Singularities Defined by $\mathfrak{s}(2, \mathbb{C})$ Invariant Polynomials and Solvability of Lie Algebras Arising From Isolated Singularities
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1. **Introduction.** Let \((V, 0)\) be an isolated singularity in \((C^n, 0)\) defined by the zero set of a holomorphic function \(f\). The moduli algebra \(A(V)\) of \((V, 0)\) is \(C\{x_1, x_2, \ldots, x_n\}/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)\). It is easy to see that \(A(V)\) is an invariant of \((V, 0)\). In [1], Mather and the author proved that the complex analytic structure of \((V, 0)\) is determined also by \(A(V)\). Thus, the above construction gives an injection map from the space of isolated singularities in \((C^n, 0)\) to the space of commutative local Artinian algebras. This raises a natural and important problem, the so called recognition problem: Give a necessary and sufficient condition for a commutative local Artinian algebra to be a moduli algebra. In [3], we define \(L(V)\) to be the algebra of derivations of \(A(V)\). Clearly \(L(V)\) is a finite dimensional Lie algebra. The main purpose of this paper is to prove that \(L(V)\) is solvable for \(n \leq 5\). Naturally one expects that a necessary condition for a commutative local Artinian algebra to be a moduli algebra is that its algebra of derivations is a solvable Lie algebra. In this paper we have used the main theorem which is available in preprint form in [4] for \(n \leq 5\). Assume that the results in [4] remain valid as expected for \(n \geq 6\). Then the proof given in this article also applies smoothly to arbitrary \(n\). In Section 2, we classify the actions of \(sl(2, C)\) on \(C[[x_1, x_2, \ldots, x_n]]\) via derivations preserving the \(m\)-adic filtration. The main point here is to get rid of the “higher order operator” (i.e., \(\Sigma_i a_i(\partial/\partial x_i)\) with multiplicity of \(a_i \geq 2\)) by means of the vanishing theorem for semisimple Lie algebra cohomology. It seems to us that the material here is not available in literature form. I would like to thank Professor H. Sah for
many useful discussions. In Section 3 we prove that the singular set of $sl(2, \mathbb{C})$ invariant polynomials of degree at least 3 in five variables is at least one dimensional. Similar results can be generalized to higher dimension without difficulty. In Section 4, we prove our main theorem that $L(V)$ is solvable for $n \leq 5$.

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2. Classification of $sl(2, \mathbb{C})$ in $\text{Der} \ C[[x_1, x_2, \ldots, x_n]]$ preserving the $m$-adic filtration.

**Proposition 2.1.** Let $L = sl(2, \mathbb{C})$ act on $C[[x_1, \ldots, x_n]]$ via derivations preserving the $m$-adic filtration i.e., $L(m^k) \subseteq m^k$ where $m$ is the maximal ideal in $C[[x_1, \ldots, x_n]]$. Then there exists a coordinate change $y_1, \ldots, y_n$ with respect to which $sl(2, \mathbb{C})$ is spanned by

$$
\begin{align*}
  h &= \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial y_j} \\
  e &= \sum_{j=1}^{n} a_{2j} \frac{\partial}{\partial y_j} \\
  f &= \sum_{j=1}^{n} a_{3j} \frac{\partial}{\partial y_j}
\end{align*}
$$

where $a_{ij}$ is a linear function in $y_1, \ldots, y_n$ variables for all $1 \leq i \leq 3$ and $1 \leq j \leq n$. Here $\{h, e, f\}$ is a standard basis for $sl(2, \mathbb{C})$ i.e., $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$.

**Proof.** Let $\sigma$ be an element of $L$. Let $U_1^{(1)}(\sigma)$ be the matrix representation of $\sigma$ on $m/m^2$ and $U_2^{(1)}(\sigma)$ be the matrix representation of $\sigma$ on $m^2/m^3$. Since $\sigma$ preserves the $m$-adic filtration, the matrix representation of $\sigma$ on $m/m^3$ is given as follows

$$
W^{(1)}(\sigma) = \begin{pmatrix}
  U_1^{(1)}(\sigma) & 0 \\
  T_1^{(1)}(\sigma) & U_2^{(1)}(\sigma)
\end{pmatrix}
$$
where \(T_{12}^{(1)}(\sigma)\) represents an element in \(\text{Hom}_C(m/m^2, m^2/m^3)\). Observe that

\[
W^{(1)}([\sigma, \tau]) = W^{(1)}(\sigma)W^{(1)}(\tau) - W^{(1)}(\tau)W^{(1)}(\sigma)
\]

\[
= \begin{pmatrix}
U_1^{(1)}(\sigma) & 0 \\
T_{12}^{(1)}(\sigma) & U_2^{(1)}(\sigma)
\end{pmatrix}
\begin{pmatrix}
U_1^{(1)}(\tau) & 0 \\
T_{12}^{(1)}(\tau) & U_2^{(1)}(\tau)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
U_1^{(1)}(\sigma)U_1^{(1)}(\tau) & 0 \\
T_{12}^{(1)}(\sigma)U_1^{(1)}(\tau) + U_2^{(1)}(\sigma)T_{12}^{(1)}(\tau) & U_2(\sigma)U_2(\tau)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
U_1^{(1)}(\sigma)U_1^{(1)}(\tau) - U_1^{(1)}(\tau)U_1^{(1)}(\sigma) & 0 \\
T_{12}^{(1)}(\sigma)U_1^{(1)}(\tau) + U_2^{(1)}(\sigma)T_{12}^{(1)}(\tau) & U_2^{(1)}(\sigma)U_2^{(1)}(\tau) - U_2^{(1)}(\tau)U_2^{(1)}(\sigma)
\end{pmatrix}
\]

\[
\Rightarrow T_{12}^{(1)}([\sigma, \tau]) = T_{12}^{(1)}(\sigma)U_1^{(1)}(\tau) + U_2^{(1)}(\sigma)T_{12}^{(1)}(\tau)
\]

Observe also that \(\text{Hom}_C(m/m^2, m^2/m^3)\) is a \(L\)-module. The action of \(L\) on \(\text{Hom}_C(m/m^2, m^2/m^3)\) is given as follows. Let \(\sigma \in L, \varphi \in \text{Hom}_C(m/m^2, m^2/m^3)\) and \(u_1 \in m/m^2\)

\[(\sigma \varphi)(u_1) = -\varphi(\sigma(u_1)) + \sigma(\varphi(u_1)).\]

We now claim that \(T_{12}^{(1)}: L \rightarrow \text{Hom}_C(m/m^2, m^2/m^3)\) is a 1-cocycle of \(L\) with coefficient in \(\text{Hom}_C(m/m^2, m^2/m^3)\). To see this, consider

\[
\delta T_{12}^{(1)}(\sigma, \tau) = \sigma \cdot T_{12}^{(1)}(\tau) - \tau \cdot T_{12}^{(1)}(\sigma) - T_{12}^{(1)}([\sigma, \tau]).
\]
For any \( v \in m/m^2 \), we have

\[
[\delta T_{12}^{(1)}(\sigma, \tau)](v) = [\sigma \cdot T_{12}^{(1)}(\tau)](v) - [\tau \cdot T_{12}^{(1)}(\sigma)](v) - T_{12}^{(1)}([\sigma, \tau])(v)
\]

\[
= -T_{12}^{(1)}(\sigma(v)) + \sigma(T_{12}^{(1)}(\tau)(v)) + T_{12}^{(1)}(\tau)(v) - \tau(T_{12}^{(1)}(\sigma)(v))
\]

\[
- T_{12}^{(1)}(\sigma)U_1^{(1)}(\tau)(v) - U_2^{(1)}(\sigma)T_{12}^{(1)}(\tau)(v)
\]

\[
+ T_{12}^{(1)}(\tau)U_1^{(1)}(\sigma)(v) + U_2^{(1)}(\tau)T_{12}^{(1)}(\sigma)(v)
\]

\[
= 0.
\]

Therefore \( T_{12}^{(1)} \) is a 1-cocycle as claimed. Since \( L \) is simple, \( H^1(L, \text{Hom}_C(m/m^2, m^2/m^3)) = 0 \). We conclude that \( T_{12}^{(1)} \) is a 1-coboundary. There exists \( \beta^{(2)} \in \text{Hom}(m/m^2, m^2/m^3) \) a 0-cochain of \( L \) with coefficients in \( \text{Hom}(m/m^2, m^2/m^3) \) such that

\[
T_{12}^{(1)}(\sigma) = (\delta\beta^{(2)})(\sigma) \quad \forall \sigma \in L
\]

\[
= \sigma\beta^{(2)}
\]

\[
\Rightarrow T_{12}^{(1)}(\sigma)(v) = \sigma\beta^{(2)}(v) \quad \forall v \in m/m^2
\]

\[
= \sigma(\beta^{(2)}(v)) - \beta^{(2)}(\sigma(v)).
\]

Let \( S^{(2)} \) be the matrix representation of \( \beta^{(2)} \). Thus we have

\[
T_{12}^{(1)}(\sigma)(v) = U_2^{(1)}(\sigma)S^{(2)}(v) - S^{(2)}U_1^{(1)}(\sigma)(v).
\]

This is equivalent to say that

\[
\begin{pmatrix}
I & 0 \\
S^{(2)} & I
\end{pmatrix}
\begin{pmatrix}
U_1^{(1)}(\sigma) & 0 \\
T_{12}^{(1)}(\sigma) & U_2^{(1)}(\sigma)
\end{pmatrix}
= \begin{pmatrix}
U_1^{(1)}(\sigma) & 0 \\
0 & U_2^{(1)}(\sigma)
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
S^{(2)} & I
\end{pmatrix}
\]

\[
\Leftrightarrow \begin{pmatrix}
I & 0 \\
S^{(2)} & I
\end{pmatrix}
\begin{pmatrix}
U_1^{(1)}(\sigma) & 0 \\
T_{12}^{(1)}(\sigma) & U_2^{(1)}(\sigma)
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
S^{(2)} & I
\end{pmatrix}^{-1}
= \begin{pmatrix}
U_1^{(1)}(\sigma) & 0 \\
0 & U_2^{(1)}(\sigma)
\end{pmatrix}
\]
The above equation means that we can make a change of variable in the following form:

\[
\begin{align*}
    y_1^{(2)} &= x_1 + q_1^{(2)}(x_1, x_2, \ldots, x_n) \\
    y_2^{(2)} &= x_2 + q_2^{(2)}(x_1, x_2, \ldots, x_n) \\
    &\vdots \\
    y_n^{(2)} &= x_n + q_n^{(2)}(x_1, x_2, \ldots, x_n)
\end{align*}
\]

where \(q_i^{(2)}\) is a homogeneous polynomial of degree 2 for \(1 \leq i \leq n\), such that with respect to such coordinate, the matrix representation of \(\sigma\) on \(m/m^3\) is given by

\[
\begin{pmatrix}
    U_1^{(2)}(\sigma) & 0 \\
    0 & U_2^{(2)}(\sigma)
\end{pmatrix} = \begin{pmatrix}
    U_1^{(1)}(\sigma) & 0 \\
    0 & U_2^{(1)}(\sigma)
\end{pmatrix}
\]

i.e., with respect to the coordinate system \(y_1^{(2)}, \ldots, y_n^{(2)}\), \(sl(2, \mathbb{C})\) is spanned by

\[
\begin{align*}
    h &= \sum_{j=1}^{n} a_{ij}^{(2)} \frac{\partial}{\partial y_j^{(2)}} + \sum_{j=1}^{n} b_{ij}^{(2)} \frac{\partial}{\partial y_j^{(2)}} \\
    e &= \sum_{j=1}^{n} a_{3j}^{(2)} \frac{\partial}{\partial y_j^{(2)}} + \sum_{j=1}^{n} b_{3j}^{(2)} \frac{\partial}{\partial y_j^{(2)}} \\
    f &= \sum_{j=1}^{n} a_{3j}^{(2)} \frac{\partial}{\partial y_j^{(2)}} + \sum_{j=1}^{n} b_{3j}^{(2)} \frac{\partial}{\partial y_j^{(2)}}
\end{align*}
\]

where \(a_{ij}^{(2)}\) is a linear function in \(y_1^{(2)}, \ldots, y_n^{(2)}\) and \(b_{ij}^{(2)}\) is a polynomial in \(y_1^{(2)}, \ldots, y_n^{(2)}\) with multiplicity at least three.

By induction, we shall assume that there exists coordinates

\[
\begin{align*}
    y_1^{(k)} &= y_1^{(k-1)} + q_1^{(k)}(y_1^{(k-1)}, y_2^{(k-1)}, \ldots, y_n^{(k-1)}) \\
    y_2^{(k)} &= y_2^{(k-1)} + q_2^{(k)}(y_1^{(k-1)}, y_2^{(k-1)}, \ldots, y_n^{(k-1)}) \\
    &\vdots \\
    y_n^{(k)} &= y_n^{(k-1)} + q_n^{(k)}(y_1^{(k-1)}, y_2^{(k-1)}, \ldots, y_n^{(k-1)})
\end{align*}
\]

where $q_i^{(k)}$ is a homogeneous polynomial of degree for $1 \leq i \leq n$ such that with respect to such coordinate, the matrix representation of $\sigma$ on $m/m^{k+1}$ is given by

$$
\begin{bmatrix}
U_1^{(k)}(\sigma) & 0 \\
0 & U_2^{(k)}(\sigma) \\
& \ddots \\
& & 0 \\
& & & U_k^{(k)}(\sigma)
\end{bmatrix}
$$

where $U_1^{(k)}(\sigma)$ is the matrix representation of $\sigma$ on $m/m^{i+1}$. This means that with respect to the coordinate system $y_1^{(k)}, \ldots, y_n^{(k)}$, $sl(2, \mathbb{C})$ is spanned by

$$
h = \sum_{j=1}^{n} a_{ij}^{(k)} \frac{\partial}{\partial y_j^{(k)}} + \sum_{j=1}^{n} b_{ij}^{(k)} \frac{\partial}{\partial y_j^{(k)}}$$

$$
e = \sum_{j=1}^{n} a_{2j}^{(k)} \frac{\partial}{\partial y_j^{(k)}} + \sum_{j=1}^{n} b_{2j}^{(k)} \frac{\partial}{\partial y_j^{(k)}}$$

$$
f = \sum_{j=1}^{n} a_{3j}^{(k)} \frac{\partial}{\partial y_j^{(k)}} + \sum_{j=1}^{n} b_{3j}^{(k)} \frac{\partial}{\partial y_j^{(k)}}$$

where $a_{ij}^{(k)}$ is a linear function in $y_1^{(k)}, \ldots, y_n^{(k)}$ and $b_{ij}^{(k)}$ is a polynomial in $y_1^{(k)}, \ldots, y_n^{(k)}$ with multiplicity at least $k+1$. The matrix representations of $\sigma$ on $m/m^{k+2}$ with respect to the coordinate system $y_1^{(k)}, \ldots, y_n^{(k)}$ is given by

$$
W^{(k)}(\sigma) =
\begin{bmatrix}
U_1^{(k)}(\sigma) & 0 \\
0 & U_2^{(k)}(\sigma) \\
& \ddots \\
& & 0 \\
& & & U_k^{(k)}(\sigma) \\
T_{1,k+1}^{(k)}(\sigma) & * & \cdots & * & U_{k+1}^{(k)}(\sigma)
\end{bmatrix}
$$
where $T_{1,k+1}^{(k)}(\sigma)$ represents an element in $\text{Hom}_C(m/m^2, m^{k+1}/m^{k+2})$. Observe that

\[
W^{(k)}([\sigma, \tau]) = W^{(k)}(\sigma)W^{(k)}(\tau) - W^{(k)}(\tau)W^{(k)}(\sigma)
\]

\[
= \begin{pmatrix}
U^{(k)}(\sigma) & 0 \\
0 & U^{(k)}(\sigma) \\
\vdots & \vdots & \ddots & \vdots \\
T_{1,k+1}^{(k)}(\sigma) & \cdots & \cdots & U_{k+1}^{(k)}(\sigma)
\end{pmatrix}
- \begin{pmatrix}
U^{(k)}(\tau) & 0 \\
0 & U^{(k)}(\tau) \\
\vdots & \vdots & \ddots & \vdots \\
T_{1,k+1}^{(k)}(\tau) & \cdots & \cdots & U_{k+1}^{(k)}(\tau)
\end{pmatrix}
- \begin{pmatrix}
U^{(k)}(\sigma) & 0 \\
0 & U^{(k)}(\sigma) \\
\vdots & \vdots & \ddots & \vdots \\
T_{1,k+1}^{(k)}(\sigma) & \cdots & \cdots & U_{k+1}^{(k)}(\sigma)
\end{pmatrix}
- \begin{pmatrix}
0 & a_{11} & \cdots & 0 \\
a_{22} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
a_{k+1,1+n} & \cdots & 0 & a_{k+1,k+n}
\end{pmatrix}
\]

where $a_{ij} = U^{(k)}(\sigma)U_{i}^{(k)}(\tau) - U_{i}^{(k)}(\tau)U^{(k)}(\sigma)$ and $a_{k+1,1} = T_{1,k+1}^{(k)}(\sigma)U_{1}^{(k)}(\tau) + U_{k+1}^{(k)}(\sigma)T_{1,k+1}^{(k)}(\tau) - T_{1,k+1}^{(k)}(\tau)U_{1}^{(k)}(\sigma) - U_{k+1}^{(k)}(\tau)T_{1,k+1}^{(k)}(\sigma)$. We now claim that $T_{1,k+1}^{(k)}: L \to \text{Hom}_C(m/m^2, m^{k+1}/m^{k+2})$ is a 1-cocycle. To see this, consider

\[
\delta T_{1,k+1}^{(k)}(\sigma, \tau) = \sigma \cdot T_{1,k+1}^{(k)}(\tau) - \tau \cdot T_{1,k+1}^{(k)}(\sigma) - T_{1,k+1}^{(k)}([\sigma, \tau]).
\]

For any $\nu \in m/m^2$, we have

\[
[\delta \cdot T_{1,k+1}^{(k)}(\sigma, \tau)](\nu) = [\sigma \cdot T_{1,k+1}^{(k)}(\tau)](\nu) - [\tau \cdot T_{1,k+1}^{(k)}(\sigma)](\nu)
- T_{1,k+1}^{(k)}([\sigma, \tau])(\nu)
= -T_{1,k+1}^{(k)}(\tau)(\sigma(\nu)) + \sigma(T_{1,k+1}^{(k)}(\tau)(\nu))
\]
\[ T_{1,k+1}^{(k)}(\sigma)(v) - \tau(T_{1,k+1}^{(k)}(\sigma)(v)) \]

\[ - T_{1,k+1}^{(k)}(\sigma)U_1^{(k)}(\tau)(v) - U_{k+1}^{(k)}(\sigma)T_{1,k+1}^{(k)}(\tau)(v) \]

\[ + T_{1,k+1}^{(k)}(\tau)U_1^{(k)}(\sigma)(v) + U_{k+1}^{(k)}(\tau)T_{1,k+1}^{(k)}(\sigma)(v) \]

\[ = 0. \]

Therefore \( T_{1,k+1}^{(k)} \) is a 1-cocycle as claimed. Since \( L \) is simple \( H^1(L, \text{Hom}(m/m^2, m^{k+1}/m^{k+2})) = 0 \). We conclude that \( T_{1,k+1}^{(k)} \) is a 1-coboundary. There exists \( \beta^{(k+1)} \in \text{Hom}(m/m^2, m^{k+1}/m^{k+2}) \) a 0-cochain of \( L \) with coefficient in \( \text{Hom}(m/m^2, m^{k+1}/m^{k+2}) \) such that

\[ T_{1,k+1}^{(k)}(\sigma) = (\delta \beta^{(k+1)})(\sigma) \quad \forall \sigma \in L \]

\[ = \sigma \beta^{(k+1)} \]

\[ \Rightarrow T_{1,k+1}^{(k)}(\sigma)(v) = \sigma \beta^{(k+1)}(v) \quad \forall v \in m/m^2 \]

\[ = \sigma(\beta^{(k+1)}(v)) - \beta^{(k+1)}(\sigma(v)) \]

Let \( S^{(k+1)} \) be the matrix representation of \( \beta^{(k+1)} \). Then we have

\[ T_{1,k+1}^{(k)}(\sigma)(v) = U_{k+1}^{(k)}(\sigma)S^{(k+1)}(v) - S^{(k+1)}U_1^{(k)}(\sigma)(v) \]

This is equivalent to say that

\[
\begin{pmatrix}
I & 0 & \cdots & 0 \\
0 & I & \cdots & 0 \\
0 & \cdots & I \\
S^{(k+1)} & \cdots & 0 & I \\
\end{pmatrix}
\begin{pmatrix}
U_1^{(k)}(\sigma) & 0 \\
0 & U_2^{(k)}(\sigma) \\
\cdots & \cdots \\
T_{1,k+1}(\sigma) & \cdots & U_{k+1}^{(k)}(\sigma) \\
\end{pmatrix}
\]
This means that we can make a change of variable in the following form

\[ y_i^{(k+1)} = y_i^{(k)} + q_i^{(k+1)}(y_1^{(k)}, \ldots, y_n^{(k)}) \]

where \( q_i^{(k+1)} \) is a homogeneous polynomial of degree \( k + 1 \) for \( 1 \leq i \leq n \) such that with respect to such coordinate, the matrix representation of \( \sigma \) on \( m/m^{k+2} \) is given by

\[
\begin{pmatrix}
U_1^{(k+1)}(\sigma) & 0 \\
U_2^{(k+1)}(\sigma) & \\
0 & \ddots \\
0 & \cdots & U_k^{(k+1)}(\sigma) \\
0 & \cdots & U_{k+1}^{(k+1)}(\sigma)
\end{pmatrix}
\]

In particular with respect to the coordinate system \( y_1^{(k+1)}, \ldots, y_n^{(k+1)} \) \( sl(2, \mathbb{C}) \) is spanned by
\[ h = \sum_{j=1}^{n} a_{ij}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} + \sum_{j=1}^{n} b_{ij}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} \]

\[ e = \sum_{j=1}^{n} a_{ij}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} + \sum_{j=1}^{n} b_{ij}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} \]

\[ f = \sum_{j=1}^{n} a_{ij}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} + \sum_{j=1}^{n} b_{ij}^{(k+1)} \frac{\partial}{\partial y_j^{(k+1)}} \]

where \( a_{ij}^{(k+1)} \) is a linear function in \( y_{1}^{(k+1)}, \ldots, y_{n}^{(k+1)} \) and \( b_{ij}^{(k+1)} \) is a function in \( y_{1}^{(k+1)}, \ldots, y_{n}^{(k+1)} \) with multiplicity at least \( k + 2 \).

By construction, for each \( \ell \in \mathbb{N} \), we have \( y_{i}^{(\ell+1)} - y_{i}^{(\ell)} \in m^{\ell+1} \) where \( m \) is the maximal ideal of \( C[[x_1, x_2, \ldots, x_n]] \). Then the limit of the coordinate system \( \{ y_{1}^{(\ell+1)}, y_{2}^{(\ell+1)}, \ldots, y_{n}^{(\ell+1)} \} \) with respect to the \( m \)-adic topology is a coordinate system \( \{ y_{1}, y_{2}, \ldots, y_{n} \} \) in \( C[[x_1, x_2, \ldots, x_n]] \) with the property that

\[ y_{i} - y_{i}^{(\ell+1)} \in m^{\ell+2} \quad \text{for all} \quad 1 \leq i \leq n. \]

By chain rule, we know that for \( 1 \leq i \leq n \)

\[
\frac{\partial}{\partial y_{i}^{(\ell+1)}} = \frac{\partial y_{1}}{\partial y_{1}^{(\ell+1)}} \frac{\partial}{\partial y_{1}} + \frac{\partial y_{2}}{\partial y_{1}^{(\ell+1)}} \frac{\partial}{\partial y_{2}} + \cdots + \frac{\partial y_{n}}{\partial y_{1}^{(\ell+1)}} \frac{\partial}{\partial y_{n}}
\]

\[ = \frac{\partial}{\partial y_{i}} + \text{operators of orders at least} \ \ell + 1 \]

\[ \ldots\ (2) \]

where operator of order \( \ell + 1 \) means operator of the form \( \Sigma_{j=1}^{n} p_{j}^{\ell+1}(\partial/\partial y_j) \) with \( p_{j}^{\ell+1} \) a homogeneous polynomial of degree \( \ell + 1 \) in \( y_{1}, \ldots, y_{n} \) variables. Now we claim that \( h, e \) and \( f \) can be written as operator of order 1 with respect to the coordinate system \( y_{1}, \ldots, y_{n} \).

Write

\[ h = D_{1,y} + D_{2,y} + D_{3,y} + \cdots \]

where \( D_{i,y} \) is an operator of order \( i \) with respect to the coordinate sys-
tem $y_1$, $y_2$, . . . , $y_n$. Suppose $D_{j,y} = 0$ for $2 \leq j \leq \ell - 1$. We are going to prove $D_{\ell,y} = 0$. In the coordinate system $y_1^{(\ell+1)}$, $y_2^{(\ell+1)}$, . . . , $y_n^{(\ell+1)}$, $h$ can be written in the form

$$h = \sum_{j=1}^{n} a_{j}^{(\ell+1)} \frac{\partial}{\partial y_j^{(\ell+1)}} + \sum_{j=1}^{n} b_{j}^{(\ell+1)} \frac{\partial}{\partial y_j^{(\ell+1)}}$$

where $a_j^{(\ell+1)}$ is a linear function in $y_1^{(\ell+1)}$, $y_2^{(\ell+1)}$, . . . , $y_n^{(\ell+1)}$ and $b_j^{(\ell+1)}$ is a function in $y_1^{(\ell+1)}$, $y_2^{(\ell+1)}$, . . . , $y_n^{(\ell+1)}$ with multiplicity at least $\ell + 2$. Put (1) and (2) in (3), we see that

$$h = \tilde{D}_{1,y} + \tilde{D}_{\ell+2,y} + \tilde{D}_{\ell+3,y} + \cdots$$

where $\tilde{D}_{j,y}$ is an operator of order $j$ in $y_1$, $y_2$, . . . , $y_n$ coordinate. This implies

$$0 = (D_{1,y} - \tilde{D}_{1,y}) + D_{\ell,y} + D_{\ell+1,y} + (D_{\ell+2,y} - \tilde{D}_{\ell+2,y})$$

$$+ (D_{\ell+3,y} - \tilde{D}_{\ell+3,y}) + \cdots$$

Thus $D_{\ell,y} = D_{\ell+1,y} = 0$. By induction, we have shown $D_{j,y} = 0$ for all $j \geq 2$. Hence $h$ is an operator of first order with respect to $y_1$, $y_2$, . . . , $y_n$ coordinate. Similarly we can prove that $e$ and $f$ are operators of first order with respect to $y_1$, $y_2$, . . . , $y_n$ coordinate.

Q.E.D.

**Theorem 2.2.** Let $st(2, \mathbb{C})$ act on the formal power series ring $\mathbb{C}[[x_1, \ldots, x_n]]$ preserving the $m$-adic filtration where $m$ is the maximal ideal in $\mathbb{C}[[x_1, \ldots, x_n]]$. Then there exists a coordinate system

$$x_1, x_2, \ldots, x_{\ell_1}, x_{\ell_1+1}, x_{\ell_1+2}, \ldots, x_{\ell_1+\ell_2}, \ldots,$$

$$x_{\ell_1+\ell_2+\cdots+\ell_s-1+1}, \ldots, x_{\ell_1+\ell_2+\cdots+\ell_s}$$

such that

$$h = D_{h,1} + \cdots + D_{h,j} + \cdots + D_{h,r}$$

$$e = D_{e,1} + \cdots + D_{e,j} + \cdots + D_{e,r}$$

$$f = D_{f,1} + \cdots + D_{f,j} + \cdots + D_{f,r}$$
where $r \leq s$ and

$$D_{h,j} = (\ell_j - 1)x_1^{\ell_1+\cdots+\ell_{j-1}+1} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$+ (\ell_j - 3)x_1^{\ell_1+\cdots+\ell_{j-1}+2} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+2}}$$

$$+ \cdots + (-(\ell_j - 3))x_1^{\ell_1+\cdots+\ell_{j-1}+1} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$+ (-)(\ell_j - 1))x_1^{\ell_1+\cdots+\ell_{j-1}+1} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$D_{e,j} = (\ell_j - 1)x_1^{\ell_1+\cdots+\ell_{j-1}+1} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$+ \cdots + i(\ell_j - i)x_1^{\ell_1+\cdots+\ell_{j-1}+i} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+i}}$$

$$+ \cdots + (-(\ell_j - 1))x_1^{\ell_1+\cdots+\ell_{j-1}+1} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$D_{f,i} = x_1^{\ell_1+\cdots+\ell_{j-1}+1} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$+ \cdots + x_1^{\ell_1+\cdots+\ell_{i-1}+i} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{i-1}+i}}$$

$$+ \cdots + x_1^{\ell_1+\cdots+\ell_{j-1}+1} \frac{\partial}{\partial x_1^{\ell_1+\cdots+\ell_{j-1}+1}}$$

**Proof.** According to Proposition 2.1, we can choose a coordinate system \(\{y_1, \ldots, y_n\}\) such that the coefficient of \(\partial/\partial y_i, 1 \leq i \leq n\), of every element in \(s\ell(2, \mathbb{C})\) are linear functions in \(y_1, \ldots, y_n\) variables. In view of the proof of complete classification of representations of \(s\ell(2, \mathbb{C})\) representations, by further change of coordinate we obtain a coordinate system \(\{x_1, x_2, \ldots, x_n\}\) such that \(s\ell(2, \mathbb{C})\) takes the form as stated in the theorem. Q.E.D.
3. Singular sets of $sl(2, \mathbb{C})$ invariants polynomials.

**Lemma 3.1.** Suppose $sl(2, \mathbb{C})$ acts on $M^5_k$ the space of homogeneous polynomials of degree $k \geq 2$ in $x_1, x_2, x_3, x_4$ and $x_5$ variables via

$$
\tau = 4x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} - 2x_4 \frac{\partial}{\partial x_4} - 4x_5 \frac{\partial}{\partial x_5}
$$

$$X_+ = 4x_1 \frac{\partial}{\partial x_2} + 6x_2 \frac{\partial}{\partial x_3} + 6x_3 \frac{\partial}{\partial x_4} + 4x_4 \frac{\partial}{\partial x_5}
$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_4}.
$$

Let $I$ be the complex vector subspace spanned by $\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4$ and $\partial f/\partial x_5$ where $f$ is a homogeneous polynomial of degree $k + 1$. If $I$ is a $sl(2, \mathbb{C})$-submodule, then the singular set of $f$ contains the $x_1$ axis and $x_5$ axis.

**Proof.** By Theorem 4 of Section 1 in [4], $f$ is necessarily an invariant $sl(2, \mathbb{C})$ polynomial in $x_1, x_2, x_3, x_4, x_5$ variables. Suppose the weight of $x_i$ is given by the corresponding coefficient in the expression of $\tau$ above i.e.,

$$wt(x_1) = 4, \quad wt(x_2) = 2, \quad wt(x_3) = 0,$$

$$wt(x_4) = -2, \quad wt(x_5) = -4.$$

Then $f$ is a polynomial of weight $0$. Let us assume on the contrary that $x_1$ axis does not lie in the singular set of $f$. Clearly the monomial $x_1^k$ appears in $\partial f/\partial x_i$ for some $1 \leq i \leq 5$. Thus the monomial $x_1^k x_i$ appears in $f$. However, since $k \geq 2$, weight of $x_1^k x_i$ is strictly bigger than zero. This gives a contradiction. Hence $x_1$ axis is contained in the singular set of $f$.

Similarly we can prove that $x_5$ axis is contained in the singular set of $f$. Q.E.D.

**Lemma 3.2.** Suppose $sl(2, \mathbb{C})$ acts on $M^5_k$ the space of homogeneous polynomials of degree $k \geq 2$ in $x_1, x_2, x_3, x_4$ and $x_5$ variables via

$$
\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} - x_5 \frac{\partial}{\partial x_5}
$$
\[
X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_5}
\]
\[
X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4}.
\]

Let \( I \) be the complex vector subspace spanned by \( \partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4 \) and \( \partial f/\partial x_5 \), where \( f \) is a homogeneous polynomial of degree \( k + 1 \). If \( I \) is a \( s\ell(2, \mathbb{C}) \)-submodule then the singular set of \( f \) contains the set \{ \( x_1, x_2, x_3, x_4, x_5 \): \( x_2^2 - 2x_1x_3 = 0 = x_4 = x_5 \) \}.

Proof. By Theorem 4 of Section 1 in [4], we may assume that \( f \) is an invariant \( s\ell(2, \mathbb{C}) \) polynomial in \( x_1, x_2, x_3, x_4 \) and \( x_5 \) variables. Suppose the weight of \( x_i \) is given by the corresponding coefficient in the expression of \( \tau \) as above i.e.,

\[
wt(x_1) = 2, \quad wt(x_2) = 0, \quad wt(x_3) = -2, \quad wt(x_4) = 1, \quad wt(x_5) = -1.
\]

Then \( f \) is a polynomial of weight 0. Write

\[
f = \sum_{\alpha \geq 0, \beta \geq 0} g_{(\alpha, \beta)}(x_1, x_2, x_3)x_4^\alpha x_5^\beta.
\]

Since weight of \( g_{(\alpha, \beta)}(x_1, x_2, x_3) \) is even, we conclude that \( g_{(1,0)}(x_1, x_2, x_3) = 0 = g_{(0,1)}(x_1, x_2, x_3) \). Therefore our lemma will follow if we can show that \( g_{(0,0)}(x_1, x_2, x_3) \) is divisible by \( (x_2^2 - 2x_1x_3)^2 \) whenever \( g_{(0,0)}(x_1, x_2, x_3) \) is nonzero. Observe that \( g_{(0,0)}(x_1, x_2, x_3) \) is a polynomial of weight 0. As \( f \) is an invariant polynomial, we have \( X_- f = X_+ f = 0 \). It follows that \( X_- g_{(0,0)}(x_1, x_2, x_3) = X_+ g_{(0,0)}(x_1, x_2, x_3) = 0 \). Hence \( g_{(0,0)}(x_1, x_2, x_3) \) is also an invariant polynomial in \( x_1, x_2 \) and \( x_3 \) variables of degree \( k + 1 \geq 3 \). Recall that the invariant polynomial in \( x_1, x_2 \) and \( x_3 \) variables must be even degree of the form \( (x_2^2 - 2x_1x_3)^\ell \) (cf. [3]). Therefore \( g_{(0,0)}(x_1, x_2, x_3) \) is divisible by \( (x_2^2 - 2x_1x_3)^2 \) as claimed.

Q.E.D.

Lemma 3.3. Suppose \( s\ell(2, \mathbb{C}) \) acts on \( M_k^5 \), the space of homogeneous polynomials of degree \( k \geq 2 \) in \( x_1, x_2, x_3, x_4 \) and \( x_5 \) variables via

\[
\tau = 3x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} - 3x_4 \frac{\partial}{\partial x_4}.
\]
Let $I$ be the complex vector subspace spanned by $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$ and $\partial f/\partial x_5$, where $f$ is a homogeneous polynomial of degree $k + 1$. If $I$ is a $sl(2, \mathbb{C})$-submodule then the singular set of $f$ contains the set $x_1$ axis and $x_4$ axis.

Proof. By Theorem 4 of Section 1 in [4], $f$ is necessary an invariant $sl(2, \mathbb{C})$ polynomial in $x_1, x_2, x_3, x_4$ and $x_5$ variables. Suppose the weight $x_i$ is given by the corresponding coefficient in the expression of $\tau$ above i.e.,

$$wt(x_1) = 3, \quad wt(x_2) = 1, \quad wt(x_3) = -1,$$

$$wt(x_4) = -3, \quad wt(x_5) = 0.$$

Then $f$ is a polynomial of weight 0. Let us assume on the contrary that $x_1$-axis does not lie in the singular set of $f$. Clearly the monomial $x_1^k$ appears in $\partial f/\partial x_i$ for some $1 \leq i \leq 5$. Thus the monomial $x_1^k x_i$ appears in $f$. However, since $k \geq 2$, weight of $x_1^k x_i$ is strictly bigger than zero. This gives a contradiction. Hence $x_1$ axis is contained in the singular set of $f$.

Similarly we can prove that $x_4$ axis is contained in the singular set of $f$. Q.E.D.

**Lemma 3.4.** Suppose $sl(2, \mathbb{C})$ acts on $M^k_4$ the space of homogeneous polynomials of degree $k \geq 2$ in $x_1, x_2, x_3$ and $x_4$ variables via

$$\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}$$

$$X_+ = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}. $$
Suppose $f$ is a $\mathfrak{s}(2, \mathbb{C})$ invariant homogeneous polynomial of degree $k + 1$ in $x_1, x_2, x_3$ and $x_4$ variables. Then $k + 1 = 2\ell$ is an even integer and $f = c(x_1x_4 - x_2x_3)^\ell$ for some constant $c$.

Proof. Suppose the weight of $x_i$ is given by the corresponding coefficient in the expression of $\tau$ above i.e.,

$$
wt(x_1) = 1, \quad wt(x_2) = -1, \quad wt(x_3) = 1, \quad wt(x_4) = -1.
$$

Then $f$ is a homogeneous polynomial of degree $k + 1$ and weight 0. Let $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\beta_1}x_4^{\beta_2}$ be a monomial appearing in $f$. Then

\begin{align*}
(4.1) \quad & \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = k + 1 \\
(4.2) \quad & \alpha_1 - \alpha_2 + \beta_1 - \beta_2 = 0 \\
(4.3) \quad & 2(\alpha_1 + \beta_1) = k + 1
\end{align*}

Therefore $k + 1$ is an even integer $2\ell$. From (4.3) and (4.2) we have $\beta_1 = \ell - \alpha_1$ and $\beta_2 = \ell - \alpha_2$. We can write $f$ in the following form

$$
f = \sum_{\alpha_1=0}^{\ell} a_{(\alpha_1, \alpha_2)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\beta_1}x_4^{\ell-\alpha_1}\ell^\ell-\alpha_2
$$

$$
X_-(f) = \sum_{\alpha_1, \alpha_2=0}^{\ell} a_{(\alpha_1, \alpha_2)} X_-(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\beta_1}x_4^{\ell-\alpha_1}\ell^\ell-\alpha_2)
$$

\begin{align*}
= & \sum_{\alpha_1=1}^{\ell-1} \sum_{\alpha_2=0}^{\ell-1} \alpha_1 a_{(\alpha_1, \alpha_2)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1}\ell^\ell-\alpha_2 \\
+ & \sum_{\alpha_1=0}^{\ell-1} \sum_{\alpha_2=0}^{\ell-1} (\ell - \alpha_1) a_{(\alpha_1, \alpha_2)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1}\ell^\ell-\alpha_2 \\
= & \sum_{\alpha_1=0}^{\ell-1} \sum_{\alpha_2=1}^{\ell+1} (\alpha_1 + 1) a_{(\alpha_1+1, \alpha_2-1)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1}\ell^\ell-\alpha_2 \\
+ & \sum_{\alpha_1=0}^{\ell-1} \sum_{\alpha_2=0}^{\ell} (\ell - \alpha_1) a_{(\alpha_1, \alpha_2)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell-\alpha_1}\ell^\ell-\alpha_2
\end{align*}
\[
= \sum_{\alpha_1=0}^{\ell-1} \sum_{\alpha_2=1}^{\ell} [(\alpha_1 + 1)a(\alpha_1 + 1, \alpha_2 - 1) + (\ell - \alpha_1)a(\alpha_1, \alpha_2)]x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\ell - \alpha_1 - 1}x_4^{\ell - \alpha_2 + 1} + \sum_{\alpha_1=0}^{\ell-1} (\alpha_1 + 1)a(\alpha_1 + 1, \ell)x_1^{\alpha_1}x_2^{\ell + 1}x_3^{\ell - \alpha_1 - 1} + \sum_{\alpha_1=0}^{\ell-1} (\ell - \alpha_1)a(\alpha_1, 0)x_1^{\alpha_1}x_2^{\ell - \alpha_1 - 1}x_4^{\ell + 1}
\]

Since \(X_\ell f = 0\), we have
\[
(4.4) \quad a(1, \ell) = 0 = a(2, \ell) = \cdots = a(\ell, \ell)
\]
\[
(4.5) \quad a(0, 0) = 0 = a(1, 0) = \cdots = a(\ell - 1, 0)
\]
\[
(4.6) \quad (\alpha_1 + 1)a(\alpha_1 + 1, \alpha_2 - 1) + (\ell - \alpha_1)a(\alpha_1, \alpha_2) = 0 \quad 0 \leq \alpha_1 \leq \ell - 1
\]

\[
1 \leq \alpha_2 \leq \ell
\]

(4.4) and (4.6) imply \(a(\alpha_1, \alpha_2) = 0\) for all \((\alpha_1, \alpha_2)\) such that \(\ell + 1 \leq \alpha_1 + \alpha_2 \leq 2\ell\), \(0 \leq \alpha_1 \leq \ell\), and \(0 \leq \alpha_2 \leq \ell\). On the other hand, (4.5) and (4.6) imply \(a(\alpha_1, \alpha_2) = 0\) for all \((\alpha_1, \alpha_2)\) such that \(0 \leq \alpha_1 + \alpha_2 \leq \ell - 1, 0 \leq \alpha_1 \leq \ell\) and \(0 \leq \alpha_2 \leq \ell\).

Therefore we conclude that the only possible nonzero \(a(\alpha_1, \alpha_2)\) has the property that \(\alpha_1 + \alpha_2 = \ell\). We shall denote \(a(\alpha_1, \ell - \alpha_1)\) by \(b_{\alpha_1}\). Then (4.6) becomes
\[
(\alpha_1 + 1)b_{\alpha_1 + 1} + (\ell - \alpha_1)b_{\alpha_1} = 0 \quad \text{for} \quad 0 \leq \alpha_1 \leq \ell - 1
\]
\[
\Rightarrow b_1 = -\binom{\ell}{1}b_0
\]
\[
b_2 = (-1)^2\binom{\ell}{2}b_0
\]
\[
\vdots
\]
\[ b_i = (-1)^i \binom{\ell}{i} b_0 \]

\[
\vdots
\]

\[ b_{\ell} = (-1)^{\ell} b_0. \]

It follows that

\[
f = b_\ell x_1^\ell x_4^\ell + b_{\ell-1} x_1^{\ell-1} x_2 x_3 x_4^{\ell-1} + \cdots + b_1 x_1 x_2^{\ell-i} x_3^{\ell-i} x_4^{i} \\
+ \cdots + b_0 x_2^\ell x_3^\ell
\]

\[
= b_0 x_2^\ell x_3^\ell - \binom{\ell}{1} b_0 x_1 x_2^{\ell-1} x_3^{\ell-1} x_4 + \cdots + (-1)^i \binom{\ell}{i} b_0 x_1 x_2^{\ell-i} x_3^{\ell-i} x_4^i \\
+ \cdots + (-1)^{\ell} b_0 x_1 x_4^\ell
\]

\[
= b_0 (x_2 x_3 - x_1 x_4)^\ell
\]

Q.E.D.

**Lemma 3.5.** Suppose \( \mathfrak{sl}(2, \mathbb{C}) \) acts on \( M_k^5 \) the space of homogeneous polynomials of degree \( k \geq 2 \) in \( x_1, x_2, x_3, x_4 \) and \( x_5 \) variables via

\[
\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}
\]

\[
X_+ = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}
\]

\[
X_- = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}
\]

Suppose \( f \) is a \( \mathfrak{sl}(2, \mathbb{C}) \) invariant homogeneous polynomial of degree \( k + 1 \) in \( x_1, x_2, x_3, x_4, \) and \( x_5 \) variables. Then \( f \) can be written in the following form

\[
f = a_0 x_5^{k+1} + a_2 x_5^{k-1} (x_1 x_4 - x_2 x_3) + a_4 x_5^{k-3} (x_1 x_4 - x_2 x_3)^2 + \cdots
\]
Proof. Write

\[ f = \sum_{i=0}^{k+1} g_{k+1-i}(x_1, x_2, x_3, x_4)x_5^i \]

where \( g_{k+1-i}(x_1, x_2, x_3, x_4) \) is a homogeneous polynomial of degree \( k + 1 - i \) in \( x_1, x_2, x_3 \) and \( x_4 \) variables.

\[ 0 = X_-(f) = \sum_{i=0}^{k+1} x_5^i X_-(g_{k+1-i}(x_1, x_2, x_3, x_4)) \]

\[ = X_-(g_{k+1-i}(x_1, x_2, x_3, x_4)) = 0. \]

By Lemma 4, \( k + 1 - i \) is an even integer \( 2\ell \) and \( g_{k+1-i}(x_1, x_2, x_3, x_4) \) is a constant multiple of \((x_1x_4 - x_2x_3)\)'s. Q.E.D.

Lemma 3.6. Suppose \( sl(2, \mathbb{C}) \) acts on \( M_5^k \) the space of homogeneous polynomials of degree \( k \geq 2 \) in \( x_1, x_2, x_3, x_4 \) and \( x_5 \) variables via

\[ \tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \]

\[ X_+ = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4} \]

\[ X_- = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}. \]

Let \( I \) be the complex vector subspace spanned by \( \partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4 \) and \( \partial f/\partial x_5 \), where \( f \) is a homogeneous polynomial of degree \( k + 1 \).

If \( I \) is a \( sl(2, \mathbb{C}) \)-submodule, then the singular set of \( f \) contains the set \( \{(x_1, x_2, x_3, x_4, x_5) : x_1x_4 - x_2x_3 = 0 = x_5\} \).

Proof. By Theorem 4 of Section 1 in [4], \( f \) is one of the following.

Case (i). \( f \) is \( sl(2, \mathbb{C}) \) invariant polynomial in \( x_1, x_2, x_3, x_4 \) and \( x_5 \) variables and \( I = (2) \oplus (2) \oplus (1) \). In view of Lemma 5 \( f \) is of the form

\[ f = a_0x_5^{k+1} + a_2x_5^{k-1}(x_1x_4 - x_2x_3) + a_4x_5^{k-3}(x_1x_4 - x_2x_3)^2 + \cdots \]
Since $k \geq 2$, it is easy to see that $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ vanish on the set $\{(x_1, x_2, x_3, x_4, x_5): x_1 x_4 - x_2 x_3 = 0 = x_5\}$.

**Case (ii).** $f$ is a $s\ell(2, \mathbb{C})$ invariant polynomial in $x_1, x_2, x_3$ and $x_4$ variables and $I = (2) \oplus (2)$. In view of Lemma 4, there exists a nonzero constant $c$ such that

$$f(x_1, x_2, x_3, x_4) = c(x_1 x_4 - x_2 x_3)^{\ell}$$

where $\ell \geq 2$. It is clear that $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ vanish on the set $\{(x_1, x_2, x_3, x_4, x_5): x_1 x_4 - x_2 x_3 = 0\}$.

**Case (iii).** $f = cx_5^{k+1}$ where $c$ is a nonzero constant. Clearly the singular set of $f$ is $\{(x_1, x_2, x_3, x_4, x_5): x_5 = 0\}$. Q.E.D.

**Lemma 3.7.** Suppose $s\ell(2, \mathbb{C})$ acts on $M_k^5$ the space of homogeneous polynomials of degree $k \geq 2$ in $x_1, x_2, x_3, x_4$ and $x_5$ variables via

$$\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3}$$

$$X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}.$$ 

Let $I$ be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$, and $\frac{\partial f}{\partial x_5}$, where $f$ is a homogeneous polynomial of degree $k + 1$. If $I$ is a $s\ell(2, \mathbb{C})$ submodule, then the singular set of $f$ contains the set $\{(x_1, x_2, x_3, x_4, x_5): x_2^2 - 2x_1 x_3 = 0 = x_4 = x_5\}$.

**Proof.** By Theorem 4 of Section 1 in [4], we may assume that $f$ is an invariant $s\ell(2, \mathbb{C})$ polynomial. Arguing similarly as in the proof of Lemma 5, we know that in view of Corollary 4.17 in [3], if $k + 1$ is even, say $k + 1 = 2\ell$, then

$$f = c_0(x_2^2 - 2x_1 x_3)^{\ell} + (x_2^2 - 2x_1 x_3)^{\ell - 1}g_2(x_4, x_5)$$

$$+ (x_2^2 - 2x_1 x_3)^{\ell - 2}g_4(x_4, x_5)$$

where...
and if $k + 1$ is odd, say $k + 1 = 2^l + 1$, then

$$f = (x_2^2 - 2x_1x_3)^l g_1(x_4, x_5) + (x_2^2 - 2x_1x_3)^{l-1} g_3(x_4, x_5)$$

$$+ (x_2^2 - 2x_1x_3)^{l-2} g_5(x_4, x_5)$$

$$+ \cdots + g_{2^l + 1}(x_4, x_5)$$

where $g_i(x_4, x_5)$ is a homogeneous polynomial of degree $i$ in $x_4$ and $x_5$ variables.

It is clear that the singular set of $f$ contains the set \{(x_1, x_2, x_3, x_4, x_5) : x_2^2 - 2x_1x_3 = 0 = x_4 = x_5\}. Q.E.D.

**Lemma 3.8.** Suppose $sl(2, \mathbb{C})$ acts on $M^5_8$ the space of homogeneous polynomials of degree $k \geq 2$ in $x_1, x_2, x_3, x_4, and x_5$ variables via

$$\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$$

$$X_+ = x_1 \frac{\partial}{\partial x_2}$$

$$X_- = x_2 \frac{\partial}{\partial x_1}.$$ 

Let $I$ be the complex vector subspace spanned by $\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4,$ and $\partial f/\partial x_5$, where $f$ is a homogeneous polynomial of degree $k + 1$. If $I$ is a $sl(2, \mathbb{C})$ submodule, then the singular set of $f$ contains the set \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 : x_3 = x_4 = x_5 = 0\}.

**Proof.** In view of Theorem 4 of Section 1 in [4], $f$ is a polynomial in $x_3, x_4$ and $x_5$ variables. Our lemma follows immediately. Q.E.D.

**4. Solvability of the Lie algebras $L(V)$.** In [3], we first established a connection between the set of isolated hypersurface singularities and the set of finite dimensional Lie algebras. Let $(V, 0)$ be an isolated singularity in $(\mathbb{C}^n, 0)$ defined by the zero set of a holomorphic function of $f$. The mo-
duili algebra $A(V)$ of $(V, 0)$ is $\mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f, \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$. We define $L(V)$ to be the algebra of derivations of $A(V)$. $A(V)$ is finite dimensional as a $\mathbb{C}$ vector space and $L(V)$ is contained in the endomorphism algebra of $A(V)$; consequently $L(V)$ is a finite dimensional Lie algebra. In [3], we proved that $L(V)$ is solvable for $n \leq 3$. It is the purpose of this section to prove $L(V)$ solvable for $n \leq 5$. In order to avoid the repetition, we shall only concentrate on the case $n = 5$.

Remark. In general, in order to prove $L(V)$ is solvable, it suffices to prove the statement with an additional assumption that multiplicity of $f$ is bigger than two. Because if the multiplicity of $f$ is two, then after a biholomorphic change of coordinates, we can assume that $f = x_n^2 - g(x_1, \ldots, x_{n-1})$. In this case $L(V) = L(W)$ which is solvable by induction hypothesis, where $W = \{(x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1} : g(x_1, \ldots, x_{n-1}) = 0\}$.

Theorem 4.1. Suppose that $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 : f(x_1, x_2, x_3, x_4, x_5) = 0\}$ has an isolated singularity at $(0, 0, 0, 0, 0)$. Then the finite dimensional Lie algebra $L(V)$ associated to the singularity is solvable.

Proof. By the Levi decomposition, if the Lie algebra is not solvable, then the Lie algebra $L(V)$ contains $s\ell(2, \mathbb{C})$ as subalgebra. By Lemma 4.3 of [3], we shall assume that $s\ell(2, \mathbb{C})$ acts on $m/m^2$ nontrivially where $m$ is the maximal ideal in $\mathfrak{o}_{\mathbb{C}^5,0}$. Write $f = \Sigma_{i=k+1}^{\infty} f_i$. According to the above remark, we shall assume without loss of generality that multiplicity of $f = k + 1 \geq 3$. By Theorem 2.2, we know that the action of $s\ell(2, \mathbb{C})$ on $\mathbb{C}[[x_1, x_2, x_3, x_4, x_5]]$ is one of the following forms.

Case 1. $s\ell(2, \mathbb{C})$ acts on $\mathbb{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$
\tau = 4x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} - 2x_4 \frac{\partial}{\partial x_4} - 4x_5 \frac{\partial}{\partial x_5}
$$

$$
X_+ = 4x_1 \frac{\partial}{\partial x_2} + 6x_2 \frac{\partial}{\partial x_3} + 6x_3 \frac{\partial}{\partial x_4} + 4x_4 \frac{\partial}{\partial x_5}
$$

$$
X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_4}.
$$

We shall prove by induction that $\partial f_i/\partial x_j$ for $i \geq k + 1$ and $1 \leq j \leq 5$ vanish along the $x_1$ axis and $x_5$ axis. Let $\Delta(f)$ denote the ideal generated by, $\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4$ and $\partial f/\partial x_5$. $(f) + \Delta(f)$ is a $s\ell(2, \mathbb{C})$ module.
Clearly $m^k/m^{k+1}$ is also a $sl(2, \mathbb{C})$ module. Let $J_k(f)$ denote the image of the canonical map $(f) + \Delta(f) \to m^k/m^{k+1}$. $J_k(f)$ is an invariant subspace spanned by $<\partial f_{k+1}/\partial x_1, \partial f_{k+1}/\partial x_1, \partial f_{k+1}/\partial x_3, \partial f_{k+1}/\partial x_4, \partial f_{k+1}/\partial x_5>$ and hence may be identified with an invariant subspace of $M^\xi_k$. By Lemma 3.1, $\partial f_{k+1}/\partial x_i$ for $1 \leq i \leq 5$ vanish along $x_1$ axis and $x_5$ axis.

Let $g = g_t + g_{t+1} + \cdots$ be a Taylor series expansion of $g$ where $g_t$ is a homogeneous polynomial of degree $i$. Then for any $D \in sl(2, \mathbb{C})$, $Dg = Dg_t + Dg_{t+1} + \cdots$ is a Taylor series expansion of $Dg$. It follows easily that

$$m^n + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_n)/m^{n+1}$$

$$+ \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_n)$$

is a $sl(2, \mathbb{C})$ module. Let $J_n$ denote the image of the canonical map $(f) + \Delta(f) \to m^n + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_n)/m^{n+1} + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_n)$. $J_n$ is an invariant subspace spanned by $<\partial f_{n+1}/\partial x_1, \partial f_{n+1}/\partial x_2, \partial f_{n+1}/\partial x_3, \partial f_{n+1}/\partial x_4, \partial f_{n+1}/\partial x_5>$ and hence may be identified with an invariant subspace of $M^\xi_n$. By Lemma 3.1, $\partial f_{n+1}/\partial x_i$ for $1 \leq i \leq 5$ vanish along $x_1$ axis and $x_5$ axis. This finishes the induction step.

Obviously $\Delta(f)$ vanishes along $x_1$ axis and $x_5$ axis. This implies that $f$ cannot have isolated singularity at the origin, a contradiction to our assumption.

**Case 2.** $sl(2, \mathbb{C})$ acts on $\mathbb{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} - x_5 \frac{\partial}{\partial x_5}$$

$$X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4}.$$

In view of Lemma 3.2, we shall obtain a contradiction by the same argument as Case 1 above.

**Case 3.** $sl(2, \mathbb{C})$ acts on $\mathbb{C}[[x_1, x_2, x_3, x_4, x_5]]$ via
\[
\tau = 3x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} - 3x_4 \frac{\partial}{\partial x_4}
\]

\[
X_+ = 3x_1 \frac{\partial}{\partial x_2} + 4x_2 \frac{\partial}{\partial x_3} + 3x_3 \frac{\partial}{\partial x_4}
\]

\[
X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}.
\]

In view of Lemma 3.3, we shall obtain a contradiction by the same argument as Case 1 above.

**Case 4.** \(\mathfrak{sl}(2, \mathbb{C})\) acts on \(\mathbb{C}[[x_1, x_2, x_3, x_4, x_5]]\) via

\[
\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}
\]

\[
X_+ = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}
\]

\[
X_- = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}
\]

In view of Lemma 3.6, we shall obtain a contradiction by the same argument as Case 1 above.

**Case 5.** \(\mathfrak{sl}(2, \mathbb{C})\) acts on \(\mathbb{C}[[x_1, x_2, x_3, x_4, x_5]]\) via

\[
\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3}
\]

\[
X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3}
\]

\[
X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}.
\]

In view of Lemma 3.7, we shall obtain a contradiction by the same argument as Case 1 above.
Case 6. $\mathfrak{sl}(2, \mathbb{C})$ acts on $\mathbb{C}[[x_1, x_2, x_3, x_4, x_5]]$ via

$$
\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}
$$

$$
X_+ = x_1 \frac{\partial}{\partial x_2}
$$

$$
X_- = x_2 \frac{\partial}{\partial x_1}.
$$

In view of Lemma 3.8, we shall obtain a contradiction by the same argument as Case 1 above. Q.E.D.

REFERENCES


