beta)$ is increasing in $\beta \in (\frac{1}{2}, \frac{3}{5})$. So $f''(\beta) > f''(\frac{1}{2}) = \frac{1001}{8} > 0$, which tells us that $f'(\beta)$ is also increasing in $\beta \in (\frac{1}{2}, \frac{3}{5})$. Thus, $f'(\beta) > f'(\frac{1}{2}) = \frac{149}{4} > 0$, for $\beta \in (\frac{1}{2}, \frac{3}{5})$. It follows that $f(\beta)$ is increasing in $\beta \in (\frac{1}{2}, \frac{3}{5})$. So $f(\beta) < f(\frac{3}{5}) = -\frac{5952}{3125} < 0$. Therefore, $\triangle_3 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq A_4 \geq \frac{2\beta}{1-\beta}$ and $\beta \in (\frac{1}{2}, \frac{3}{5})$.

For case (2c), the proof is nearly the same as subcase (3c). As we know, in this case $P_5 = P_4(1) + P_4(2) + P_4(3) > 0$, thus $(1, 1, 1, 1, 3)$ is the smallest positive integral solution for the level $k = 2$, i.e. $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4} \leq 1-\frac{3}{a_5} \triangleq \gamma$, $\gamma \in (\frac{1}{4}, \frac{2}{5})]$, since $a_5 \in (4, 5)$. Also, let $A_i = a_i\gamma$, $i = 1, 2, 3, 4$. The new range of γ helps us to improve the condition (2.14) to the following one:

$$A_1 \geq 4,\ A_2 \geq 3,\ A_3 \geq 2, A_4 \geq \frac{3\gamma}{1-\gamma}, \tag{2.23}$$

since $A_i = a_i\gamma \geq a_5\gamma = \frac{3\gamma}{1-\gamma}$. With $\gamma \in (\frac{1}{4}, \frac{2}{5})$, it is easy to check that $1 < \frac{3\gamma}{1-\gamma} < 2$. Therefore, it is sufficient to show that $\triangle_4 > 0$, for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq \frac{3\gamma}{1-\gamma}$ and $\gamma \in (\frac{1}{4}, \frac{2}{5})$. Notice that in the proof of Lemma 2.4 all the partial derivatives of A_i are positive for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\gamma \in (0, \frac{2}{5})$ until the last step to compute $\triangle_4|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ only for $\gamma \in (0, \frac{1}{4}]$. Thus, we need to take condition (2.23) instead of the rough estimate (2.14) of A_i, $i = 1, 2, 3, 4$.

$$\begin{aligned}&\triangle_4|_{A_1=4,A_2=3,A_3=2,A_4=\frac{3\gamma}{1-\gamma}}\\ =&\frac{1}{9}\gamma(-1+\gamma)(-544+1560\gamma-1572\gamma^2+539\gamma^3+2349\gamma^4-1353\gamma^5-2974\gamma^6+5640\gamma^7) > 0,\end{aligned}$$

for $\gamma \in (\frac{1}{4}, \frac{2}{5})$. Indeed, $-544+1560\gamma-1572\gamma^2+539\gamma^3+2349\gamma^4-1353\gamma^5-2974\gamma^6+5640\gamma^7 < -544+1560\gamma+(-1572+539\times\frac{2}{5}+2349\times(\frac{2}{5})^2-1353\times(\frac{1}{4})^3-2974\times(\frac{1}{4})^4+5640\times(\frac{2}{5})^5)\gamma^2 = -544+1560\gamma-\frac{76445137}{80000}\gamma^2 \triangleq f(\gamma)$. And $f(\gamma) = -\frac{76445137}{80000}(\gamma-\frac{62400000}{76445137})^2+\frac{7085845472}{76445137}$. So $f(\gamma) < f(\frac{2}{5}) = -\frac{36445137}{500000} < 0$, for $\gamma \in (\frac{1}{4}, \frac{2}{5})$. Therefore, $\triangle_4 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq \frac{3\gamma}{1-\gamma}$ and $\gamma \in (\frac{1}{4}, \frac{2}{5})$.

For subcase (2d), $P_4(4) > 0$ which implies that $(1, 1, 1, 1, 4)$ is the smallest positive integer solution to the level $k = 4$. So we have $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4} \leq 1-\frac{4}{a_5} \triangleq \delta$, $\delta \in (0, \frac{1}{5})$, since $a_5 \in (4, 5)$. Let $A_i = a_i\delta$, $i = 1, 2, 3, 4$. Also notice that condition (2.14) still holds here.

(2.13) can be written as

$$
\begin{aligned}
5!P_5 =&5!(P_4(1)+P_4(2)+P_4(3)+P_4(4))\\
\le&5[(A_1\frac{3+\delta}{4\delta}-1)(A_2\frac{3+\delta}{4\delta}-1)(A_3\frac{3+\delta}{4\delta}-1)(A_4\frac{3+\delta}{4\delta}-1)\\
&-(A_4\frac{3+\delta}{4\delta}-1)^4+A_4\frac{3+\delta}{4\delta}(A_4\frac{3+\delta}{4\delta}-1)(A_4\frac{3+\delta}{4\delta}-2)(A_4\frac{3+\delta}{4\delta}-3)\\
&+(A_1\frac{1+\delta}{2\delta}-1)(A_2\frac{1+\delta}{2\delta}-1)(A_3\frac{1+\delta}{2\delta}-1)(A_4\frac{1+\delta}{2\delta}-1)\\
&-(A_4\frac{1+\delta}{2\delta}-1)^4+A_4\frac{1+\delta}{2\delta}(A_4\frac{1+\delta}{2\delta}-1)(A_4\frac{1+\delta}{2\delta}-2)(A_4\frac{1+\delta}{2\delta}-3)\\
&+(A_1\frac{1+3\delta}{4\delta}-1)(A_2\frac{1+3\delta}{4\delta}-1)(A_3\frac{1+3\delta}{4\delta}-1)(A_4\frac{1+3\delta}{4\delta}-1)\\
&-(A_4\frac{1+3\delta}{4\delta}-1)^4+A_4\frac{1+3\delta}{4\delta}(A_4\frac{1+3\delta}{4\delta}-1)(A_4\frac{1+3\delta}{4\delta}-2)(A_4\frac{1+3\delta}{4\delta}-3)\\
&+(A_1-1)(A_2-1)(A_3-1)(A_4-1)-(A_4-1)^4+A_4(A_4-1)(A_4-2)(A_4-3)].
\end{aligned}
\tag{2.24}
$$

It is sufficient to show that R.H.S. of (1.8) is strictly larger than R.H.S. of (2.24).

Lemma 2.5 *When* $4 < a_5 < 5$*, R.H.S. of (1.8)* $>$ *R.H.S. of (2.24).*

Proof. Substitute $a_i = \frac{A_i}{\delta}$, $i = 1, 2, 3, 4$ and $a_5 = \frac{4}{1-\delta}$ to R.H.S. of (1.8), subtract that by R.H.S. of (2.24), and multiply $(1-\delta)^5\delta^4$, we get

$$
\begin{aligned}
\triangle_5 \triangleq& A_1A_2A_3A_4(\frac{139}{128}-\frac{643}{128}\delta+\frac{283}{32}\delta^2-\frac{207}{32}\delta^3-\frac{219}{64}\delta^4+\frac{1499}{64}\delta^5\\
&\qquad-\frac{1585}{32}\delta^6+\frac{1765}{32}\delta^7-\frac{3965}{128}\delta^8+\frac{885}{128}\delta^9)\\
&+(A_1A_2A_3+A_1A_2A_4+A_1A_3A_4+A_2A_3A_4)\\
&\qquad\cdot(-\frac{3}{16}\delta+\frac{13}{8}\delta^2-\frac{37}{8}\delta^3+\frac{73}{8}\delta^4-24\delta^5+\frac{407}{8}\delta^6-\frac{475}{8}\delta^7+\frac{275}{8}\delta^8-\frac{125}{16}\beta^9)\\
&+(A_1A_2+A_1A_3+A_1A_4+A_2A_3+A_2A_4+A_3A_4)\\
&\qquad\cdot(-\frac{11}{8}\delta^2+\frac{37}{8}\delta^3-\frac{63}{8}\delta^4+\frac{177}{8}\delta^5-\frac{433}{8}\delta^6+\frac{543}{8}\delta^7-\frac{325}{8}\delta^8+\frac{75}{8}\delta^9)\\
&+(A_1+A_2+A_3)(\frac{9}{2}\delta^3-14\delta^4-\frac{3}{2}\delta^5+56\delta^6-\frac{173}{2}\delta^7+54\delta^8-\frac{25}{2}\delta^9)\\
&+A_4^3(\frac{45}{8}\delta-\frac{75}{4}\delta^2+\frac{75}{4}\delta^3+\frac{25}{4}\delta^4-50\delta^5+\frac{415}{4}\delta^6-\frac{475}{4}\delta^7+\frac{275}{4}\delta^8-\frac{125}{8}\delta^9)\\
&+A_4^2(-\frac{175}{8}\delta^2+\frac{625}{8}\delta^3-\frac{875}{8}\delta^4+\frac{1125}{8}\delta^5-\frac{2125}{8}\delta^6+\frac{2675}{8}\delta^7-\frac{1625}{8}\delta^8+\frac{375}{8}\delta^9)\\
&+A_4(\frac{39}{2}\delta^3-64\delta^4+\frac{47}{2}\delta^5+156\delta^6-\frac{523}{2}\delta^7+164\delta^8-\frac{75}{2}\delta^9)\\
&+(-240\delta^4-320\delta^5+160\delta^6+320\delta^7+80\delta^8)
\end{aligned}
$$

The idea is to show that for all $\delta \in (0, \frac{1}{5})$, the minimum of $\triangle_5$ in $A_1 \geq 4,\ A_2 \geq 3,\ A_3 \geq 2$ and $A_4 \geq 1$ occurs at $A_1 = 4,\ A_2 = 3,\ A_3 = 2$ and $A_4 = 1$ and $\triangle_5 \mid_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$, for all $\delta \in (0, \frac{1}{5})$.

$$\frac{\partial^4 \triangle_5}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} = \frac{1}{128}(-1+\delta)^4(139 - 87\delta - 50\delta^2 + 50\delta^3 - 425\delta^4 + 885\delta^5) > 0,$$

for $\delta \in (0, \frac{1}{5})$, since $139 - 87\delta - 50\delta^2 + 50\delta^3 - 425\delta^4 + 885\delta^5 > 139 - 87 \times \frac{1}{5} - 50 \times (\frac{1}{5})^2 + (50 - 425 \times \frac{1}{5})\delta^3 + 885\delta^5 = \frac{598}{5} - 35\delta^3 + 885\delta^5 > \frac{2983}{25} + 885\delta^5 > 0$, for $\delta \in (0, \frac{1}{5})$. It follows that $\frac{\partial^3 \triangle_5}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 for $\delta \in (0, \frac{1}{5})$, $A_4 \geq 1$. Hence the minimum of $\frac{\partial^3 \triangle_5}{\partial A_1 \partial A_2 \partial A_3}$ occurs at $A_4 = 1$,

$$\begin{aligned}
&\frac{\partial^3 \triangle_5}{\partial A_1 \partial A_2 \partial A_3}\bigg|_{A_4=1} \\
&= \bigg[A_4(\frac{139}{128} - \frac{643}{128}\delta + \frac{283}{32}\delta^2 - \frac{207}{32}\delta^3 - \frac{219}{64}\delta^4 + \frac{1499}{64}\delta^5 \\
&\qquad - \frac{1585}{32}\delta^6 + \frac{1765}{32}\delta^7 - \frac{3965}{128}\delta^8 + \frac{885}{128}\delta^9) \\
&\qquad + (-\frac{3}{16}\delta + \frac{13}{8}\delta^2 - \frac{37}{8}\delta^3 + \frac{73}{8}\delta^4 - 24\delta^5 + \frac{407}{8}\delta^6 - \frac{475}{8}\delta^7 + \frac{275}{8}\delta^8 - \frac{125}{16}\beta^9)\bigg]\bigg|_{A_4=1} \\
&= -\frac{1}{128}(-1+\delta^5)(139 + 28\delta + 90\delta^2 + 140\delta^3 + 115\delta^4) > 0,
\end{aligned}$$

for $\delta \in (0, \frac{1}{5})$. It follows that $\frac{\partial^3 \triangle_5}{\partial A_1 \partial A_2 \partial A_3} > 0$ for $A_4 \geq 1$, $\delta \in (0, \frac{1}{5})$. Note that $\frac{\partial^2 \triangle_5}{\partial A_1 \partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3 \triangle_5}{\partial A_1 \partial A_2 \partial A_4} > 0$, for $A_3 \geq 1$, $\delta \in (0, \frac{1}{5})$. Moreover, we have $\frac{\partial^2 \triangle_5}{\partial A_1 \partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. The minimum of $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_2}$ occurs at $A_3 = A_4 = 1$,

$$\begin{aligned}
&\frac{\partial^2 \triangle_5}{\partial A_1 \partial A_2}\bigg|_{A_3=A_4=1} \\
&= \bigg[A_3 A_4(\frac{139}{128} - \frac{643}{128}\delta + \frac{283}{32}\delta^2 - \frac{207}{32}\delta^3 - \frac{219}{64}\delta^4 + \frac{1499}{64}\delta^5 \\
&\qquad - \frac{1585}{32}\delta^6 + \frac{1765}{32}\delta^7 - \frac{3965}{128}\delta^8 + \frac{885}{128}\delta^9) \\
&\quad + (A_3 + A_4)(-\frac{3}{16}\delta + \frac{13}{8}\delta^2 - \frac{37}{8}\delta^3 + \frac{73}{8}\delta^4 - 24\delta^5 + \frac{407}{8}\delta^6 - \frac{475}{8}\delta^7 + \frac{275}{8}\delta^8 - \frac{125}{16}\beta^9) \\
&\quad + (-\frac{11}{8}\delta^2 + \frac{37}{8}\delta^3 - \frac{63}{8}\delta^4 + \frac{177}{8}\delta^5 - \frac{433}{8}\delta^6 + \frac{543}{8}\delta^7 - \frac{325}{8}\delta^8 + \frac{75}{8}\delta^9)\bigg]\bigg|_{A_3=A_4=1} \\
&= \frac{1}{128}(-1+\delta)^6(139 + 143\delta + 145\delta^2 + 85\delta^3) > 0,
\end{aligned}$$

for $\delta \in (0, \frac{1}{5})$. It follows that $\frac{\partial^2 \triangle_5}{\partial A_1 \partial A_2} > 0$, for $A_3 \geq A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. From the property that $\frac{\partial \triangle_5}{\partial A_1}$ is symmetric with respect to A_2, A_3 and A_4, we also get $\frac{\partial^2 \triangle_5}{\partial A_1 \partial A_3} > 0$, for $A_2 \geq A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$ and $\frac{\partial^2 \triangle_5}{\partial A_1 \partial A_4} > 0$, for $A_2 \geq A_3 \geq 1$ and $\delta \in (0, \frac{1}{5})$. Therefore, we have $\frac{\partial \triangle_5}{\partial A_1}$ is an increasing function of A_2, A_3 and A_4 for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. Hence the minimum of $\frac{\partial \triangle_5}{\partial A_1}$ occurs at $A_2 = A_3 = A_4 = 1$,

$$
\begin{aligned}
&\frac{\partial \triangle_5}{\partial A_1}\bigg|_{A_2=A_3=A_4=1} \\
=&\bigg[A_2A_3A_4(\frac{139}{128} - \frac{643}{128}\delta + \frac{283}{32}\delta^2 - \frac{207}{32}\delta^3 - \frac{219}{64}\delta^4 + \frac{1499}{64}\delta^5 \\
&\qquad - \frac{1585}{32}\delta^6 + \frac{1765}{32}\delta^7 - \frac{3965}{128}\delta^8 + \frac{885}{128}\delta^9) \\
&+ (A_2A_3 + A_2A_4 + A_3A_4) \\
&\qquad \cdot (-\frac{3}{16}\delta + \frac{13}{8}\delta^2 - \frac{37}{8}\delta^3 + \frac{73}{8}\delta^4 - 24\delta^5 + \frac{407}{8}\delta^6 - \frac{475}{8}\delta^7 + \frac{275}{8}\delta^8 - \frac{125}{16}\beta^9) \\
&+ (A_2 + A_3 + A_4)(-\frac{11}{8}\delta^2 + \frac{37}{8}\delta^3 - \frac{63}{8}\delta^4 + \frac{177}{8}\delta^5 - \frac{433}{8}\delta^6 + \frac{543}{8}\delta^7 - \frac{325}{8}\delta^8 + \frac{75}{8}\delta^9) \\
&+(\frac{9}{2}\delta^3 - 14\delta^4 - \frac{3}{2}\delta^5 + 56\delta^6 - \frac{173}{2}\delta^7 + 54\delta^8 - \frac{25}{2}\delta^9)\bigg]\bigg|_{A_2=A_3=A_4=1} \\
=& - \frac{1}{128}(-1+\delta)^7(139 + 258\delta + 115\delta^2) > 0,
\end{aligned}
$$

for $\delta \in (0, \frac{1}{5})$. It follows that $\frac{\partial \triangle_5}{\partial A_1} > 0$, for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. By the property that $\triangle_5$ is symmetric with respect to A_1, A_2 and A_3. We also have $\frac{\partial \triangle_5}{\partial A_2} > 0$, for $A_1 \geq A_3 \geq A_4 \geq 1$ and $\frac{\partial \triangle_5}{\partial A_3} > 0$, for $A_1 \geq A_2 \geq A_4 \geq 1$. Meanwhile,

$$
\frac{\partial^3 \triangle_5}{\partial A_4^3} = -\frac{15}{4}(-1+\delta)^5\delta(3+5\delta)(3+5\delta^2) > 0,
$$

for $\delta \in (0, \frac{1}{5})$. It follows that $\frac{\partial^2 \triangle_5}{\partial A_4^2}$ is an increasing function of A_4, for $A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. Thus, the minimum of $\frac{\partial^2 \triangle_5}{\partial A_4^2}$ occurs at $A_4 = 1$,

$$
\begin{aligned}
&\frac{\partial^2 \triangle_5}{\partial A_4^2}\bigg|_{A_4=1} \\
=&\bigg[6A_4(\frac{45}{8}\delta - \frac{75}{4}\delta^2 + \frac{75}{4}\delta^3 + \frac{25}{4}\delta^4 - 50\delta^5 + \frac{415}{4}\delta^6 - \frac{475}{4}\delta^7 + \frac{275}{4}\delta^8 - \frac{125}{8}\delta^9) \\
&+2(-\frac{175}{8}\delta^2 + \frac{625}{8}\delta^3 - \frac{875}{8}\delta^4 + \frac{1125}{8}\delta^5 - \frac{2125}{8}\delta^6 + \frac{2675}{8}\delta^7 - \frac{1625}{8}\delta^8 + \frac{375}{8}\delta^9)\bigg]\bigg|_{A_4=1} \\
=&\frac{5}{4}(-1+\delta)^5\delta(-27 - 10\delta + 5\delta^2) > 0,
\end{aligned}
$$

for $\delta \in (0, \frac{1}{5})$. In fact, let $f(\delta) \triangleq -27 - 10\delta + 5\delta^2 = 5(\delta-1)^2 - 32$. Thus, $f(\delta) < f(0) = -27 < 0$, for $\gamma \in (0, \frac{2}{5})$. It follows that $\frac{\partial^2 \triangle_5}{\partial A_4^2} > 0$, for $A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. Thus, $\frac{\partial \triangle_5}{\partial A_4}$ is an increasing function of A_4, for $A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. Moreover, it's an increasing function with respect to A_1, A_2, A_3 and A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$, $\delta \in (0, \frac{1}{5})$, since $\frac{\partial \triangle_5}{\partial A_4}$ is symmetric with respect to A_1, A_2 and A_3. The minimum of $\frac{\partial \triangle_5}{\partial A_4}$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$,

$$\begin{aligned}
&\frac{\partial \triangle_5}{\partial A_4}\Big|_{A_1=A_2=A_3=A_4=1} \\
=&\Big[A_1A_2A_3\big(\frac{139}{128} - \frac{643}{128}\delta + \frac{283}{32}\delta^2 - \frac{207}{32}\delta^3 - \frac{219}{64}\delta^4 + \frac{1499}{64}\delta^5 \\
&\qquad - \frac{1585}{32}\delta^6 + \frac{1765}{32}\delta^7 - \frac{3965}{128}\delta^8 + \frac{885}{128}\delta^9\big) \\
&+ (A_1A_2 + A_1A_3 + A_2A_3) \\
&\quad \cdot \big(-\frac{3}{16}\delta + \frac{13}{8}\delta^2 - \frac{37}{8}\delta^3 + \frac{73}{8}\delta^4 - 24\delta^5 + \frac{407}{8}\delta^6 - \frac{475}{8}\delta^7 + \frac{275}{8}\delta^8 - \frac{125}{16}\beta^9\big) \\
&+ (A_1 + A_2 + A_3)\big(-\frac{11}{8}\delta^2 + \frac{37}{8}\delta^3 - \frac{63}{8}\delta^4 + \frac{177}{8}\delta^5 - \frac{433}{8}\delta^6 + \frac{543}{8}\delta^7 - \frac{325}{8}\delta^8 + \frac{75}{8}\delta^9\big) \\
&+ 3A_4^2\big(\frac{45}{8}\delta - \frac{75}{4}\delta^2 + \frac{75}{4}\delta^3 + \frac{25}{4}\delta^4 - 50\delta^5 + \frac{415}{4}\delta^6 - \frac{475}{4}\delta^7 + \frac{275}{4}\delta^8 - \frac{125}{8}\delta^9\big) \\
&+ 2A_4\big(-\frac{175}{8}\delta^2 + \frac{625}{8}\delta^3 - \frac{875}{8}\delta^4 + \frac{1125}{8}\delta^5 \\
&\qquad -\frac{2125}{8}\delta^6 + \frac{2675}{8}\delta^7 - \frac{1625}{8}\delta^8 + \frac{375}{8}\delta^9\big)\Big]\Big|_{A_1=A_2=A_3=A_4=1} \\
=&\frac{1}{128}(-1+\delta)^5(-139 - 2140\delta + 2262\delta^2 + 2452\delta^3 + 2685\delta^4) > 0,
\end{aligned}$$

for $\delta \in (0, \frac{1}{5})$, since $-139 - 2140\delta + 2262\delta^2 + 2452\delta^3 + 2685\delta^4 < -139 + (-2140 + 2262 \times \frac{1}{5} + 2452 \times (\frac{1}{5})^2 + 2685 \times (\frac{1}{5})^3)\delta = -139 - \frac{39201}{25}\delta < 0$, for $\delta \in (0, \frac{1}{5})$. It follows that $\frac{\partial \triangle_5}{\partial A_4} > 0$, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. Therefore, $\triangle_5$ is an increasing function of A_1, A_2, A_3 and A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. Thus, the minimum of $\triangle_5$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$ and take condition (2.14) into consideration,

$$\begin{aligned}
&\triangle_5|_{A_1=4, A_2=3, A_3=2, A_4=1} \\
=&\frac{1}{16}\delta(417 - 1572\delta + 1704\delta^2 + 620\delta^3 - 6334\delta^4 - 7660\delta^5 - 5760\delta^6 - 2140\delta^7 + 245\delta^8) > 0,
\end{aligned}$$

for $\delta \in (0, \frac{1}{5})$, since $417 - 1572\delta + 1704\delta^2 + 620\delta^3 - 6334\delta^4 - 7660\delta^5 - 5760\delta^6 - 2140\delta^7 + 245\delta^8 > 417 - 1572 \times \frac{1}{5} + 1704\delta^2 + (620 - 6334 \times \frac{1}{5} - 7660 \times (\frac{2}{5})^2 - 5760 \times (\frac{2}{5})^3 - 2140 \times (\frac{2}{5})^4)\delta^3 + 245\delta^8 = \frac{513}{5} + 1704\delta^2 - \frac{125338}{125}\delta^3 + 245\delta^8 > \frac{513}{5} + (1704 - \frac{125338}{125} \times \frac{1}{5})\delta^2 + 245\delta^8 = \frac{513}{5} + \frac{939662}{625}\delta^2 + 245\delta^8 > 0$, for $\delta \in (0, \frac{1}{5})$. It follows that $\triangle_5 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\delta \in (0, \frac{1}{5})$. $\square$

2.2 Proof of Theorem 1.3

As we state in introduction, the estimate of Dickman-De Bruijn function $\psi(x,y)$ is equivalent to a sharp estimate of Q_n (or P_n by (1.2)). We've already got the estimate of P_n, $n \le 5$, thus, apply this to our estimate of $\psi(x,y)$. In detail, Let $p_1 < p_2 < \cdots < p_n$ be five prime numbers up to y. It is clear that $p_1^{l_1} p_2^{l_2} \cdots p_5^{l_5} \le x$ if and only if $\frac{l_1}{\frac{\log x}{\log p_1}} + \frac{l_2}{\frac{\log x}{\log p_2}} \cdots \frac{l_n}{\frac{\log x}{\log p_n}} \le 1$. Therefore, $\psi(x,y)$ is precisely the number Q_n of (1.1) with $a_i = \frac{\log x}{\log p_i}$, $1 \le i \le n$. Moreover, by (1.2), $\psi(x,y)$ is also precisely the number $P(a_1(1+a), a_2(1+a), \cdots, a_n(1+a))$, where $a = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$.

According to the number of prime numbers up to y, we split the proof of Theorem 1.3 into three cases:

Case (i): $5 \le y < 7$;

Case (ii): $7 \le y < 11$;

Case (iii): $11 \le y < 13$.

Case (i) and (ii) have been proven in [18]. For Case (iii), we have five prime numbers $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$ and $p_5 = 11$, thus $a = \frac{\log 2 + \log 3 + \log 5 + \log 7 + \log 11}{\log x}$. Therefore,

$$\begin{aligned}
\psi(x,y) &= Q_5 \\
&= P\Big(\frac{\log x}{\log 2}\Big(1 + \frac{\log(2\times3\times5\times7\times11)}{\log x}\Big), \frac{\log x}{\log 3}\Big(1 + \frac{\log(2\times3\times5\times7\times11)}{\log x}\Big), \\
&\quad \frac{\log x}{\log 5}\Big(1 + \frac{\log(2\times3\times5\times7\times11)}{\log x}\Big), \frac{\log x}{\log 7}\Big(1 + \frac{\log(2\times3\times5\times7\times11)}{\log x}\Big), \\
&\quad \frac{\log x}{\log 11}\Big(1 + \frac{\log(2\times3\times5\times7\times11)}{\log x}\Big)\Big) \\
&\le \frac{1}{5!}\Big\{\Big(\frac{\log x}{\log 2} + \frac{\log(3\times5\times7\times11)}{\log 2}\Big)\Big(\frac{\log x}{\log 3} + \frac{\log(2\times5\times7\times11)}{\log 3}\Big) \\
&\quad \cdot \Big(\frac{\log x}{\log 5} + \frac{\log(2\times3\times7\times11)}{\log 5}\Big)\Big(\frac{\log x}{\log 7} + \frac{\log(2\times3\times5\times11)}{\log 7}\Big) \\
&\quad \cdot \Big(\frac{\log x}{\log 11} + \frac{\log(2\times3\times5\times7)}{\log 11}\Big) \\
&\quad - \Big[\Big(\frac{\log x}{\log 11} + \frac{\log(2\times3\times5\times7)}{\log 11}\Big)^5 \\
&\qquad - \Big(\frac{\log x}{\log 11} + 1 + \frac{\log(2\times3\times5\times7)}{\log 11}\Big)\Big(\frac{\log x}{\log 11} + \frac{\log(2\times3\times5\times7)}{\log 11}\Big) \\
&\qquad \cdot \Big(\frac{\log x}{\log 11} + \frac{\log(2\times3\times5\times7)}{\log 11} - 1\Big)\Big(\frac{\log x}{\log 11} + \frac{\log(2\times3\times5\times7)}{\log 11} - 2\Big) \\
&\qquad \cdot \Big(\frac{\log x}{\log 11} + \frac{\log(2\times3\times5\times7)}{\log 11} - 3\Big)
\end{aligned}$$

$$=\frac{1}{120}\{\frac{1}{\log 2\log 3\log 5\log 7\log 11}(\log x+\log 1155)(\log x+\log 770)(\log x+\log 462)$$

$$\cdot(\log x+\log 330)(\log x+\log 210)$$

$$-\frac{1}{\log^5 11}[(\log x+\log 210)^5$$

$$-(\log x+\log 11+\log 210)(\log x+\log 210)(\log x+\log 210-\log 11)$$

$$\cdot(\log x+\log 210-2\log 11)(\log x+\log 210-3\log 11)]\}.$$

□

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