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Source: *American Journal of Mathematics*, Vol. 113, No. 5 (Oct., 1991), pp. 773-778
Published by: The Johns Hopkins University Press
Stable URL: http://www.jstor.org/stable/2374785
Accessed: 14/09/2011 23:45

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SOLVABILITY OF LIE ALGEBRAS ARISING FROM ISOLATED SINGULARITIES AND NONISOLATEDNESS OF SINGULARITIES DEFINED BY $s\ell(2, \mathbf{C})$ INVARIANT POLYNOMIALS

Dedicated to Professor J.-I. Igusa and Professor J. H. Sampson on their sixty-fifth birthdays

By Stephen S.-T. Yau

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1. Introduction. Let $(V, 0)$ be an isolated singularity in $(\mathbf{C}^n, 0)$ defined by the zero set of a holomorphic function f. The moduli algebra $A(V)$ of $(V, 0)$ is $\mathbf{C}\{x_1, x_2, \ldots, x_n\}/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. It is easy to see that $A(V)$ is a finite dimensional $\mathbf{C}$-algebra and is an invariant of $(V, 0)$. In [Ma-Ya], Mather and the author proved that the complex analytic structure of $(V, 0)$ is determined also by $A(V)$. Thus, the above construction gives an injection map from the space of isolated singularities in $(\mathbf{C}^n, 0)$ to the space of commutative local Artinian algebras. This raises a natural and important problem, the so called recognition problem: Give a necessary and sufficient condition for a commutative local Artinian algebra to be a moduli algebra. In [Ya1], we define $L(V)$ to be the algebra of derivations of $A(V)$. Clearly $L(V)$ is a finite dimensional Lie algebra. In [Se-Ya], Seeley and the present author have shown that this Lie algebra is an interesting invariant in the following sense. It can be used to study moduli problem of singularities. Namely, we have constructed continuous numerical invariants for isolated singularities by means of these Lie algebras. In [Ya2] and [Ya3], the Lie algebra $L(V)$ was shown to be solvable for $n \le 3$ and $n \le 5$ respectively. Naturally one expects that a necessary condition for a commutative local Artinian algebra to be a moduli algebra is that its algebra of derivations is a solvable Lie algebra. In fact, we were encouraged by both of the referees of [Ya2] and [Ya3] to attack the general problem. It is the

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Manuscript received 19 April 1990.
Research partially supported by N.S.F. grant.
American Journal of Mathematics 113 (1991), 773–778.

purpose of this paper to report that $L(V)$ is indeed solvable for general n. The proof depends on two main ingredients. The first one is the classification of the $s\ell(2, \mathbf{C})$ actions on $\mathbf{C}[[x_1, x_2, \ldots, x_n]]$ via derivations preserving the m-adic filtration, which was done in our previous work [Ya3]. The second one is the recent work of Sampson, Yu and the author [Sa-Ya-Yu] on classification of gradient spaces invariant by $s\ell(2, \mathbf{C})$ actions, which was done by [Ya4] only for the case $n \leq 5$. Actually what we need is the following special statement in [Sa-Ya-Yu]. Let f be a homogeneous polynomial in n variables of degree $k + 1$, $k \geq 2$. Then the vector space $I(f)$ spanned by $\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n$ is a $s\ell(2, \mathbf{C})$-submodule if and only if $I(f) = \langle \partial g/\partial x_1, \partial g/\partial x_2, \ldots, \partial g/\partial x_n \rangle$ for some $s\ell(2, \mathbf{C})$ invariant polynomial g. This special statement was proved independently by Kempf [Ke]. In fact he proved that this special statement is true also for other semisimple Lie algebras as well. The main observation in this paper is that once we fix a $s\ell(2, \mathbf{C})$ action, the singular sets of varieties defined by $s\ell(2, \mathbf{C})$ invariant polynomials of degree ≥ 3 have to contain a one dimensional set independent of the invariant polynomials.

The author would like to thank the hospitality of the Mathematics Department of Johns Hopkins University while he was visiting in 1989–1990 academic year.

2. Proof of the main theorems.

THEOREM 1. *Let $s\ell(2, \mathbf{C})$ act on M_n^k, the space of homogeneous polynomials of degree $k \geq 2$ in*

$$x_1, x_2, \ldots, x_{\ell_1}; x_{\ell_1+1}, \ldots, x_{\ell_1+\ell_2}; \ldots; x_{\ell_1+\ell_2+\cdots+\ell_{r-1}+1}, \ldots, x_{\ell_1+\ell_2+\cdots+\ell_r}; x_{\ell_1+\cdots+\ell_r+1}, \ldots, x_n$$

via

$$\tau = D_{\tau,1} + \cdots + D_{\tau,j} + \cdots + D_{\tau,r}$$

$$X_+ = D_{X_+,1} + \cdots + D_{X_+,j} + \cdots + D_{X_+,r}$$

$$X_- = D_{X_-,1} + \cdots + D_{X_-,j} + \cdots + D_{X_-,r}$$

where

$$D_{\tau, j} = (\ell_j - 1)x_{\ell_1+\cdots+\ell_{j-1}+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+1}}$$

$$+ (\ell_j - 3)x_{\ell_1+\cdots+\ell_{j-1}+2}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+2}} + \cdots$$

$$+ (-(\ell_j - 3))x_{\ell_1+\cdots+\ell_{j-1}-1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}-1}}$$

$$+ (-(\ell_j - 1))x_{\ell_1+\cdots+\ell_j}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_j}}$$

$$D_{X_+, j} = (\ell_j - 1)x_{\ell_1+\cdots+\ell_{j-1}+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+2}} + \cdots$$

$$+ i(\ell_j - i)x_{\ell_1+\cdots+\ell_{j-1}+i}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+i+1}} + \cdots$$

$$+ (\ell_j - 1)x_{\ell_1+\cdots+\ell_j-1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_j}}$$

$$D_{X_-, j} = x_{\ell_1+\cdots+\ell_{j-1}+2}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+1}} + \cdots$$

$$+ x_{\ell_1+\cdots+\ell_{j-1}+i+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{j-1}+i}} + \cdots$$

$$+ x_{\ell_1+\cdots+\ell_j}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_j-1}}$$

and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r \geq 2$. *Let I be the complex vector subspace spanned by* $\partial f/\partial x_1$, $\partial f/\partial x_2, \ldots, \partial f/\partial x_n$ *where f is a homogeneous polynomial of degree* $k + 1$. *If I is a* $s\ell(2, \mathbf{C})$*-submodule, then the singular set of f contains the* x_1*-axis and the* x_{ℓ_1}*-axis.*

Proof. Since I is a $s\ell(2, \mathbf{C})$-submodule, by the result of [Sa-Ya-Yu] and [Ke], there exists a $s\ell(2, \mathbf{C})$ invariant g in M_n^{k+1} such that the complex vector space spanned by $\partial g/\partial x_1, \partial g/\partial x_2, \ldots, \partial g/\partial x_n$ is exactly I. Therefore the singular set of f and the singular set of g are exactly the same. Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above, i.e.

$$Wt(x_{\ell_1+\cdots+\ell_{j-1}+i}) = \ell_j - (2i - 1) \qquad \text{for } 1 \le i \le \ell_j \text{ and } 1 \le j \le r$$

and

$$Wt(x_\alpha) = 0 \qquad \text{if } \alpha > \ell_1 + \cdots + \ell_r$$

Then g is a polynomial of weight 0. Let us assume on the contrary that the x_1-axis does not lie in the singular set of g. Clearly the monomial x_1^k appears in $\partial g/\partial x_i$ for some $1 \le i \le n$. Thus the monomial $x_1^k x_i$ appears in g. Observe that $Wt(x_1) \ge |Wt(x_i)|$ for all $1 \le i \le n$. Since $k \ge 2$, weight of $x_1^k x_i$ is strictly greater than zero. This gives a contradiction. Hence the x_1-axis is contained in the singular set of g which is the same as the singular set of f.

Similarly we can prove that the x_{ℓ_1}-axis is contained in the singular set of f. Q.E.D.

In [Ya1], we first established a connection between the set of isolated hypersurface singularities and the set of finite dimensional Lie algebras. Let $(V, 0)$ be an isolated singularity in $(\mathbf{C}^n, 0)$ defined by the zero set of a holomorphic function of f. The moduli algebra $A(V)$ of $(V, 0)$ is $\mathbf{C}\{x_1, x_2, \ldots, x_n\}/(f, \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$. We define $L(V)$ to be the algebra of derivations of $A(V)$. $A(V)$ is finite dimensional as a $\mathbf{C}$-vector space and $L(V)$ is contained in the endomorphism algebra of $A(V)$; consequently $L(V)$ is a finite dimensional Lie algebra. In [Ya2], we proved that $L(V)$ is solvable for $n \le 3$. Later, we generalized it to the case where $n \le 5$ [Ya3]. It is the purpose here to prove $L(V)$ is solvable for general n.

Remark. In general, in order to prove $L(V)$ is solvable, it suffices to prove the statement with an additional assumption that the multiplicity of f is bigger than two. Because if the multiplicity of f is two, then after a biholomorphic change of coordinates, we can assume that $f = x^2 - g(x_1, \ldots, x_{n-1})$. In this case $L(V) = L(W)$ which is solvable by induction hypothesis, where $W = \{(x_1, \ldots, x_{n-1}) \in \mathbf{C}^{n-1} : g(x_1, \ldots, x_{n-1}) = 0\}$.

THEOREM 2. *Suppose that* $V = \{(x_1, \ldots, x_n) \in \mathbf{C}^n : f(x_1, \ldots, x_n) = 0\}$ *has an isolated singularity at* $(0, 0, \ldots, 0)$. *Then the finite dimensional Lie algebra* $L(V)$ *associated to the singularity is solvable.*

Proof. By the Levi decomposition, if the Lie algebra is not solvable, then the Lie algebra $L(V)$ contains $s\ell(2, \mathbf{C})$ as a subalgebra. By Lemma 4.3 of [Ya2], we shall assume that $s\ell(2, \mathbf{C})$ acts on m/m^2 nontrivially where m is the maximal ideal in $\mathcal{O}_{\mathbf{C}^n,0}$. Write $f = \Sigma_{i=k+1}^{\infty} f_i$. According to the above remark, we shall assume without loss of generality that the multiplicity of $f = k + 1 \geq 3$. By Theorem 2.2 of [Ya3], we know that the action of $s\ell(2, \mathbf{C})$ on $\mathbf{C}[[X_1, \ldots, x_n]]$ is of the form given in the statement of Theorem 1 above. We shall prove by induction that $\partial f_i/\partial x_j$ for $i \geq k + 1$ and $1 \leq j \leq n$ vanish along the x_1-axis. Let $\Delta(f)$ denote the ideal generated by $\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n$. The moduli ideal $(f) + \Delta(f)$ is an $s\ell(2, \mathbf{C})$-module. Clearly m^k/m^{k-1} is also an $s\ell(2, \mathbf{C})$ module. Let $J_k(f)$ denote the image of the canonical map $(f) + \Delta(f) \to m^k/m^{k+1}$. $J_k(f)$ is an invariant subspace spanned by $\langle \partial f_{k+1}/\partial x_1, \partial f_{k+1}/\partial x_2, \ldots, \partial f_{k+1}/\partial x_n \rangle$ and hence may be identified with an invariant subspace of M_n^k. By Theorem 1, $\partial f_{k+1}/\partial x_i$ for $1 \leq i \leq n$ vanish along the x_1-axis.

Let $g = \Sigma_{i\geq 0}\, g_i$ be a Taylor series expansion of g where g_i is a homogeneous polynomial of degree i. Then for any $D \in s\ell(2, \mathbf{C})$, $Dg = \Sigma_{i\geq 0}\, Dg_i$ is also a Taylor series expansion of Dg. It follows easily that $m^\ell + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_\ell)/m^{\ell+1} + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_\ell)$ is an $s\ell(2, \mathbf{C})$ module. Let J_ℓ denote the image of the canonical map $(f) + \Delta(f) \to m^\ell + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_\ell)/m^{\ell-1} + \Delta(f_{k+1}) + \Delta(f_{k+2}) + \cdots + \Delta(f_\ell)$. J_ℓ is an invariant subspace spanned by $\langle \partial f_{\ell+1}/\partial x_1, \partial f_{\ell+1}/\partial x_2, \ldots, \partial f_{\ell+1}/\partial x_n \rangle$ and hence may be identified with an invariant subspace of M_n^ℓ. By Theorem 1, $\partial f_\ell/\partial x_i$ for $1 \leq i \leq n$ vanish along the x_1-axis. This finishes the induction step.

Obviously $\Delta(f)$ vanishes along the x_1-axis. This implies that f cannot have an isolated singularity at the origin, a contradiction to our assumption. Q.E.D.

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