SINGULARITIES DEFINED BY $s\ell(2, \mathbf{C})$ INVARIANT POLYNOMIALS AND SOLVABILITY OF LIE ALGEBRAS ARISING FROM ISOLATED SINGULARITIES

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1. Introduction. Let $(V, 0)$ be an isolated singularity in $(\mathbf{C}^n, 0)$ defined by the zero set of a holomorphic function f. The moduli algebra $A(V)$ of $(V, 0)$ is $\mathbf{C}\{x_1, x_2, \ldots, x_n\}/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. It is easy to see that $A(V)$ is an invariant of $(V, 0)$. In [1], Mather and the author proved that the complex analytic structure of $(V, 0)$ is determined also by $A(V)$. Thus, the above construction gives an injection map from the space of isolated singularities in $(\mathbf{C}^n, 0)$ to the space of commutative local Artinian algebras. This raises a natural and important problem, the so called recognition problem: Give a necessary and sufficient condition for a commutative local Artinian algebra to be a moduli algebra. In [3], we define $L(V)$ to be the algebra of derivations of $A(V)$. Clearly $L(V)$ is a finite dimensional Lie algebra. The main purpose of this paper is to prove that $L(V)$ is solvable for $n \leq 5$. Naturally one expects that a necessary condition for a commutative local Artinian algebra to be a moduli algebra is that its algebra of derivations is a solvable Lie algebra. In this paper we have used the main theorem which is available in preprint form in [4] for $n \leq 5$. Assume that the results in [4] remain valid as expected for $n \geq 6$. Then the proof given in this article also applies smoothly to arbitrary n. In Section 2, we classify the actions of $s\ell(2, \mathbf{C})$ on $\mathbf{C}[[x_1, x_2, \ldots, x_n]]$ via derivations preserving the m-adic filtration. The main point here is to get rid of the "higher order operator" (i.e., $\Sigma_i\, a_i(\partial/\partial x_i)$ with multiplicity of $a_i \geq 2$) by means of the vanishing theorem for semisimple Lie algebra cohomology. It seems to us that the material here is not available in literature form. I would like to thank Professor H. Sah for

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many useful discussions. In Section 3 we prove that the singular set of $s\ell(2, \mathbf{C})$ invariant polynomials of degree at least 3 in five variables is at least one dimensional. Similar results can be generalized to higher dimension without difficulty. In Section 4, we prove our main theorem that $L(V)$ is solvable for $n \leq 5$.

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2. Classification of $s\ell(2, \mathbf{C})$ in Der $\mathbf{C}[[x_1, x_2, \ldots, x_n]]$ preserving the m-adic filtration.

PROPOSITION 2.1. *Let $L = s\ell(2, \mathbf{C})$ act on $\mathbf{C}[[x_1, \ldots, x_n]]$ via derivations preserving the m-adic filtration i.e., $L(m^k) \subseteq m^k$ where m is the maximal ideal in $\mathbf{C}[[x_1, \ldots, x_n]]$. Then there exists a coordinate change $y_1, \ldots, y_n$ with respect to which $s\ell(2, \mathbf{C})$ is spanned by*

$$h = \sum_{j=1}^{n} a_{1j} \frac{\partial}{\partial y_j}$$

$$e = \sum_{j=1}^{n} a_{2j} \frac{\partial}{\partial y_j}$$

$$f = \sum_{j=1}^{n} a_{3j} \frac{\partial}{\partial y_j}$$

where a_{ij} is a linear function in $y_1, \ldots, y_n$ variables for all $1 \leq i \leq 3$ and $1 \leq j \leq n$. Here $\{h, e, f\}$ is a standard basis for $s\ell(2, \mathbf{C})$ i.e., $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$.

Proof. Let σ be an element of L. Let $U_1^{(1)}(\sigma)$ be the matrix representation of σ on m/m^2 and $U_2^{(1)}(\sigma)$ be the matrix representation of σ on m^2/m^3. Since σ preserves the m-adic filtration, the matrix representation of σ on m/m^3 is given as follows

$$W^{(1)}(\sigma) = \begin{pmatrix} U_1^{(1)}(\sigma) & 0 \\ T_{12}^{(1)}(\sigma) & U_2^{(1)}(\sigma) \end{pmatrix}$$

where $T_{12}^{(1)}(\sigma)$ represents an element in $\mathrm{Hom}_{\mathbf{C}}(m/m^2, m^2/m^3)$. Observe that

$$W^{(1)}([\sigma, \tau]) = W^{(1)}(\sigma)W^{(1)}(\tau) - W^{(1)}(\tau)W^{(1)}(\sigma)$$

$$= \begin{pmatrix} U_1^{(1)}(\sigma) & 0 \\ T_{12}^{(1)}(\sigma) & U_2^{(1)}(\sigma) \end{pmatrix} \begin{pmatrix} U_1^{(1)}(\tau) & 0 \\ T_{12}^{(1)}(\tau) & U_2^{(1)}(\tau) \end{pmatrix}$$

$$- \begin{pmatrix} U_1^{(1)}(\tau) & 0 \\ T_{12}^{(1)}(\tau) & U_2^{(1)}(\tau) \end{pmatrix} \begin{pmatrix} U_1^{(1)}(\sigma) & 0 \\ T_{12}^{(1)}(\sigma) & U_2^{(1)}(\sigma) \end{pmatrix}$$

$$= \begin{pmatrix} U_1^{(1)}(\sigma)U_1^{(1)}(\tau) & 0 \\ T_{12}^{(1)}(\sigma)U_1^{(1)}(\tau) + U_2^{(1)}(\sigma)T_{12}^{(1)}(\tau) & U_2(\sigma)U_2(\tau) \end{pmatrix}$$

$$- \begin{pmatrix} U_1^{(1)}(\tau)U_1^{(1)}(\sigma) & 0 \\ T_{12}^{(1)}(\tau)U_1^{(1)}(\sigma) + U_2^{(1)}(\tau)T_{12}^{(1)}(\sigma) & U_2^{(1)}(\tau)U_2^{(1)}(\sigma) \end{pmatrix}$$

$$= \begin{pmatrix} U_1^{(1)}(\sigma)U_1^{(1)}(\tau) - U_1^{(1)}(\tau)U_1^{(1)}(\sigma) & 0 \\ T_{12}^{(1)}(\sigma)U_1^{(1)}(\tau) + U_2^{(1)}(\sigma)T_{12}^{(1)}(\tau) & U_2^{(1)}(\sigma)U_2^{(1)}(\tau) - U_2^{(1)}(\tau)U_2^{(1)}(\sigma) \\ - T_{12}^{(1)}(\tau)U_1^{(1)}(\sigma) - U_2^{(1)}(\tau)T_{12}^{(1)}(\sigma) & \end{pmatrix}$$

$$\Rightarrow T_{12}^{(1)}([\sigma, \tau]) = T_{12}^{(1)}(\sigma)U_1^{(1)}(\tau) + U_2^{(1)}(\sigma)T_{12}^{(1)}(\tau) - T_{12}^{(1)}(\tau)U_1^{(1)}(\sigma) - U_2^{(1)}(\tau)T_{12}^{(1)}(\sigma).$$

Observe also that $\mathrm{Hom}_{\mathbf{C}}(m/m^2, m^2/m^3)$ is a L-module. The action of L on $\mathrm{Hom}_{\mathbf{C}}(m/m^2, m^2/m^3)$ is given as follows. Let $\sigma \in L$, $\varphi \in \mathrm{Hom}_{\mathbf{C}}(m/m^2, m^2/m^3)$ and $u_1 \in m/m^2$

$$(\sigma\varphi)(u_1) = -\varphi(\sigma(u_1) + \sigma(\varphi(u_1)).$$

We now claim that $T_{12}^{(1)}: L \to \mathrm{Hom}_{\mathbf{C}}(m/m^2, m^2/m^3)$ is a 1-cocycle of L with coefficient in $\mathrm{Hom}_{\mathbf{C}}(m/m^2, m^2/m^3)$. To see this, consider

$$\delta T_{12}^{(1)}(\sigma, \tau) = \sigma \cdot T_{12}^{(1)}(\tau) - \tau \cdot T_{12}^{(1)}(\sigma) - T_{12}^{(1)}([\sigma, \tau]).$$

For any $v \in m/m^2$, we have

$$\begin{aligned}[\delta T_{12}^{(1)}(\sigma, \tau)](v) &= [\sigma \cdot T_{12}^{(1)}(\tau)](v) - [\tau \cdot T_{12}^{(1)}(\sigma)](v) - T_{12}^{(1)}([\sigma, \tau])(v) \\ &= -T_{12}^{(1)}(\tau)(\sigma(v)) + \sigma(T_{12}^{(1)}(\tau)(v)) + T_{12}^{(1)}(\sigma)(\tau(v)) - \tau(T_{12}^{(1)}(\sigma)(v)) \\ &\quad - T_{12}^{(1)}(\sigma)U_1^{(1)}(\tau)(v) - U_2^{(1)}(\sigma)T_{12}^{(1)}(\tau)(v) \\ &\quad + T_{12}^{(1)}(\tau)U_1^{(1)}(\sigma)(v) + U_2^{(1)}(\tau)T_{12}^{(1)}(\sigma)(v) \\ &= 0.\end{aligned}$$

Therefore $T_{12}^{(1)}$ is a 1-cocycle as claimed. Since L is simple, $H^1(L, \mathrm{Hom}_{\mathbf{C}}(m/m^2, m^2/m^3)) = 0$. We conclude that $T_{12}^{(1)}$ is a 1-coboundary. There exists $\beta^{(2)} \in \mathrm{Hom}(m/m^2, m^2/m^3)$ a 0-cochain of L with coefficients in $\mathrm{Hom}(m/m^2, m^2/m^3)$ such that

$$\begin{aligned}T_{12}^{(1)}(\sigma) &= (\delta\beta^{(2)})(\sigma) \qquad \forall \sigma \in L \\ &= \sigma\beta^{(2)} \\ \Rightarrow T_{12}^{(1)}(\sigma)(v) &= \sigma\beta^{(2)}(v) \qquad \forall v \in m/m^2 \\ &= \sigma(\beta^{(2)}(v)) - \beta^{(2)}(\sigma(v)).\end{aligned}$$

Let $S^{(2)}$ be the matrix representation of $\beta^{(2)}$. Thus we have

$$T_{12}^{(1)}(\sigma)(v) = U_2^{(1)}(\sigma)S^{(2)}(v) - S^{(2)}U_1^{(1)}(\sigma)(v).$$

This is equivalent to say that

$$\begin{pmatrix} I & 0 \\ S^{(2)} & I \end{pmatrix}\begin{pmatrix} U_1^{(1)}(\sigma) & 0 \\ T_{12}^{(1)}(\sigma) & U_2^{(1)}(\sigma) \end{pmatrix} = \begin{pmatrix} U_1^{(1)}(\sigma) & 0 \\ 0 & U_2^{(1)}(\sigma) \end{pmatrix}\begin{pmatrix} I & 0 \\ S^{(2)} & I \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} I & 0 \\ S^{(2)} & I \end{pmatrix}\begin{pmatrix} U_1^{(1)}(\sigma) & 0 \\ T_{12}^{(1)}(\sigma) & U_2^{(1)}(\sigma) \end{pmatrix}\begin{pmatrix} I & 0 \\ S^{(2)} & I \end{pmatrix}^{-1} = \begin{pmatrix} U_1^{(1)}(\sigma) & 0 \\ 0 & U_2^{(1)}(\sigma) \end{pmatrix}$$

The above equation means that we can make a change of variable in the following form

$$y_1^{(2)} = x_1 + q_1^{(2)}(x_1, x_2, \ldots, x_n)$$
$$y_2^{(2)} = x_2 + q_2^{(2)}(x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$y_n^{(2)} = x_n + q_n^{(2)}(x_1, x_2, \ldots, x_n)$$

where $q_i^{(2)}$ is a homogeneous polynomial of degree 2 for $1 \leq i \leq n$, such that with respect to such coordinate, the matrix representation of σ on m/m^3 is given by

$$\begin{pmatrix} U_1^{(2)}(\sigma) & 0 \\ 0 & U_2^{(2)}(\sigma) \end{pmatrix} = \begin{pmatrix} U_1^{(1)}(\sigma) & 0 \\ 0 & U_2^{(1)}(\sigma) \end{pmatrix}$$

i.e., with respect to the coordinate system $y_1^{(2)}, \ldots, y_n^{(2)}$, $s\ell(2, \mathbf{C})$ is spanned by

$$h = \sum_{j=1}^{n} a_{1j}^{(2)} \frac{\partial}{\partial y_j^{(2)}} + \sum_{j=1}^{n} b_{1j}^{(2)} \frac{\partial}{\partial y_j^{(2)}}$$

$$e = \sum_{j=1}^{n} a_{2j}^{(2)} \frac{\partial}{\partial y_j^{(2)}} + \sum_{j=1}^{n} b_{2j}^{(2)} \frac{\partial}{\partial y_j^{(2)}}$$

$$f = \sum_{j=1}^{n} a_{3j}^{(2)} \frac{\partial}{\partial y_j^{(2)}} + \sum_{j=1}^{n} b_{3j}^{(2)} \frac{\partial}{\partial y_j^{(2)}}$$

where $a_{ij}^{(2)}$ is a linear function in $y_1^{(2)}, \ldots, y_n^{(2)}$ and $b_{ij}^{(2)}$ is a polynomial in $y_1^{(2)}, \ldots, y_n^{(2)}$ with multiplicity at least three.

By induction, we shall assume that there exists coordinates

$$y_1^{(k)} = y_1^{(k-1)} + q_1^{(k)}(y_1^{(k-1)}, y_2^{(k-1)}, \ldots, y_n^{(k-1)})$$
$$y_2^{(k)} = y_2^{(k-1)} + q_2^{(k)}(y_1^{(k-1)}, y_2^{(k-1)}, \ldots, y_n^{(k-1)})$$
$$\vdots$$
$$y_n^{(k)} = y_n^{(k-1)} + q_n^{(k)}(y_1^{(k-1)}, y_2^{(k-1)}, \ldots, y_n^{(k-1)})$$

where $q_i^{(k)}$ is a homogeneous polynomial of degree for $1 \leq i \leq n$ such that with respect to such coordinate, the matrix representation of σ on m/m^{k+1} is given by

$$\begin{pmatrix} U_1^{(k)}(\sigma) & & & \mathbf{0} \\ & U_2^{(k)}(\sigma) & & \\ \mathbf{0} & & \ddots & \\ & & & U_k^{(k)}(\sigma) \end{pmatrix}$$

where $U_i^{(k)}(\sigma)$ is the matrix representation of σ on m^i/m^{i+1}. This means that with respect to the coordinate system $y_1^{(k)}, \ldots, y_n^{(k)}$, $s\ell(2, \mathbf{C})$ is spanned by

$$h = \sum_{j=1}^{n} a_{1j}^{(k)} \frac{\partial}{\partial y_j^{(k)}} + \sum_{j=1}^{n} b_{1j}^{(k)} \frac{\partial}{\partial y_j^{(k)}}$$

$$e = \sum_{j=1}^{n} a_{2j}^{(k)} \frac{\partial}{\partial y_j^{(k)}} + \sum_{j=1}^{n} b_{2j}^{(k)} \frac{\partial}{\partial y_j^{(k)}}$$

$$f = \sum_{j=1}^{n} a_{3j}^{(k)} \frac{\partial}{\partial y_j^{(k)}} + \sum_{j=1}^{n} b_{3j}^{(k)} \frac{\partial}{\partial y_j^{(k)}}$$

where $a_{ij}^{(k)}$ is a linear function in $y_1^{(k)}, \ldots, y_n^{(k)}$ and $b_{ij}^{(k)}$ is a polynomial in $y_1^{(k)}, \ldots, y_n^{(k)}$ with multiplicity at least $k + 1$. The matrix representations of σ on m/m^{k+2} with respect to the coordinate system $y_1^{(k)}, \ldots, y_n^{(k)}$ is given by

$$W^{(k)}(\sigma) = \begin{pmatrix} U_1^{(k)}(\sigma) & & & & \mathbf{0} \\ & U_2^{(k)}(\sigma) & & & \\ \mathbf{0} & & \ddots & & \\ & & & U_k^{(k)}(\sigma) & \\ T_{1,k+1}^{(k)}(\sigma) & * & \cdots & * & U_{k+1}^{(k)}(\sigma) \end{pmatrix}$$

where $T^{(k)}_{1,k+1}(\sigma)$ represents an element in $\mathrm{Hom}_{\mathbf{C}}(m/m^2,\ m^{k+1}/m^{k+2})$. Observe that

$$W^{(k)}([\sigma,\tau]) = W^{(k)}(\sigma)W^{(k)}(\tau) - W^{(k)}(\tau)W^{(k)}(\sigma)$$

$$= \begin{pmatrix} U_1^{(k)}(\sigma) & & & & \mathbf{0} \\ & U_2^{(k)}(\sigma) & & & \\ \mathbf{0} & & \ddots & & \\ & & & U_k^{(k)}(\sigma) & \\ T^{(k)}_{1,k+1}(\sigma) & * & \cdots & * & U^{(k)}_{k+1}(\sigma) \end{pmatrix} \begin{pmatrix} U_1^{(k)}(\tau) & & & & \mathbf{0} \\ & U_2^{(k)}(\tau) & & & \\ \mathbf{0} & & \ddots & & \\ & & & U_k^{(k)}(\tau) & \\ T^{(k)}_{1,k+1}(\tau) & * & \cdots & * & U^{(k)}_{k+1}(\tau) \end{pmatrix}$$

$$- \begin{pmatrix} U_1^{(k)}(\tau) & & & & \mathbf{0} \\ & U_2^{(k)}(\tau) & & & \\ \mathbf{0} & & \ddots & & \\ & & & U_k^{(k)}(\tau) & \\ T^{(k)}_{1,k+1}(\tau) & * & \cdots & * & U^{(k)}_{k+1}(\tau) \end{pmatrix} \begin{pmatrix} U_1^{(k)}(\sigma) & & & & \mathbf{0} \\ & U_1^{(k)}(\sigma) & & & \\ \mathbf{0} & & \ddots & & \\ & & & U_k^{(k)}(\sigma) & \\ T^{(k)}_{1,k+1}(\sigma) & * & \cdots & * & U^{(k)}_{k+1}(\sigma) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & & & \mathbf{0} \\ & a_{22} & & \\ \mathbf{0} & & \ddots & \\ & & & \ddots \\ a_{k+1,1} & & & a_{k+1,k+1} \end{pmatrix}$$

where $a_{ii} = U_i^{(k)}(\sigma)U_i^{(k)}(\tau) - U_i^{(k)}(\tau)U_i^{(k)}(\sigma)$ and $a_{k+1,1} = T^{(k)}_{1,k+1}(\sigma)U_1^{(k)}(\tau) + U^{(k)}_{k+1}(\sigma)T^{(k)}_{1,k+1}(\tau) - T^{(k)}_{1,k+1}(\tau)U_1^{(k)}(\sigma) - U^{(k)}_{k+1}(\tau)T^{(k)}_{1,k+1}(\sigma)$. We now claim that $T^{(k)}_{1,k+1}\colon L \to \mathrm{Hom}_{\mathbf{C}}(m/m^2,\ m^{k+1}/m^{k+2})$ is a 1-cocycle. To see this, consider

$$\delta T^{(k)}_{1,k+1}(\sigma,\tau) = \sigma\cdot T^{(k)}_{1,k+1}(\tau) - \tau\cdot T^{(k)}_{1,k+1}(\sigma) - T^{(k)}_{1,k+1}([\sigma,\tau]).$$

For any $v \in m/m^2$, we have

$$\begin{aligned}[\delta\cdot T^{(k)}_{1,k+1}(\sigma,\tau)](v) &= [\sigma\cdot T^{(k)}_{1,k+1}(\tau)](v) - [\tau\cdot T^{(k)}_{1,k+1}(\sigma)](v) \\ &\quad - T^{(k)}_{1,k+1}([\sigma,\tau])(v) \\ &= -T^{(k)}_{1,k+1}(\tau)(\sigma(v)) + \sigma(T^{(k)}_{1,k+1}(\tau)(v))\end{aligned}$$

$$+ T^{(k)}_{1,k+1}(\sigma)(\tau(v)) - \tau(T^{(k)}_{1,k+1}(\sigma)(v))$$

$$- T^{(k)}_{1,k+1}(\sigma)U^{(k)}_1(\tau)(v) - U^{(k)}_{k+1}(\sigma)T^{(k)}_{1,k+1}(\tau)(v)$$

$$+ T^{(k)}_{1,k+1}(\tau)U^{(k)}_1(\sigma)(v) + U^{(k)}_{k+1}(\tau)T^{(k)}_{1,k+1}(\sigma)(v)$$

$$= 0.$$

Therefore $T^{(k)}_{1,k+1}$ is a 1-cocycle as claimed. Since L is simple $H^1(L, \mathrm{Hom}_{\mathbf{C}}(m/m^2, m^{k+1}/m^{k+2})) = 0$. We conclude that $T^{(k)}_{1,k+1}$ is a 1-coboundary. There exists $\beta^{(k+1)} \in \mathrm{Hom}(m/m^2, m^{k+1}/m^{k+2})$ a 0-cochain of L with coefficient in $\mathrm{Hom}(m/m^2, m^{k+1}/m^{k+2})$ such that

$$\begin{aligned} T^{(k)}_{1,k+1}(\sigma) &= (\delta\beta^{(k+1)})(\sigma) \qquad \forall \sigma \in L \\ &= \sigma\beta^{(k+1)} \\ &\Rightarrow T^{(k)}_{1,k+1}(\sigma)(v) = \sigma\beta^{(k+1)}(v) \qquad \forall v \in m/m^2 \\ &\qquad = \sigma(\beta^{(k+1)}(v)) - \beta^{(k+1)}(\sigma(v)) \end{aligned}$$

Let $S^{(k+1)}$ be the matrix representation of $\beta^{(k+1)}$. Then we have

$$T^{(k)}_{1,k+1}(\sigma)(v) = U^{(k)}_{k+1}(\sigma)S^{(k+1)}(v) - S^{(k+1)}U^{(k)}_1(\sigma)(v)$$

This is equivalent to say that

$$\begin{pmatrix} I & & & & \mathbf{0} \\ & I & & & \\ \mathbf{0} & & \ddots & & \\ & & & I & \\ S^{(k+1)} & * & \cdots & * & I \end{pmatrix} \begin{pmatrix} U^{(k)}_1(\sigma) & & & & \mathbf{0} \\ & U^{(k)}_2(\sigma) & & & \\ \mathbf{0} & & \ddots & & \\ & & & U^{(k)}_k(\sigma) & \\ T_{1,k+1}(\sigma) & * & \cdots & * & U^{(k)}_{(k+1)}(\sigma) \end{pmatrix}$$

$$= \begin{bmatrix} U_1^{(k)}(\sigma) & & & & 0 \\ & U_2^{(k)}(\sigma) & & & \\ 0 & & \ddots & & \\ & & & U_k^{(k)}(\sigma) & \\ 0 & 0 & \cdots & 0 & U_{k+1}^{(k)}(\sigma) \end{bmatrix} \begin{bmatrix} I & & & & 0 \\ & I & & & \\ 0 & & \ddots & & \\ & & & I & \\ S^{(k+1)} & * & \cdots & * & I \end{bmatrix}$$

This means that we can make a change of variable in the following form

$$\begin{aligned} y_1^{(k+1)} &= y_1^{(k)} + q_1^{(k+1)}(y_1^{(k)}, \ldots, y_n^{(k)}) \\ y_2^{(k+1)} &= y_2^{(k)} + q_2^{(k+1)}(y_1^{(k)}, \ldots, y_n^{(k)}) \\ &\vdots \\ y_n^{(k+1)} &= y_n^{(k)} + q_n^{(k+1)}(y_1^{(k)}, \ldots, y_n^{(k)}) \end{aligned}$$

where $q_i^{(k+1)}$ is a homogeneous polynomial of degree $k + 1$ for $1 \leq i \leq n$ such that with respect to such coordinate, the matrix representation of σ on m/m^{k+2} is given by

$$\begin{bmatrix} U_1^{(k+1)}(\sigma) & & & & 0 \\ & U_2^{(k+1)}(\sigma) & & & \\ 0 & & \ddots & & \\ & & & U_k^{(k+1)}(\sigma) & \\ & & & & U_{k+1}^{(k+1)}(\sigma) \end{bmatrix}$$

$$= \begin{bmatrix} U_1^{(k)}(\sigma) & & & & 0 \\ & U_2^{(k)}(\sigma) & & & \\ 0 & & \ddots & & \\ & & & U_k^{(k)}(\sigma) & \\ & & & & U_{k+1}^{(k)}(\sigma) \end{bmatrix}$$

In particular with respect to the coordinate system $y_1^{(k+1)}, \ldots, y_n^{(k+1)}$ $s\ell(2, \mathbf{C})$ is spanned by

$$h = \sum_{j=1}^{n} a_{1j}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} + \sum_{j=1}^{n} b_{1j}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}}$$

$$e = \sum_{j=1}^{n} a_{2j}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} + \sum_{j=1}^{n} b_{2j}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}}$$

$$f = \sum_{j=1}^{n} a_{3j}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} + \sum_{j=1}^{n} b_{3j}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}}$$

where $a_{ij}^{(k+1)}$ is a linear function in $y_1^{(k+1)}, \ldots, y_n^{(k+1)}$ and $b_{ij}^{(k+1)}$ is a function in $y_1^{(k+1)}, \ldots, y_n^{(k+1)}$ with multiplicity at least $k + 2$.

By construction, for each $\ell \in \mathbf{N}$, we have $y_i^{(\ell+1)} - y_i^{(\ell)} \in m^{\ell+1}$ where m is the maximal ideal of $\mathbf{C}[[x_1, x_2, \ldots, x_n]]$. Then the limit of the coordinate system $\{y_1^{(\ell+1)}, y_2^{(\ell+1)}, \ldots, y_n^{(\ell+1)}\}$ with respect to the m-adic topology is a coordinate system $\{y_1, y_2, \ldots, y_n\}$ in $\mathbf{C}[[x_1, x_2, \ldots, x_n]]$ with the property that

$$y_i - y_i^{(\ell+1)} \in m^{\ell+2} \quad \text{for all} \quad 1 \le i \le n.$$

By chain rule, we know that for $1 \le i \le n$

$$\frac{\partial}{\partial y_i^{(\ell+1)}} = \frac{\partial y_1}{\partial y_1^{(\ell+1)}} \frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial y_1^{(\ell+1)}} \frac{\partial}{\partial y_2} + \cdots + \frac{\partial y_n}{\partial y_1^{(\ell+1)}} \frac{\partial}{\partial y_n}$$

$$= \frac{\partial}{\partial y_i} + \text{operators of orders at least } \ell + 1 \qquad \ldots (2)$$

where operator of order $\ell + 1$ means operator of the form $\Sigma_{j=1}^{n} p_j^{\ell+1}(\partial/\partial y_j)$ with $p_j^{\ell+1}$ a homogeneous polynomial of degree $\ell + 1$ in $y_1, \ldots, y_n$ variables. Now we claim that h, e and f can be written as operator of order 1 with respect to the coordinate system $y_1, \ldots, y_n$. Write

$$h = D_{1,y} + D_{2,y} + D_{3,y} + \cdots$$

where $D_{i,y}$ is an operator of order i with respect to the coordinate system $y_1, y_2, \ldots, y_n$. Suppose $D_{j,y} = 0$ for $2 \leq j \leq \ell - 1$. We are going to prove $D_{\ell,y} = 0$. In the coordinate system $y_1^{(\ell+1)}, y_2^{(\ell+1)}, \ldots, y_n^{(\ell+1)}$, h can be written in the form

$$h = \sum_{j=1}^{n} a_{1j}^{(\ell+1)} \frac{\partial}{\partial y_i^{(\ell+1)}} + \sum_{j=1}^{n} b_{1j}^{(\ell+1)} \frac{\partial}{\partial y_j^{(\ell+1)}} \qquad \cdots (3)$$

where $a_{1j}^{(\ell+1)}$ is a linear function in $y_1^{(\ell+1)}, y_2^{(\ell+1)}, \ldots, y_n^{(\ell+1)}$ and $b_{1j}^{(\ell+1)}$ is a function in $y_1^{(\ell+1)}, y_2^{(\ell+1)}, \ldots, y_n^{(\ell+1)}$ with multiplicity at least $\ell + 2$. Put (1) and (2) in (3), we see that

$$h = \tilde{D}_{1,y} + \tilde{D}_{\ell+2,y} + \tilde{D}_{\ell+3,y} + \cdots$$

where $\tilde{D}_{i,y}$ is an operator of order j in $y_1, y_2, \ldots, y_n$ coordinate. This implies

$$0 = (D_{1,y} - \tilde{D}_{1,y}) + D_{\ell,y} + D_{\ell+1,y} + (D_{\ell+2,y} - \tilde{D}_{\ell+2,y})$$

$$+ (D_{\ell+3,y} - \tilde{D}_{\ell+3,y}) + \cdots$$

Thus $D_{\ell,y} = D_{\ell+1,y} = 0$. By induction, we have shown $D_{j,y} = 0$ for all $j \geq 2$. Hence h is an operator of first order with respect to $y_1, y_2, \ldots, y_n$ coordinate. Similarly we can prove that e and f are operators of first order with respect to $y_1, y_2, \ldots, y_n$ coordinate. Q.E.D.

THEOREM 2.2. *Let $s\ell(2, \mathbf{C})$ act on the formal power series ring $\mathbf{C}[[x_1, \ldots, x_n]]$ preserving the m-adic filtration where m is the maximal ideal in $\mathbf{C}[[x_1, \ldots, x_n]]$. Then there exists a coordinate system*

$$x_1, x_2, \ldots, x_{\ell_1}, x_{\ell_1+1}, x_{\ell_1+2}, \ldots, x_{\ell_1+\ell_2}, \ldots,$$

$$x_{\ell_1+\ell_2+\cdots+\ell_{s-1}+1}, \ldots, x_{\ell_1+\ell_2+\cdots+\ell_s}$$

such that

$$h = D_{h,1} + \cdots + D_{h,j} + \cdots + D_{h,r}$$

$$e = D_{e,1} + \cdots + D_{e,j} + \cdots + D_{e,r}$$

$$f = D_{f,1} + \cdots + D_{f,j} + \cdots + D_{f,r}$$

where $r \leq s$ and

$$D_{h,j} = (\ell_j - 1)x_{\ell_1+\cdots+\ell_{j-1}+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$+ (\ell_j - 3)x_{\ell_1+\cdots+\ell_{j-1}+2}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+2}}$$

$$+ \cdots + (-(\ell_j - 3))x_{\ell_1+\cdots+\ell_j-1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_j-1}}$$

$$+ (-(\ell_j - 1))x_{\ell_1+\cdots+\ell_j}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_j}}$$

$$D_{e,j} = (\ell_j - 1)x_{\ell_1+\cdots+\ell_{j-1}+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+2}}$$

$$+ \cdots + i(\ell_j - i)x_{\ell_1+\cdots+\ell_{j-1}+i}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+i+1}}$$

$$+ \cdots + (-(\ell_{j-1})x_{\ell_1+\cdots+\ell_j-1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_j}}$$

$$D_{f,j} = x_{\ell_1+\cdots+\ell_{j-1}+2}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$+ \cdots + x_{\ell_1+\cdots+\ell_{j-1}+i+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+i}}$$

$$+ \cdots + x_{\ell_1+\cdots+\ell_j}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_j-1}}$$

Proof. According to Proposition 2.1, we can choose a coordinate system $\{y_1, \ldots, y_n\}$ such that the coefficient of $\partial/\partial y_i$, $1 \leq i \leq n$, of every element in $s\ell(2, \mathbf{C})$ are linear functions in $y_1, \ldots, y_n$ variables. In view of the proof of complete classification of representations of $s\ell(2, \mathbf{C})$ representations, by further change of coordinate we obtain a coordinate system $\{x_1, x_2, \ldots, x_n\}$ such that $s\ell(2, \mathbf{C})$ takes the form as stated in the theorem. Q.E.D.

3. Singular sets of $s\ell(2, \mathbf{C})$ invariants polynomials.

LEMMA 3.1. *Suppose $s\ell(2, \mathbf{C})$ acts on M_5^k the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via*

$$\tau = 4x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} - 2x_4 \frac{\partial}{\partial x_4} - 4x_5 \frac{\partial}{\partial x_5}$$

$$X_+ = 4x_1 \frac{\partial}{\partial x_2} + 6x_2 \frac{\partial}{\partial x_3} + 6x_3 \frac{\partial}{\partial x_4} + 4x_4 \frac{\partial}{\partial x_5}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_4}.$$

Let I be the complex vector subspace spanned by $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$ and $\partial f/\partial x_5$ where f is a homogeneous polynomial of degree $k+1$. If I is a $s\ell(2, \mathbf{C})$-submodule, then the singular set of f contains the x_1 axis and x_5 axis.

Proof. By Theorem 4 of Section 1 in [4], f is necessarily an invariant $s\ell(2, \mathbf{C})$ polynomial in x_1, x_2, x_3, x_4, x_5 variables. Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.,

$$wt(x_1) = 4, \qquad wt(x_2) = 2, \qquad wt(x_3) = 0,$$

$$wt(x_4) = -2, \qquad wt(x_5) = -4.$$

Then f is a polynomial of weight 0. Let us assume on the contrary that x_1 axis does not lie in the singular set of f. Clearly the monomial x_1^k appears in $\partial f/\partial x_i$ for some $1 \leq i \leq 5$. Thus the monomial ${x_1}^k x_i$ appears in f. However, since $k \geq 2$, weight of ${x_1}^k x_i$ is strictly bigger than zero. This gives a contradiction. Hence x_1 axis is contained in the singular set of f.

Similarly we can prove that x_5 axis is contained in the singular set of f. Q.E.D.

LEMMA 3.2. *Suppose $s\ell(2, \mathbf{C})$ acts on M_5^k the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via*

$$\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} - x_5 \frac{\partial}{\partial x_5}$$

$$X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_5}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4}.$$

Let I be the complex vector subspace spanned by $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$ *and* $\partial f/\partial x_5$, *where f is a homogeneous polynomial of degree* $k+1$. *If I is a* $s\ell(2, \mathbf{C})$*-submodule then the singular set of f contains the set* $\{(x_1, x_2, x_3, x_4, x_5): x_2^2 - 2x_1x_3 = 0 = x_4 = x_5\}$.

Proof. By Theorem 4 of Section 1 in [4], we may assume that f is an invariant $s\ell(2, \mathbf{C})$ polynomial in x_1, x_2, x_3, x_4 and x_5 variables. Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ as above i.e.,

$$wt(x_1) = 2, \qquad wt(x_2) = 0, \qquad wt(x_3) = -2,$$

$$wt(x_4) = 1, \qquad wt(x_5) = -1.$$

Then f is a polynomial of weight 0. Write

$$f = \sum_{\alpha \geq 0, \beta \geq 0} g_{(\alpha,\beta)}(x_1, x_2, x_3) x_4^{\alpha} x_5^{\beta}.$$

Since weight of $g_{(\alpha,\beta)}(x_1, x_2, x_3)$ is even, we conclude that $g_{(1,0)}(x_1, x_2, x_3) = 0 = g_{(0,1)}(x_1, x_2, x_3)$. Therefore our lemma will follow if we can show that $g_{(0,0)}(x_1, x_2, x_3)$ is divisible by $(x_2^2 - 2x_1x_3)^2$ whenever $g_{(0,0)}(x_1, x_2, x_3)$ is nonzero. Observe that $g_{(0,0)}(x_1, x_2, x_3)$ is a polynomial of weight 0. As f is an invariant polynomial, we have $X_- f = X_+ f = 0$. It follows that $X_- g_{(0,0)}(x_1, x_2, x_3) = X_+ g_{(0,0)}(x_1, x_2, x_3) = 0$. Hence $g_{(0,0)}(x_1, x_2, x_3)$ is also an invariant polynomial in x_1, x_2 and x_3 variables of degree $k + 1 \geq 3$. Recall that the invariant polynomial in x_1, x_2 and x_3 variables must be even degree of the form $(x_2^2 - 2x_1x_3)^{\ell}$ (cf. [3]). Therefore $g_{(0,0)}(x_1, x_2, x_3)$ is divisible by $(x_2^2 - 2x_1x_3)^2$ as claimed. Q.E.D.

LEMMA 3.3. *Suppose* $s\ell(2, \mathbf{C})$ *acts on* M_5^k *the space of homogeneous polynomials of degree* $k \geq 2$ *in* x_1, x_2, x_3, x_4 *and* x_5 *variables via*

$$\tau = 3x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} - 3x_4 \frac{\partial}{\partial x_4}$$

$$X_+ = 3x_1 \frac{\partial}{\partial x_2} + 4x_2 \frac{\partial}{\partial x_3} + 3x_3 \frac{\partial}{\partial x_4}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}.$$

Let I be the complex vector subspace spanned by $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$ and $\partial f/\partial x_5$, where f is a homogeneous polynomial of degree $k + 1$. If I is a $s\ell(2, \mathbf{C})$-submodule then the singular set of f contains the set x_1 axis and x_4 axis.

Proof. By Theorem 4 of Section 1 in [4], f is necessary an invariant $s\ell(2, \mathbf{C})$ polynomial in x_1, x_2, x_3, x_4 and x_5 variables. Suppose the weight x_i is given by the corresponding coefficient in the expression of τ above i.e.,

$$wt(x_1) = 3, \qquad wt(x_2) = 1, \qquad wt(x_3) = -1,$$

$$wt(x_4) = -3, \qquad wt(x_5) = 0.$$

Then f is a polynomial of weight 0. Let us assume on the contrary that x_1-axis does not lie in the singular set of f. Clearly the monomial ${x_1}^k$ appears in $\partial f/\partial x_i$ for some $1 \le i \le 5$. Thus the monomial ${x_1}^k x_i$ appears in f. However, since $k \ge 2$, weight of ${x_1}^k x_i$ is strictly bigger than zero. This gives a contradiction. Hence x_1 axis is contained in the singular set of f.

Similarly we can prove that x_4 axis is contained in the singular set of f. Q.E.D.

LEMMA 3.4. *Suppose $s\ell(2, \mathbf{C})$ acts on M_4^k the space of homogeneous polynomials of degree $k \ge 2$ in x_1, x_2, x_3 and x_4 variables via*

$$\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}$$

$$X_+ = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}.$$

Suppose f is a $s\ell(2, \mathbf{C})$ invariant homogeneous polynomial of degree $k+1$ in x_1, x_2, x_3 and x_4 variables. Then $k+1=2\ell$ is an even integer and $f = c(x_1x_4 - x_2x_3)^{\ell}$ for some constant c.

Proof. Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.,

$$wt(x_1) = 1, \qquad wt(x_2) = -1, \qquad wt(x_3) = 1, \qquad wt(x_4) = -1.$$

Then f is a homogeneous polynomial of degree $k+1$ and weight 0. Let $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\beta_1}x_4^{\beta_2}$ be a monomial appearing in f. Then

$$\begin{cases} \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = k+1 & (4.1) \\ \alpha_1 - \alpha_2 + \beta_1 - \beta_2 = 0 & (4.2) \end{cases}$$

$$\Rightarrow 2(\alpha_1 + \beta_1) = k+1 \tag{4.3}$$

Therefore $k+1$ is an even integer 2ℓ. From (4.3) and (4.2) we have $\beta_1 = \ell - \alpha_1$ and $\beta_2 = \ell - \alpha_2$. We can write f in the following form

$$f = \sum_{\alpha_1,\alpha_2=0}^{\ell} a_{(\alpha_1,\alpha_2)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1}x_4^{\ell-\alpha_2}$$

$$X_-(f) = \sum_{\alpha_1,\alpha_2=0}^{\ell} a_{(\alpha_1,\alpha_2)} X_-(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1}x_4^{\ell-\alpha_2})$$

$$= \sum_{\alpha_1=1}^{\ell}\sum_{\alpha_2=0}^{\ell} \alpha_1 a_{(\alpha_1,\alpha_2)} x_1^{\alpha_1-1}x_2^{\alpha_2+1}x_3^{\ell-\alpha_1}x_4^{\ell-\alpha_2}$$

$$+ \sum_{\alpha_1=0}^{\ell-1}\sum_{\alpha_2=0}^{\ell} (\ell-\alpha_1) a_{(\alpha_1,\alpha_2)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1-1}x_4^{\ell-\alpha_2+1}$$

$$= \sum_{\alpha_1=0}^{\ell-1}\sum_{\alpha_2=1}^{\ell+1} (\alpha_1+1) a_{(\alpha_1+1,\alpha_2-1)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1-1}x_4^{\ell-\alpha_2+1}$$

$$+ \sum_{\alpha_1=0}^{\ell-1}\sum_{\alpha_2=0}^{\ell} (\ell-\alpha_1) a_{(\alpha_1,\alpha_2)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1-1}x_4^{\ell-\alpha_2+1}$$

$$= \sum_{\alpha_1=0}^{\ell-1} \sum_{\alpha_2=1}^{\ell} [(\alpha_1 + 1)a_{(\alpha_1+1,\alpha_2-1)}$$

$$+ (\ell - \alpha_1)a_{(\alpha_1,\alpha_2)}]x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1-1}x_4^{\ell-\alpha_2+1}$$

$$+ \sum_{\alpha_1=0}^{\ell-1} (\alpha_1 + 1)a_{(\alpha_1+1,\ell)}x_1^{\alpha_1}x_2^{\ell+1}x_3^{\ell-\alpha_1-1}$$

$$+ \sum_{\alpha_1=0}^{\ell-1} (\ell - \alpha_1)a_{(\alpha_1,0)}x_1^{\alpha_1}x_3^{\ell-\alpha_1-1}x_4^{\ell+1}$$

Since $X_-f = 0$, we have

$$a_{(1,\ell)} = 0 = a_{(2,\ell)} = \cdots = a_{(\ell,\ell)} \tag{4.4}$$

$$a_{(0,0)} = 0 = a_{(1,0)} = \cdots = a_{(\ell-1,0)} \tag{4.5}$$

$$(\alpha_1 + 1)a_{(\alpha_1+1,\alpha_2-1)} + (\ell - \alpha_1)a_{(\alpha_1,\alpha_2)} = 0 \qquad 0 \le \alpha_1 \le \ell - 1, \quad 1 \le \alpha_2 \le \ell \tag{4.6}$$

(4.4) and (4.6) imply $a_{(\alpha_1,\alpha_2)} = 0$ for all (α_1, α_2) such that $\ell + 1 \le \alpha_1 + \alpha_2 \le 2\ell$, $0 \le \alpha_1 \le \ell$, and $0 \le \alpha_2 \le \ell$. On the other hand, (4.5) and (4.6) imply $a_{(\alpha_1,\alpha_2)} = 0$ for all (α_1, α_2) such that $0 \le \alpha_1 + \alpha_2 \le \ell - 1$, $0 \le \alpha_1 \le \ell$ and $0 \le \alpha_2 \le \ell$.

Therefore we conclude that the only possible nonzero $a_{(\alpha_1,\alpha_2)}$ has the property that $\alpha_1 + \alpha_2 = \ell$. We shall denote $a_{(\alpha_1,\ell-\alpha_1)}$ by b_{α_1}. Then (4.6) becomes

$$(\alpha_1 + 1)b_{\alpha_1+1} + (\ell - \alpha_1)b_{\alpha_1} = 0 \quad \text{for} \quad 0 \le \alpha_1 \le \ell - 1$$

$$\Rightarrow b_1 = -\binom{\ell}{1}b_0$$

$$b_2 = (-1)^2\binom{\ell}{2}b_0$$

$$\vdots$$

$$b_i = (-1)^i\binom{\ell}{i}b_0$$

$$\vdots$$

$$b_\ell = (-1)^\ell b_0.$$

It follows that

$$f = b_\ell x_1{}^\ell x_4{}^\ell + b_{\ell-1}x_1{}^{\ell-1}x_2x_3x_4{}^{\ell-1} + \cdots + b_i x_1{}^i x_2{}^{\ell-i}x_3{}^{\ell-i}x_4{}^i + \cdots + b_0x_2{}^\ell x_3{}^\ell$$

$$= b_0x_2{}^\ell x_3{}^\ell - \binom{\ell}{1}b_0x_1x_2{}^{\ell-1}x_3{}^{\ell-1}x_4 + \cdots + (-1)^i\binom{\ell}{i}b_0x_1{}^ix_2{}^{\ell-i}x_3{}^{\ell-i}x_4{}^i$$

$$+ \cdots + (-1)^\ell b_0x_1{}^\ell x_4{}^\ell$$

$$= b_0(x_2x_3 - x_1x_4)^\ell$$ Q.E.D.

LEMMA 3.5. *Suppose $s\ell(2, \mathbf{C})$ acts on M_5^k the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via*

$$\tau = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4}$$

$$X_+ = x_1\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_4}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_3}$$

Suppose f is a $s\ell(2, \mathbf{C})$ invariant homogeneous polynomial of degree $k+1$ in x_1, x_2, x_3, x_4, and x_5 variables. Then f can be written in the following form

$$f = a_0x_5^{k+1} + a_2x_5^{k-1}(x_1x_4 - x_2x_3) + a_4x_5^{k-3}(x_1x_4 - x_2x_3)^2 + \cdots$$

Proof. Write

$$f = \sum_{i=0}^{k+1} g_{k+1-i}(x_1, x_2, x_3, x_4)x_5^i$$

where $g_{k+1-i}(x_1, x_2, x_3, x_4)$ is a homogeneous polynomial of degree $k + 1 - i$ in x_1, x_2, x_3 and x_4 variables.

$$0 = X_-(f) = \sum_{i=0}^{k+1} x_5^i X_-(g_{k+1-i}(x_1, x_2, x_3, x_4))$$

$$\Rightarrow X_-(g_{k+1-i}(x_1, x_2, x_3, x_4)) = 0.$$

By Lemma 4, $k + 1 - i$ is an even integer 2ℓ and $g_{k+1-i}(x_1, x_2, x_3, x_4)$ is a constant multiple of $(x_1x_4 - x_2x_3)^{\ell}$. Q.E.D.

LEMMA 3.6. *Suppose $s\ell(2, \mathbf{C})$ acts on M_5^k the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via*

$$\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}$$

$$X_+ = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}.$$

Let I be the complex vector subspace spanned by $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$ and $\partial f/\partial x_5$, where f is a homogeneous polynomial of degree $k + 1$. If I is a $s\ell(2, \mathbf{C})$-submodule, then the singular set of f contains the set $\{(x_1, x_2, x_3, x_4, x): x_1x_4 - x_2x_3 = 0 = x_5\}$.

Proof. By Theorem 4 of Section 1 in [4], f is one of the following.

Case (i). f is $s\ell(2, \mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables and $I = (2) \oplus (2) \oplus (1)$. In view of Lemma 5 f is of the form

$$f = a_0x_5^{k+1} + a_2x_5^{k-1}(x_1x_4 - x_2x_3) + a_4x_5^{k-3}(x_1x_4 - x_2x_3)^2 + \cdots$$

Since $k \geq 2$, it is easy to see that $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$ and $\partial f/\partial x_5$ vanish on the set $\{(x_1, x_2, x_3, x_4, x_5): x_1x_4 - x_2x_3 = 0 = x_5\}$.

Case (ii). f is a $s\ell(2, \mathbf{C})$ invariant polynomial in x_1, x_2, x_3 and x_4 variables and $I = (2) \oplus (2)$. In view of Lemma 4, there exists a nonzero constant c such that

$$f(x_1, x_2, x_3, x_4) = c(x_1x_4 - x_2x_3)^{\ell}$$

where $\ell \geq 2$. It is clear that $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$ and $\partial f/\partial x_5$ vanish on the set $\{(x_1, x_2, x_3, x_4, x_5): x_1x_4 - x_2x_3 = 0\}$.

Case (iii). $f = cx_5^{k+1}$ where c is a nonzero constant. Clearly the singular set of f is $\{(x_1, x_2, x_3, x_4, x_5): x_5 = 0\}$. Q.E.D.

LEMMA 3.7. *Suppose $s\ell(2, \mathbf{C})$ acts on M_5^k the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via*

$$\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3}$$

$$X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}.$$

Let I be the complex vector subspace spanned by $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$, and $\partial f/\partial x_5$, where f is a homogeneous polynomial of degree $k + 1$. If I is a $s\ell(2, \mathbf{C})$ submodule, then the singular set of f contains the set $\{(x_1, x_2, x_3, x_4, x_5): x_2^2 - 2x_1x_3 = 0 = x_4 = x_5\}$.

Proof. By Theorem 4 of Section 1 in [4], we may assume that f is an invariant $s\ell(2, \mathbf{C})$ polynomial. Arguing similarly as in the proof of Lemma 5, we know that in view of Corollary 4.17 in [3], if $k + 1$ is even, say $k + 1 = 2\ell$, then

$$f = c_0(x_2^2 - 2x_1x_3)^{\ell} + (x_2^2 - 2x_1x_3)^{\ell-1}g_2(x_4, x_5)$$
$$+ (x_2^2 - 2x_1x_3)^{\ell-2}g_4(x_4, x_5)$$

$$+ \cdots + g_{2\ell}(x_4, x_5)$$

and if $k + 1$ is odd, say $k + 1 = 2\ell + 1$, then

$$f = (x_2^2 - 2x_1x_3)^\ell g_1(x_4, x_5) + (x_2^2 - 2x_1x_3)^{\ell-1} g_3(x_4, x_5)$$

$$+ (x_2^2 - 2x_1x_3)^{\ell-2} g_5(x_4, x_5)$$

$$+ \cdots + g_{2\ell+1}(x_4, x_5)$$

where $g_i(x_4, x_5)$ is a homogeneous polynomial of degree i in x_4 and x_5 variables.

It is clear that the singular set of f contains the set $\{(x_1, x_2, x_3, x_4, x_5): x_2^2 - 2x_1x_3 = 0 = x_4 = x_5\}$. Q.E.D.

LEMMA 3.8. *Suppose $s\ell(2, \mathbf{C})$ acts on M_5^k the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4, and x_5 variables via*

$$\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$$

$$X_+ = x_1 \frac{\partial}{\partial x_2}$$

$$X_- = x_2 \frac{\partial}{\partial x_1}.$$

Let I be the complex vector subspace spanned by $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$, and $\partial f/\partial x_5$, where f is a homogeneous polynomial of degree $k + 1$. If I is a $s\ell(2, \mathbf{C})$ submodule, then the singular set of f contains the set $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{C}^5: x_3 = x_4 = x_5 = 0\}$.

Proof. In view of Theorem 4 of Section 1 in [4], f is a polynomial in x_3, x_4 and x_5 variables. Our lemma follows immediately. Q.E.D.

4. Solvability of the Lie algebras $L(V)$. In [3], we first established a connection between the set of isolated hypersurface singularities and the set of finite dimensional Lie algebras. Let $(V, 0)$ be an isolated singularity in $(\mathbf{C}^n, 0)$ defined by the zero set of a holomorphic function of f. The moduli algebra $A(V)$ of $(V, 0)$ is $\mathbf{C}\{x_1, x_2, \ldots, x_n\}/(f, \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$. We define $L(V)$ to be the algebra of derivations of $A(V)$. $A(V)$ is finite dimensional as a $\mathbf{C}$ vector space and $L(V)$ is contained in the endomorphism algebra of $A(V)$; consequently $L(V)$ is a finite dimensional Lie algebra. In [3], we proved that $L(V)$ is solvable for $n \leq 3$. It is the purpose of this section to prove $L(V)$ solvable for $n \leq 5$. In order to avoid the repetition, we shall only concentrate on the case $n = 5$.

Remark. In general, in order to prove $L(V)$ is solvable, it suffices to prove the statement with an additional assumption that multiplicity of f is bigger than two. Because if the multiplicity of f is two, then after a biholomorphic change of coordinates, we can assume that $f = x_n^2 - g(x_1, \ldots, x_{n-1})$. In this case $L(V) = L(W)$ which is solvable by induction hypothesis, where $W = \{(x_1, \ldots, x_{n-1}) \in \mathbf{C}^{n-1}: g(x_1, \ldots, x_{n-1}) = 0\}$.

THEOREM 4.1. *Suppose that* $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{C}^5: f(x_1, x_2, x_3, x_4, x_5) = 0\}$ *has an isolated singularity at* $(0, 0, 0, 0, 0)$. *Then the finite dimensional Lie algebra* $L(V)$ *associated to the singularity is solvable.*

Proof. By the Levi decomposition, if the Lie algebra is not solvable, then the Lie algebra $L(V)$ contains $s\ell(2, \mathbf{C})$ as subalgebra. By Lemma 4.3 of [3], we shall assume that $s\ell(2, \mathbf{C})$ acts on m/m^2 nontrivially where m is the maximal ideal in $\mathcal{O}_{\mathbf{C}^5,0}$. Write $f = \Sigma_{i=k+1}^{\infty} f_i$. According to the above remark, we shall assume without loss of generality that multiplicity of $f = k + 1 \geq 3$. By Theorem 2.2, we know that the action of $s\ell(2, \mathbf{C})$ on $\mathbf{C}[[x_1, x_2, x_3, x_4, x_5]]$ is one of the following forms.

Case 1. $s\ell(2, \mathbf{C})$ acts on $\mathbf{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$\tau = 4x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} - 2x_4 \frac{\partial}{\partial x_4} - 4x_5 \frac{\partial}{\partial x_5}$$

$$X_+ = 4x_1 \frac{\partial}{\partial x_2} + 6x_2 \frac{\partial}{\partial x_3} + 6x_3 \frac{\partial}{\partial x_4} + 4x_4 \frac{\partial}{\partial x_5}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_4}.$$

We shall prove by induction that $\partial f_i/\partial x_j$ for $i \geq k + 1$ and $1 \leq j \leq 5$ vanish along the x_1 axis and x_5 axis. Let $\Delta(f)$ denote the ideal generated by, $\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4$ and $\partial f/\partial x_5$. $(f) + \Delta(f)$ is a $s\ell(2, \mathbf{C})$ module.

Clearly m^k/m^{k+1} is also a $s\ell(2, \mathbf{C})$ module. Let $J_k(f)$ denote the image of the canonical map $(f) + \Delta(f) \to m^k/m^{k+1}$. $J_k(f)$ is an invariant subspace spanned by $\langle \partial f_{k+1}/\partial x_1, \partial f_{k+1}/\partial x_1, \partial f_{k+1}/\partial x_3, \partial f_{k+1}/\partial x_4, \partial f_{k+1}/\partial x_5 \rangle$ and hence may be identified with an invariant subspace of M_5^k. By Lemma 3.1, $\partial f_{k+1}/\partial x_i$ for $1 \leq i \leq 5$ vanish along x_1 axis and x_5 axis.

Let $g = g_\ell + g_{\ell+1} + \cdots$ be a Taylor series expansion of g where g_i is a homogeneous polynomial of degree i. Then for any $D \in s\ell(2, \mathbf{C})$, $Dg = Dg_\ell + Dg_{\ell+1} + \cdots$ is a Taylor series expansion of Dg. It follows easily that

$$m^n + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_n)/m^{n+1} + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_n)$$

is a $s\ell(2, \mathbf{C})$ module. Let J_n denote the image of the canonical map $(f) + \Delta(f) \to m^n + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_n)/m^{n+1} + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_n)$. J_n is an invariant subspace spanned by $\langle \partial f_{n+1}/\partial x_1, \partial f_{n+1}/\partial x_2, \partial f_{n+1}/\partial x_3, \partial f_{n+1}/\partial x_4, \partial f_{n+1}/\partial x_5 \rangle$ and hence may be identified with an invariant subspace of M_5^n. By Lemma 3.1, $\partial f_{n+1}/\partial x_i$ for $1 \leq i \leq 5$ vanish along x_1 axis and x_5 axis. This finishes the induction step.

Obviously $\Delta(f)$ vanishes along x_1 axis and x_5 axis. This implies that f cannot have isolated singularity at the origin, a contradiction to our assumption.

Case 2. $s\ell(2, \mathbf{C})$ acts on $\mathbf{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} - x_5 \frac{\partial}{\partial x_5}$$

$$X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_5}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4}.$$

In view of Lemma 3.2, we shall obtain a contradiction by the same argument as Case 1 above.

Case 3. $s\ell(2, \mathbf{C})$ acts on $\mathbf{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$\tau = 3x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} - 3x_4 \frac{\partial}{\partial x_4}$$

$$X_+ = 3x_1 \frac{\partial}{\partial x_2} + 4x_2 \frac{\partial}{\partial x_3} + 3x_3 \frac{\partial}{\partial x_4}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}.$$

In view of Lemma 3.3, we shall obtain a contradiction by the same argument as Case 1 above.

Case 4. $s\ell(2, \mathbf{C})$ acts on $\mathbf{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}$$

$$X_+ = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}$$

In view of Lemma 3.6, we shall obtain a contradiction by the same argument as Case 1 above.

Case 5. $s\ell(2, \mathbf{C})$ acts on $\mathbf{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3}$$

$$X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}.$$

In view of Lemma 3.7, we shall obtain a contradiction by the same argument as Case 1 above.

Case 6. $s\ell(2, \mathbf{C})$ acts on $\mathbf{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$$

$$X_+ = x_1 \frac{\partial}{\partial x_2}$$

$$X_- = x_2 \frac{\partial}{\partial x_1}.$$

In view of Lemma 3.8, we shall obtain a contradiction by the same argument as Case 1 above. Q.E.D.

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