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The Signature of Milnor Fibres and Duality Theorem for Strongly Pseudoconvex Manifolds

Stephen S.-T. Yau*

School of Mathematics, The Institute For Advanced Study
Princeton, New Jersey 08540, USA
and Mathematics Department, Harvard University, Cambridge, Mass. 02138, USA

§ 0. Introduction

Let M be a complex manifold. A real-valued C^∞-functions φ on M is said to be strongly plurisubharmonic if and only if the hermitian form

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} d z_i d \bar{z}_j$$

is positive definite with respect to any system of local coordinates $(z_1, \ldots, z_n)$. The complex manifold M is said to be strongly pseudoconvex if there is a compact subset $B \subset M$, and a continuous real valued function φ on M, which is strongly plurisubharmonic outside B, and such that for each $c \in \mathbf{R}$, the set

$$B_c = \{x \in M : \varphi(x) < c\}$$

is relatively compact in M. Note that a strongly pseudoconvex manifold is a modification of a Stein space at finitely many points. From now on we assume that M is a strongly pseudoconvex manifold of dimension n. Let A be the maximal compact analytic subset in M. Then A has a finite number of connected components A_α $(\alpha = 1, \ldots, \rho)$. Each A_α consists of a finite number of irreducible components X_{α_i} $(i = 1, \ldots, n_\alpha)$.

Duality Theorems for compact complex manifolds (such as Serre duality) are well known. Serre duality is still true for open manifolds but one has to use the cohomology with compact supports. It is a natural question to ask for a duality Theorem for strongly pseudoconvex manifolds without using cohomology with compact support. The following theorem is the first theorem in this direction. Let Ω^p be the sheaf of germs of holomorphic p-forms on M. Let $\chi^p(M) = \sum_{i=1}^{n} (-1)^i \dim H^i(M, \Omega^p)$.

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Remark. The only difference between this definition and the classical definition of Euler Poincaré characteristic of a coherent sheaf on a compact complex analytic space is that we do not consider $\dim H^0(M,\Omega^p)$ because it may be infinite dimensional.

Theorem A. *Let M be a strongly pseudoconvex manifold of dimension $n \geqq 4$. Suppose the maximal compact analytic subset in M can be blown down to isolated hypersurface singularities. Then*

$$\chi^p(M) = (-1)^n \chi^{n-p}(M), \qquad 2 \leqq p \leqq n-2.$$

Let $f\colon (\mathbf{C}^{2n+1}, 0) \to (\mathbf{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. For $\varepsilon > 0$ suitably small and δ yet smaller, the space $V' = f^{-1}(\delta) \cap D_\varepsilon$ (where D_ε denotes the closed disk of radius ε about 0) is a real oriented $4n$-manifold with boundary whose diffeomorphism type depends only on f. In fact, Milnor [18] proved that V' has the homotopy type of a wedge of $2n$-spheres. In case of two dimensional singularities, various signature formulae are known. For example, Hirzebruch-Mayer [11] have a formula for the signature when f is of the type $x^a + y^b + z^c$ and Steenbrink [27] has a formula for the signature when $f(x,y,z)$ is weighted homogeneous. For the double points, i.e. $f(x,y,z)$ is of the type $g(x,y) + z^2$, it is known that the intersection pairing of V' is the same as a symmetrized Seifert matrix of the compound torus link defined by g. Furthermore, the signature of a link is by definition the signature of this symmetrized Seifert matrix. In [25] Shinohara had a simple formula for the signature of a compound knot; hence if $g^{-1}(0)$ is irreducible, a simple formula for the signature of V' in terms of the Puiseux pairs of g is available. In case $g^{-1}(0)$ has several branches at the origin, Murasugi [19] developed a method to find a Seifert matrix for the link defined by g and compute the signature. Recently, Durfee [4] has given an interesting formula for the signature σ of V' in terms of topological invariants of a resolution of the singularity at 0 of the complex surface $f^{-1}(0)$. It is a natural question to ask for a formula for the signature σ of V' for higher dimensional singularities. In the following theorem, the signature of even dimensional singularities is given in terms of topological and analytic invariants of a resolution of the singularity.

Definition 0.1. Let $\pi\colon M \to V = \{x \in \mathbf{C}^{2n+1} : f(x) = 0\}$ be a resolution of the singularity. The signature of M, denoted by $\sigma(M)$, is defined to be the signature of $\pi^{-1}(D_\varepsilon \cap V)$ where D_ε denotes the closed ball of radius ε about 0, i.e. the signature of a closed tubular neighborhood of the exceptional set $A = \pi^{-1}(0)$.

Theorem B. *Let $f(z_0, z_1, \ldots, z_{2n})$ be holomorphic in $N \subseteq \mathbf{C}^{2n+1}$, $n > 1$, a Stein neighborhood of $(0,0,\ldots,0)$ with $f(0,0,\ldots,0) = 0$. Let $V = N \cap f^{-1}(0)$ have $(0,0,\ldots,0)$ as its only singular point. Let σ be the signature of V'. Let $\pi\colon M \to V$ be a resolution of V.* Then

$$\sigma = -\sum_{p=q}^{2n-q} \chi^p(M) - 2\sum_{p=0}^{q-1} \chi^p(M) + \sigma(M)$$

for $2 \leqq q \leqq n$.

In case of surface singularities, we have the following inequality.

Theorem C. *Let $f(x,y,z)$ be holomorphic in N, a Stein neighborhood of $(0,0,0)$ with $f(0,0,0)=0$. Let $V=N\cap f^{-1}(0)$ have the origin as its only singular point. Let σ be the signature of V'. Let $\pi\colon M\to V$ be a resolution of V and s be the number of irreducible components of the exceptional set $A=\pi^{-1}(0,0,0)$. Then*

$$\sigma \leqq -s+2\dim H^1(M,\mathcal{O})+\dim H^1(M,\Omega^1).$$

Our presentation goes as follows. In § 1, we recall some definitions and notations and describe some invariants of a resolution. In § 2, we study the higher dimensional singularities. In particular, Theorem A and B are proved. In § 3, we concentrate on surface singularities. We prove that Laufer's formula [14] can be obtained without using Riemann Roch Theorem. Moreover, one can also obtain Durfee's formula for signature (Theorem 1.5 of [4]) without using his method of signature defect.

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§ 1. Preliminaries

The signature σ of an arbitrary oriented $2n$-manifold W with or without boundary is defined as follows: There is a symmetric bilinear intersection pairing (,) on $H_n(W;\mathbf{R})$ (this form is symmetric only if n is even) defined by setting

$$(x,y)=(x'\cup y')[W]$$

where x' and y' in $H^n(W,\partial W;\mathbf{R})$ are Lefschetz duals to x and y in $H_n(W;\mathbf{R})$ and $[W]\in H_{2n}(W,\partial W;\mathbf{R})$ is the orientation class. This bilinear form may be diagonalized, with diagonal entries $+1$, 0 and -1. The signature σ of W is the signature of this bilinear form, namely, the number of positive minus the number of negative diagonal entries.

Now let us describe the invariants of a resolution. Let V be Stein space with p as its only singularity. It follows from Hironaka's work [9] that a resolution $\pi\colon M\to V$ always exists. By Lemma 3.1 of [13], we know that $\dim H^i(M,\mathcal{O})=\dim(R^i\pi_*\mathcal{O})_p$ for $i>0$. The genus p of the singularity of V is defined as

$$p=\dim_C H^{n-1}(M,\mathcal{O}).$$

This number is independent of the resolution. (See Theorem A of [29].)

In case V has a surface singularity, we may write the compact complex one-dimensional exceptional set $A=\pi^{-1}(0)$ as the union of its irreducible components:

$$A=A_1\cup\cdots\cup A_s.$$

The intersection matrix $[A_i\cdot A_j]$, where $A_i\cdot A_j$ is defined as the number of points of intersection of A_i and A_j for $i\neq j$, and the self-intersection of A_i for $i=j$ (the first Chern class of the normal bundle to A_i in M), is known to be negative definite.

Recall that V is Gorenstein if there is a non-zero holomorphic two-form on its regular points $V \setminus \{0\}$. We remarked that hypersurface singularities and complete intersections are Gorenstein singularities.

Definition 1.1. Suppose that V is Gorenstein. Let g_i be the geometric genus of A_i, i.e. the genus of the desingularization of A_i. The canonical divisor $K = \sum k_i A_i$ is the divisor on M uniquely determined by the equations

$$A_i \cdot K = -A_i \cdot A_i + 2g_i - 2 + 2\delta_i$$

where δ_i is the "number" of nodes and cusps on A_i. Each singular point on A_i other than a node or cusp counts as at least two nodes.

The self-intersection of K is the number

$$K^2 = \sum_{i,j} k_i k_j A_i \cdot A_j$$

which is a topological invariant of the resolution π. Finally recall a well-known result of Samuel [23].

Theorem 1.2. *Let $f\colon U \subset \mathbf{C}^{n+1} \to \mathbf{C}$ be an analytic function on a neighborhood U of 0 in $\mathbf{C}^{n+1}$. Suppose $f(0)=0$ and 0 is an isolated critical point. Then there exists a polynomial $f_0\colon \mathbf{C}^{n+1} \to \mathbf{C}$ with an isolated critical point at 0 and an analytic isomorphism of a neighborhood U_1 of 0 onto a neighborhood U_2 of 0 which sends the points of $f=0$ on points of $f_0=0$.*

§ 2. Proof of Theorem A and Theorem B

To begin with, let us recall some techniqual lemma in [1].

Lemma 2.1. *Let*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A'_2 \oplus A''_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A'_5 \oplus A''_5 & \longrightarrow & A_6 \\
 & & \uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle \pi'_2} \ \uparrow{\scriptstyle \pi''_2} & & \uparrow{\scriptstyle \pi_3} & & \uparrow{\scriptstyle \pi_4} & & \uparrow{\scriptstyle \pi'_5} \ \uparrow{\scriptstyle \pi''_5} & & \uparrow{\scriptstyle \pi_6} \\
0 & \longrightarrow & B_1 & \longrightarrow & B'_2 \oplus B''_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B'_5 \oplus B''_5 & \longrightarrow & B_6
\end{array}$$

$$\begin{array}{cccccccc}
\longrightarrow & A_7 & \longrightarrow & A'_8 \oplus A''_8 & \longrightarrow & A_9 & \longrightarrow & \cdots \\
 & \uparrow{\scriptstyle \pi_7} & & \uparrow{\scriptstyle \pi'_8} \ \uparrow{\scriptstyle \pi''_8} & & \uparrow{\scriptstyle \pi_9} & & \\
\longrightarrow & B_7 & \longrightarrow & B'_8 \oplus B''_8 & \longrightarrow & B_9 & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{cccccccccc}
\longrightarrow & A_{3n-2} & \longrightarrow & A'_{3n-1} \oplus A''_{3n-1} & \longrightarrow & A_{3n} & \longrightarrow & A_{3n+1} & \longrightarrow & 0 \\
 & \uparrow{\scriptstyle \pi_{3n-2}} & & \uparrow{\scriptstyle \pi'_{3n-1}} \ \uparrow{\scriptstyle \pi''_{3n-1}} & & \uparrow{\scriptstyle \pi_{3n}} & & \uparrow{\scriptstyle \pi_{3n+1}} & & \\
\longrightarrow & B_{3n-2} & \longrightarrow & B'_{3n-1} \oplus B''_{3n-1} & \longrightarrow & B_{3n} & \longrightarrow & B_{3n+1} & \longrightarrow & 0
\end{array}$$

be a commutative diagram with exact rows. Suppose π_1, π'_2 and π''_{3i-1}, π_{3i}, $1 \leqq i \leqq n$ are isomorphisms and all the vector spaces are finite dimensional except possibly $A'_2, B'_2, A''_{3i-1}, A_{3i}, B''_{3i-1}, B_{3i}$, $1 \leqq i \leqq n$. Suppose also that $B'_{3i+2}=0$ for $1 \leqq i \leqq n-1$. Then

$$\sum_{i=0}^{n} (-1)^i \dim A_{3i+1} = \sum_{i=0}^{n} (-1)^i \dim B_{3i+1} + \sum_{i=1}^{n-1} (-1)^i \dim A'_{3i+2}.$$

If we assume that π_1, π'_2 are monomorphisms instead of isomorphisms, then

$$\sum_{i=0}^{n} (-1)^i \dim A_{3i+1} \geqq \sum_{i=0}^{n} (-1)^i \dim B_{3i+1} + \sum_{i=1}^{n-1} (-1)^i \dim A'_{3i+2}.$$

In fact,

$$\begin{aligned}\sum_{i=0}^{n} (-1)^i \dim A_{3i+1} - \dim \operatorname{coker} \pi_1 &= \dim \operatorname{coker} \pi'_2 \\ &= \sum_{i=0}^{n} (-1)^i \dim B_{3i+1} + \sum_{i=1}^{n-1} (-1)^i \dim A'_{3i+2}.\end{aligned}$$

Proof of Theorem B. By Theorem 1.2 any holomorphic function which agrees with f to sufficiently high order defines a holomorphically equivalent singularity at $(0, 0, \ldots, 0)$, [23, 8]. So we may take f to be a polynomial. Compactify $\mathbf{C}^{2n+1}$ to $\mathbf{P}^{2n+1}$. Let $\bar{V}_t$ be the closure in $\mathbf{P}^{2n+1}$ of

$$V_t = \{(z_0, z_1, \ldots, z_{2n}) \in \mathbf{C}^{2n+1} : f(z_0, z_1, \ldots, z_{2n}) = t\}.$$

By adding a suitably general high order homogeneous term of degree e to the polynomial f, we may additionally assume that $\bar{V}_0$ has $(0, 0, \ldots, 0) \in \mathbf{C}^{2n+1}$ as its only singularity and that $\bar{V}_t$ is non-singular for small $t \neq 0$. We may also assume that the highest order terms of f define, in homogeneous coordinates, a nonsingular hypersurface of order e in $\mathbf{P}^{2n} = \mathbf{P}^{2n+1} - \mathbf{C}^{2n+1}$. $\bar{V}_t$ is then necessarily irreducible for all small t. Without loss of generality, we take $N = \mathbf{C}^{2n+1}$. Then $V = V_0$.

For any $4k$-dimensional topological manifold S, the signature of S is denoted by σ_S. Let B_ε be an open Milnor ball of radius ε. Let $\bar{M}$ be the resolution of $\bar{V}$ which has M as an open subset. Then for small t,

$$\begin{aligned}\sigma &= \sigma_{\bar{V}_t} - \sigma_{\bar{V}_t - B_\varepsilon \cap \bar{V}_t} \\ &= \sigma_{\bar{V}_t} - \sigma_{\bar{V}_0 - B_\varepsilon \cap \bar{V}_0}\end{aligned}$$

since the family $\{\bar{V}_t\}$ is differentiably trivial away from B_ε. Hence

$$\begin{aligned}\sigma &= \sigma_{\bar{V}_t} - \sigma_{\bar{M} - \pi^{-1}(B_\varepsilon \cap \bar{V}_0)} \\ &= \sigma_{\bar{V}_t} - (\sigma_{\bar{M}} - \sigma_{\pi^{-1}(B_\varepsilon \cap \bar{V}_0)}) \\ &= \sigma_{\bar{V}_t} - \sigma_{\bar{M}} + \sigma(M). \qquad (2.1)\end{aligned}$$

In order to calculate $\sigma_{\bar{V}_t} - \sigma_{\bar{M}}$, we need the following lemmas and propositions.

Let X be a complex analytic subvariety of $\mathbf{C}^n$ and $\mathscr{I}_X$ be the ideal sheaf of X in $\mathbf{C}^n$. Let $\Omega^p_{\mathbf{C}^n}$ be the sheaf of germs of holomorphic p-forms, on $\mathbf{C}^n$. There are two ways to associate a sheaf of germs of holomorphic p-forms on X, namely,

$$\tilde{\Omega}_X^p = \Omega_{\mathbf{C}^n}^p / \{f\alpha + dg \wedge \beta : f, g \in \mathscr{I}_X,\ \alpha \in \Omega_{\mathbf{C}^n}^p,\ \beta \in \Omega_{\mathbf{C}^n}^{p-1}\}$$

and

$$\Omega_X^p = \Omega_{\mathbf{C}^n}^p / \{\omega \in \Omega_{\mathbf{C}^n}^p : \omega / X^* = 0\}$$

where X^* is the regular part of X. Ω_X^p is obtained by taking the sheaf of germs of holomorphic p forms in the ambient space modulo those germs of holomorphic p-forms whose restriction on regular part of X is zero. This is a coherent sheaf [5].

Observe that $\tilde{\Omega}_X^p$ coincide with Ω_X^p at regular parts of X and we have a natural surjective map $\varphi: \tilde{\Omega}_X^p \to \Omega_X^p$. Let K^p be the kernel of this mapping. Then

$$0 \to K^p \to \tilde{\Omega}_X^p \to \Omega_X^p \to 0$$

is exact. K^p is supported on singular part of X since φ is isomorphic off singular part of X.

Lemma 2.2. *Let* $\mathfrak{t}$ *be a coherent analytic sheaf on* X. *Let* s *be a section of* $\mathfrak{t}$ *over* X. *If* supp $s = Y$ *is a nowhere dense proper analytic subvariety of* X, *then* s *is a section of the torsion subsheaf of* $\mathfrak{t}$.

Proof. Consider the sheaf homomorphism

$$\alpha: \mathcal{O}_X \to \mathfrak{t}$$

defined by multiplication by s. Let $\mathscr{I}$ be the kernel of α which is a coherent subsheaf. We claim that for each $x \in Y$, $\mathscr{I}_x \neq 0$. Suppose on the contrary that $\mathscr{I}_x = 0$ for some $x \in Y$. Then there exists an open neighborhood U of x such that $\mathscr{I}/U = 0$ since the support of $\mathscr{I}$ is closed. This means that α is injective on U. However, since $Y \cap U$ is nowhere dense in U and s is zero on $U \setminus Y$, α is a zero map on $U \setminus Y$. This leads to a contradiction. Q.E.D.

Proposition 2.3. *If* X *is irreducible at every point in singular part of* X, *then* K^p $=$ *torsion of* $\tilde{\Omega}_X^p$.

Proof. By the Lemma 2.2 and the Remark before Lemma 2.2, we need only to prove torsion of $\tilde{\Omega}_X^p$ is in K^p. It is clear that the support of torsion subsheaf of $\tilde{\Omega}_X^p$ is in singular part of X. Let $\omega \in \tilde{\Omega}_{X,x}^p$ be a torsion element. Let $\tilde{\omega}$ be a section of $\tilde{\Omega}_X^p$ over U, a neighborhood of x such the germ of $\tilde{\omega}$ at x is ω. Since ω is a torsion element, there exists $0 \neq g \in \mathcal{O}_{X,x}$ such that $g \cdot \omega = 0$. Let $\tilde{g}$ be a section of $\mathcal{O}_X$ over U such that the germ of $\tilde{g}$ at x is g. Then $\tilde{g} \cdot \tilde{\omega} = 0$ in a perhaps smaller neighborhood of x. In particular $\tilde{g} \cdot \tilde{\omega} = 0$ in a regular part of X near x. Since $\tilde{g} \not\equiv 0$ on regular part of X and x is irreducible at x, $\tilde{\omega} = 0$ on regular part of X near x. Therefore $\omega \in K^p$. Q.E.D.

From now on, we assume that X is a hypersurface,

$$X = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : f(z_1, \ldots, z_n) = 0\} \subseteq \mathbf{C}^n$$

and $SX =$ singular part of $X = \{0\}$.

Proposition 2.4. *Let* $\tilde{\omega} \in \Omega_{\mathbf{C}^n, x}^p$ *have image* $\omega \in \tilde{\Omega}_{X,x}^p$; *then* $\omega \in (K^p)_x \Leftrightarrow \exists \alpha \in \Omega_{X,x}^{p+1}$ *such that* $df \wedge \tilde{\omega} = f\alpha$ $(0 \leqq p \leqq n-1)$.

Proof. Since X is a hypersurface with only isolated singularity, X is normal. In particular, X is irreducible at each point.

"$\Rightarrow$" Suppose $\omega \in (K^p)_x$. Then there exists $g \neq 0$ in $\mathcal{O}_{X,x}$ such that $g \cdot \omega = 0$. Let $\tilde{g}$ and $\tilde{\omega}$ be the representatives of g and ω in the ambient space. There exist a p-form α_1 and $p-1$ form β such that

$$\tilde{g}\tilde{\omega} = f\alpha_1 + df \wedge \beta$$
$$\Rightarrow \tilde{g}\, df \wedge \tilde{\omega} = f\, df \wedge \alpha_1.$$

Suppose $df \wedge \tilde{\omega} = \sum_I a_I\, dZ_I$ and $df \wedge \alpha_1 = \sum_I b_I\, dZ_I$ where $dZ_I = dz_{i_1} \wedge \cdots \wedge dz_{i_{p+1}}$, $I = (i_1, \ldots, i_{p+1})$ and $i_1 < \cdots < i_{p+1}$. Hence $g a_I = f b_I$ for all I. Since $g \not\equiv 0$ on X, f does not divide g. But f is irreducible, so $a_I = a'_I f$ for some holomorphic germs a'_I in the ambient space for all I. Hence

$$\begin{aligned} df \wedge \tilde{\omega} &= \sum_I a_I\, dZ_I \\ &= f\Big(\sum_I a'_I\, dZ_I\Big) \\ &= f \cdot \alpha' \quad \text{where} \quad \alpha' = \sum_I a'_I\, dZ_I. \end{aligned}$$

Let α be the corresponding element of α' in $\tilde{\Omega}^{p+1}_{X,x}$. Then we have

$$df \wedge \omega = f\alpha.$$

"$\Leftarrow$" Conversely, we need Coleff's residue theory. Let $\tilde{\gamma}$ be a p-form in a neighborhood U of the origin which is polar in X, i.e. $\tilde{\gamma} = \frac{\gamma}{f}$ where γ is a regular p-form in U. For any C^∞ $2n-p-1$ form α with compact support

$$\lim_{\varepsilon \to 0} \int_{\{|f| = \varepsilon\}} \tilde{\gamma} \wedge \alpha$$

exists. The residue $R_X(\tilde{\gamma})$ of $\tilde{\gamma}$ along X is a $\mathbf{C}$-linear functional on the C^∞ $2n-p-1$ forms with compact supports which is defined as follows.

$$R_X(\tilde{\gamma}) \colon A_c^{2n-p-1}(U) \to \mathbf{C}$$
$$R_X(\tilde{\gamma})(\theta) = \lim_{\varepsilon \to 0} \int_{\{|f| = \varepsilon\}} \tilde{\gamma} \wedge \theta.$$

This residue $R_X(\tilde{\gamma})$ of $\tilde{\gamma}$ is actually continuous in the sense of distribution. We need the following two basic properties

(1) If $\tilde{\gamma}$ is regular, then $R_X(\tilde{\gamma}) = 0$

(2) If $\tilde{\gamma} = \frac{df}{f} \wedge \eta$, then

$$R_X\left(\frac{df}{f} \wedge \eta\right)(\theta) = 2\pi i \int_X \eta \wedge \theta, \qquad \theta \in A_c^{2n-p-1}(U).$$

Now the converse part of Proposition 2.4 follows easily from (1) and (2). First choose $\tilde{\omega}$ and $\tilde{\alpha}$ be the representatives of ω and α in the open neighborhood U or origin such that

$$df \wedge \tilde{\omega} = f\tilde{\alpha} + f\tilde{\beta} \qquad \text{where } \tilde{\beta} \text{ is a } (p+1)\text{-form on } U.$$
$$\Rightarrow \frac{df}{f} \wedge \tilde{\omega} = \tilde{\alpha} + \tilde{\beta}.$$

Let X^* be the regular part of X. Then

$$R_{X^*}\left(\frac{df}{f} \wedge \tilde{\omega}\right) = R_{X^*}(\tilde{\alpha} + \tilde{\beta}) = 0$$
$$\Rightarrow R_{X^*}\left(\frac{df}{f} \wedge \tilde{\omega}\right)(\theta) = 2\pi i \int_{X^*} \tilde{\omega} \wedge \theta$$
$$= 0 \quad \text{for all } \theta \in A_c^{2n-p-1}(U)$$
$$\Rightarrow \tilde{\omega} = 0 \quad \text{on } X^*.$$

By Lemma 2.2, $\omega \in K^p$. Q.E.D.

Remark. Actually the converse part of Proposition 2.4 is not needed for the rest of our work.

Theorem 2.5. $K^p = 0$ *for* $0 \leqq p \leqq n-2$, *i.e.* $\tilde{\Omega}_X^p$ *is torsion free for* $0 \leqq p \leqq n-2$.

Before we prove this theorem, let us recall a beautiful Theorem due to Saito. Let R be a noetherian commutative ring with unit. The depth of an ideal I of R is the maximal length q of prime sequences $a_1, \ldots, a_q \in I$ with:

i) a_1 is a non-zero-divisor of R

ii) a_i is a non-zero-divisor of $R/a_1 R + \cdots + a_{i-1} R$, $i = 2, \ldots, q$. Let M be a free R-module of finite rank n. We denote by $\bigwedge^p M$ the p-th exterior product of M (with $\bigwedge^0 M = R$ and $\bigwedge^{-1} M = 0$).

Let $\omega_1, \ldots, \omega_k$ be given elements of M, and $(e_1, \ldots, e_n)$ be a free basis of M.

$$\omega_1 \wedge \cdots \wedge \omega_k = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} a_{i_1 \ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Let I be the ideal of R generated by the coefficients $a_{i_1 \ldots i_k}$, $1 \leqq i_1 < \cdots < i_k \leqq n$. (We put $I = R$, when $k = 0$). Let

$$Z^p := \{\omega \in \bigwedge^p M : \omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0\}, \qquad p = 0, 1, 2, \ldots$$
$$H^p := Z^p \Big/ \sum_{i=1}^{k} \left(\omega_i \wedge \bigwedge^{p-1} M\right), \qquad p = 0, 1, 2, \ldots.$$

In the case when $k = 0$, we understand $Z^p = 0$, $H^p = 0$ for $p = 0, 1, 2, \ldots$.

Theorem 2.6 (Saito). (i) *There exists an integer* $m \geqq 0$ *such that*

$$I^m H^p = 0 \quad \textit{for} \ \ p = 0, 1, 2, \ldots, n.$$

(ii) $H^p = 0$ *for* $0 \leq p < prof(I)$.

Now we prove the Theorem 2.5. Let $A = \mathcal{O}_{\mathbf{C}^n, 0}$ be the ring of germs of holomorphic functions at origin. This is a noetherian regular local ring with unit. In particular, it is Cohen-Macaulay of dimension n. Let $I = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ be the ideal in A. Since X has an isolated singularity, $\left\{\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right\}$ forms a system of parameters of A by Hilbert Nullstellensatz. $\left\{\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right\}$ is a prime sequence because of Corollary 1 of Appendix 6 of [30]. Hence $\mathrm{prof}(I) = n$. By Theorem 2.6, we have for any $\theta \in \Omega^p_{\mathbf{C}^n, 0}$, $1 \leq p \leq n-1$

$$\theta \wedge df = 0 \Leftrightarrow \theta = df \wedge \theta_1 \tag{2.2}$$

for some $\theta_1 \in \Omega^{p-1}_{\mathbf{C}^n, 0}$.

Let $\omega \in (K^p)_0$ where $0 \leq p \leq n-2$. Let $\tilde{\omega}$ be a representative of ω. Then by the proof of Proposition 2.4, there exists α in $\Omega^{p+1}_{\mathbf{C}^n, 0}$ such that

$$\begin{aligned}
& df \wedge \tilde{\omega} = f\alpha \\
\Rightarrow\ & df \wedge f\alpha = df \wedge df \wedge \tilde{\omega} = 0 \\
\Rightarrow\ & df \wedge \alpha = 0 \\
\Rightarrow\ & \alpha = df \wedge \alpha_1 \quad \text{for some } \alpha_1 \in \Omega^p_{\mathbf{C}^n, 0} \text{ since } p+1 \leq n-1 \\
\Rightarrow\ & df \wedge \tilde{\omega} = f df \wedge \alpha_1 \\
\Rightarrow\ & df \wedge (\tilde{\omega} - f\alpha_1) = 0 \\
\Rightarrow\ & \tilde{\omega} - f\alpha_1 = df \wedge \beta_1 \quad \text{for some } \beta_1 \in \Omega^{p-1}_{\mathbf{C}^n, 0} \\
\Rightarrow\ & \tilde{\omega} = f\alpha_1 + df \wedge \beta_1 \\
\Rightarrow\ & \omega = 0 \quad \text{in } \tilde{\Omega}^p_{X, 0}. \quad \text{Q.E.D.}
\end{aligned}$$

Now we return to the proof of Theorem B. Let $\tilde{V}$ be the hypersurface defined by

$$z^e_{2n+1} f\left(\frac{z_0}{z_{2n+1}}, \frac{z_1}{z_{2n+1}}, \ldots, \frac{z_{2n}}{z_{2n+1}}\right) - t z^e_{2n+1} = 0$$

in $\mathbf{P}^{2n+1} \times D_\varepsilon$ where D_ε is a disk of radius ε in $\mathbf{C}$. By Theorem 2.5, we know that $\tilde{\Omega}^p_{\tilde{V}}$ is torsion free for $0 \leq p \leq 2n$. We claim that

$$\tilde{\Omega}^p_{\tilde{V}/D_\varepsilon} := \tilde{\Omega}^p_{\tilde{V}} / dt \wedge \tilde{\Omega}^{p-1}_{\tilde{V}}$$

the sheaf of relative p-forms of $\pi: \tilde{V} \to D_\varepsilon$, where π is the natural projection is also torsion free. To prove this it is enough to consider the point $((0, 0, \ldots, 0); 0)$ where $(0, 0, \ldots, 0)$ denotes the origin of $\mathbf{C}^{2n+1}$, the original affine space. Let us denote this point by 0. Let $\omega \in (\tilde{\Omega}^p_{\tilde{V}/D_\varepsilon})_0$. Suppose there exists $0 \neq g \in \mathcal{O}_{\tilde{V}, 0}$ such that $g\omega = 0$. Let $\omega' \in \tilde{\Omega}^p_{\tilde{V}, 0}$ such that the image of ω' in $(\tilde{\Omega}^p_{\tilde{V}/D_\varepsilon})_0$ is ω. There exists $\alpha \in \tilde{\Omega}^{p-1}_{\tilde{V}, 0}$ such that $g\omega' = dt \wedge \alpha$. Let $\tilde{g}$, $\tilde{\omega}'$ and $\tilde{\alpha}$ be elements in $\mathcal{O}_{\mathbf{P}^{2n+1} \times D_\varepsilon, 0}$, $\Omega^p_{\mathbf{P}^{2n+1} \times D_\varepsilon, 0}$ and $\Omega^{p-1}_{\mathbf{P}^{2n+1} \times D_\varepsilon, 0}$ such that their image in $\mathcal{O}_{\tilde{V}, 0}$, $\tilde{\Omega}^p_{\tilde{V}, 0}$, and $\tilde{\Omega}^{p-1}_{\tilde{V}, 0}$ are g, ω, and α respectively.

There exist $h \in \mathcal{O}_{\mathbf{P}^{2n+1} \times D_\varepsilon, 0}$, β_1, $\tilde{\omega}'_1 \in \Omega^p_{\mathbf{P}^{2n+1} \times D_\varepsilon, 0}$, γ_1, β_2, δ_1, $\delta_2 \in \Omega^{p-1}_{\mathbf{P}^{2n+1} \times D_\varepsilon, 0}$, γ_2, $\delta_3 \in \Omega^{p-2}_{\mathbf{P}^{2n+1} \times D_\varepsilon, 0}$ such that $\tilde{\omega}'_1$, β_1 and γ_1 do not involve dt, $\tilde{\omega}' = \tilde{\omega}'_1 + dt \wedge \delta_1$ and

$$\begin{aligned}
&(\tilde{g} + (f-t)h)[\tilde{\omega}' + (f-t)(\beta_1 + dt \wedge \delta_2) + (df - dt)(\gamma_1 + dt \wedge \delta_3)] \\
&\quad = dt \wedge [\tilde{\alpha} + (f-t)\beta_2 + (df - dt)\gamma_2] \\
&\quad \Rightarrow (\tilde{g} + (f-t)h)[\tilde{\omega}'_1 + (f-t)\beta_1 + df \wedge \gamma_1] = 0.
\end{aligned}$$

Since $\tilde{\Omega}^p_{\bar{V}}$ is torsion free and $g \neq 0$ in $\mathcal{O}_{V,0}$, the image of $\tilde{\omega}'_1 + (f-t)\beta_1 + df \wedge \gamma_1$ in $\tilde{\Omega}^p_{V,0}$ is zero. Hence

$$\begin{aligned}
&\text{zero of } \tilde{\Omega}^p_{V,0} \\
&\quad = \text{the image of } \tilde{\omega}'_1 + (f-t)\beta_1 + df \wedge \gamma_1 \quad \text{in } \tilde{\Omega}^p_{V,0} \\
&\quad = \text{the image of } \tilde{\omega}'_1 + (f-t)\beta_1 + (df - dt) \wedge \gamma_1 + dt \wedge \gamma_1 \quad \text{in } \tilde{\Omega}^p_{V,0} \\
&\quad = \omega'_1 + dt \wedge \gamma \quad \text{in } \tilde{\Omega}^p_{V,0}
\end{aligned}$$

where γ is the image of γ_1 in $\tilde{\Omega}^{p-1}_{V,0}$ and ω'_1 is the image of $\tilde{\omega}'_1$ in $\tilde{\Omega}^p_{V,0}$.

$$\begin{aligned}
&\omega \text{ in } (\tilde{\Omega}^p_{V/D_\varepsilon})_0 \\
&\quad = \text{image of } \omega'_1 + dt \wedge \gamma \quad \text{in } \tilde{\Omega}^p_{V/D_\varepsilon, 0} \\
&\quad = 0 \quad \text{in } (\tilde{\Omega}^p_{V/D_\varepsilon})_0.
\end{aligned}$$

This proves that the sheafs of relative p-forms $\tilde{\Omega}^p_{V/D_\varepsilon}$, $0 \leq p \leq 2n$, is also torsion free. In particular, $\tilde{\Omega}^p_{V/D_\varepsilon}$, $0 \leq p \leq 2n$, is a π-flat coherent analytic sheaf [p. 154, 6]. The analytic restriction of $\tilde{\Omega}^p_{\bar{V}/D_\varepsilon}$ on the fibre $\bar{V}_t$ gives the sheaf of holomorphic forms $\tilde{\Omega}^p_{\bar{V}_t}$ of the fibre. Therefore the Euler-Poincaré characteristic

$$\chi^p(\bar{V}_t) = \sum_{q=0}^{\infty} (-1)^q \dim_{\mathbf{C}} H^q(\bar{V}_t, \tilde{\Omega}^p_{\bar{V}_t})$$

of $\tilde{\Omega}^p_{\bar{V}_t}$ is equal to the Euler-Poincaré characteristic

$$\chi^p(\bar{V}_0) = \sum_{q=0}^{\infty} (-1)^q \dim_{\mathbf{C}} H^q(\bar{V}_0, \tilde{\Omega}^p_{\bar{V}_0})$$

of $\tilde{\Omega}^p_{\bar{V}_0}$ for $0 \leq p \leq 2n$. By (2.1) and Hodge index Theorem, we have

$$\begin{aligned}
\sigma &= \sum_{p=0}^{2n} \chi^p(\bar{V}_t) - \sum_{p=0}^{2n} \chi^p(\bar{M}) + \sigma(M) \\
&= \sum_{p=q}^{2n-q} \chi^p(\bar{V}_0) + 2\sum_{p=0}^{q-1} \chi^p(\bar{V}_0) - \sum_{p=q}^{2n-q} \chi^p(\bar{M}) - 2\sum_{p=0}^{q-1} \chi^p(\bar{M}) + \sigma(M) \qquad (2.3)
\end{aligned}$$

by Serre duality.

In order to calculate $\chi^p(\bar{V}_0) - \chi^p(\bar{M})$, we need the following Theorem.

Theorem 2.7. *Let $V = \{(z_1, \ldots, z_n) : f(z_1, \ldots, z_n) = 0\} \subseteq \mathbf{C}^n$ be a hypersurface with origin as its only singularity. Let ω be a holomorphic p-form on a deleted neighborhood $U \setminus \{0\}$ of 0 in $\mathbf{C}^n$ where $0 \leq p \leq n-3$. Then ω has a holomorphic extension to the* *origin, i.e. there exists $\omega' \in \Gamma(U, \tilde{\Omega}^p_V)$ such that the restriction of ω' to $U \setminus \{0\}$ are equal to ω.*

Proof. Let $\mathscr{D}^p = f\Omega^p_{\mathbf{C}^n} + df \wedge \Omega^{p-1}_{\mathbf{C}^n}$. We claim that for $0 \leqq p \leqq n-1$, the following sequence

$$\begin{aligned} 0 \to \Omega^0_{\mathbf{C}^n} \xrightarrow{\tau_p} \Omega^1_{\mathbf{C}^n} \oplus \Omega^0_{\mathbf{C}^n} \to \cdots \to \Omega^{p-i}_{\mathbf{C}^n} \oplus \Omega^{p-i-1}_{\mathbf{C}^n} \xrightarrow{\tau_i} \Omega^{p-i+1}_{\mathbf{C}^n} \oplus \Omega^{p-i}_{\mathbf{C}^n} \to \cdots \\ \to \Omega^{p-1}_{\mathbf{C}^n} \oplus \Omega^{p-2}_{\mathbf{C}^n} \xrightarrow{\tau_1} \Omega^p_{\mathbf{C}^n} \oplus \Omega^{p-1}_{\mathbf{C}^n} \xrightarrow{\psi} \mathscr{D}^p \to 0 \end{aligned} \tag{2.4}$$

is exact at 0 in $\mathbf{C}^n$ where

$$\begin{aligned} &\psi(\alpha, \beta) = f\alpha + df \wedge \beta \qquad (\alpha, \beta) \in \Omega^p_{\mathbf{C}^n} \oplus \Omega^{p-1}_{\mathbf{C}^n} \\ &\tau_i(\alpha, \beta) = (df \wedge \alpha, df \wedge \beta + (-1)^i f\alpha) \qquad (\alpha, \beta) \in \Omega^{p-i}_{\mathbf{C}^n} \oplus \Omega^{p-i-1}_{\mathbf{C}^n} \quad 1 \leqq i \leqq p-1 \\ &\tau_p(\alpha) = (df \wedge \alpha, (-1)^p f\alpha) \qquad \alpha \in \Omega^0_{\mathbf{C}^n} \end{aligned}$$

are $\mathcal{O}$-linear.

Obviously, by the definition of ψ, (2.4) is exact at the stage $\mathscr{D}^p$. Let $(\alpha, \beta) \in \Omega^{p-1}_{\mathbf{C}^n,0} \oplus \Omega^{p-2}_{\mathbf{C}^n,0}$. Then

$$\begin{aligned} \psi \circ \tau_1(\alpha, \beta) &= \psi(df \wedge \alpha, df \wedge \beta - f\alpha) \\ &= f df \wedge \alpha + df \wedge (df \wedge \beta - f\alpha) \\ &= 0. \end{aligned}$$

Let $(\gamma, \delta) \in \Omega^p_{\mathbf{C}^n,0} \oplus \Omega^{p-1}_{\mathbf{C}^n,0}$ such that $\psi(\gamma, \delta) = 0$, i.e.

$$\begin{aligned} &f\gamma + df \wedge \delta = 0 \\ \Rightarrow\ &f df \wedge \gamma = -df \wedge df \wedge \delta = 0 \\ \Rightarrow\ &df \wedge \gamma = 0 \\ \Rightarrow\ &\gamma = df \wedge \alpha \qquad \text{for some } \alpha \in \Omega^{p-1}_{\mathbf{C}^n,0} \text{ by (2.2)} \\ \Rightarrow\ &df \wedge (f\alpha + \delta) = 0 \\ \Rightarrow\ &f\alpha + \delta = df \wedge \beta \qquad \text{for some } \beta \in \Omega^{p-2}_{\mathbf{C}^n,0} \text{ by (2.2)} \\ \Rightarrow\ &\delta = df \wedge \beta - f\alpha. \end{aligned}$$

Hence $\tau_1(\alpha, \beta) = (\gamma, \delta)$. The sequence (2.3) is exact at the stage $\Omega^p_{\mathbf{C}^n,0} \oplus \Omega^{p-1}_{\mathbf{C}^n,0}$. We are going to show that the sequence (2.3) is exact at the stage $\Omega^{p-i}_{\mathbf{C}^n,0} \oplus \Omega^{p-i-1}_{\mathbf{C}^n,0}$ for $1 \leqq i \leqq p-1$. Let (α, β) be an element in $\Omega^{p-i-1}_{\mathbf{C}^n,0} \oplus \Omega^{p-i-2}_{\mathbf{C}^n,0}$. Then

$$\begin{aligned} \tau_i \circ \tau_{i+1}(\alpha, \beta) &= \tau_i(df \wedge \alpha, df \wedge \beta + (-1)^{i+1} f\alpha) \\ &= (df \wedge df \wedge \alpha, df \wedge (df \wedge \beta + (-1)^{i+1} f\alpha) + (-1)^i f df \wedge \alpha) \\ &= (0, 0). \end{aligned}$$

Conversely, let $(\gamma, \delta) \in \Omega^{p-i}_{\mathbf{C}^n,0} \oplus \Omega^{p-i-1}_{\mathbf{C}^n,0}$ such that $\tau_i(\gamma, \delta) = 0$, i.e.

$$\begin{aligned} &(df \wedge \gamma, df \wedge \delta + (-1)^i f\gamma) = (0, 0) \\ \Rightarrow\ &\gamma = df \wedge \alpha \qquad \text{for some } \alpha \in \Omega^{p-i-1}_{\mathbf{C}^n,0} \text{ by (2.2)} \\ \Rightarrow\ &df \wedge (\delta + (-1)^i f\alpha) = 0 \\ \Rightarrow\ &\delta + (-1)^i f\alpha = df \wedge \beta \qquad \text{for some } \beta \in \Omega^{p-i-2}_{\mathbf{C}^n,0} \text{ by (2.2)} \\ \Rightarrow\ &d = df \wedge \beta + (-1)^{i+1} f\alpha. \end{aligned}$$

Hence $\tau_{i+1}(\alpha, \beta)=(\gamma, \delta)$ and the sequence (2.3) is exact at the stage $\Omega^{p-i}_{\mathbf{C}^n,0}\oplus\Omega^{p-i-1}_{\mathbf{C}^n,0}$ for $1\leqq i\leqq p-1$. The sequence (2.3) is also exact at the stage $\Omega^0_{\mathbf{C}^n,0}$ because τ_p is injective. Since (2.3) is a complex of coherent sheaves and it is exact at origin in $\mathbf{C}^n$, there exist polydisc Δ containing 0 such that (2.3) is exact on Δ. We have the following sheaf exact sequences on Δ

$$\begin{array}{lllll} 0\to & \Omega^0_{\mathbf{C}^n} & \xrightarrow{\tau_p} \Omega^1_{\mathbf{C}^n}\oplus\Omega^0_{\mathbf{C}^n} & \to \operatorname{Im}\tau_{p-1} & \to 0\\ 0\to \operatorname{Im}\tau_{p-1} & \to & \Omega^2_{\mathbf{C}^n}\oplus\Omega^1_{\mathbf{C}^n} & \to \operatorname{Im}\tau_{p-2} & \to 0\\ & & \vdots & & \\ 0\to \operatorname{Im}\tau_2 & \to & \Omega^{p-1}_{\mathbf{C}^n}\oplus\Omega^{p-2}_{\mathbf{C}^n} & \to \operatorname{Im}\tau_1 & \to 0\\ 0\to \operatorname{Im}\tau_1 & \to & \Omega^p_{\mathbf{C}^n}\oplus\Omega^{p-1}_{\mathbf{C}^n} & \to \mathscr{D}^p & \to 0. \end{array}$$

Consider the corresponding long cohomology exact sequence on $\Delta\setminus\{0\}$. We see inductively that

$$\begin{aligned} &H^i(\Delta\setminus\{0\}, \operatorname{Im}\tau_{p-1})=0 \quad \text{for } 1\leqq i\leqq n-3\\ &H^i(\Delta\setminus\{0\}, \operatorname{Im}\tau_{p-2})=0 \quad \text{for } 1\leqq i\leqq n-4\\ &H^i(\Delta\setminus\{0\}, \operatorname{Im}\tau_1)\ =0 \quad \text{for } 1\leqq i\leqq n-p-1 \end{aligned}$$

and

$$H^i(\Delta\setminus\{0\}, \mathscr{D}^p) \quad =0 \quad \text{for } 1\leqq i\leqq n-p-2.$$

We have the exact sheaf sequence

$$0\to\mathscr{D}^p\to\Omega^p_{\mathbf{C}^n}\to\tilde{\Omega}^p_V\to 0 \tag{2.5}$$

(2.5) yields the usual exact long cohomology sequence. Since $H^1(\Delta-\{0\}, \mathscr{D}^p)=0$ for $0\leqq p\leqq n-3$, every holomorphic p-forms $\omega\in\Gamma(\Delta\cap V-\{0\}, \tilde{\Omega}^p_V)=\Gamma(\Delta\setminus\{0\}, \tilde{\Omega}^p_V)$ will be the restriction of a holomorphic p-form $\tilde{\omega}\in\Gamma(\Delta\setminus\{0\}, \Omega^p_{\mathbf{C}^n})$. $\tilde{\omega}$ extends to be holomorphic in Δ by Hartog's theorem and then restricts back to V to give a holomorphic extension of ω. Q.E.D.

Corollary 2.8. *Let $V=\{(z_1, \dots, z_n): f(z_1, \dots, z_n)=0\}\subseteq\mathbf{C}^n$ be a hypersurface with origin as its only singularity. Let $\pi\colon M\to V$ be a resolution of the singularity of V. Then*

$$\pi^*\colon \Gamma(V, \tilde{\Omega}^p_V)\to\Gamma(M, \Omega^p_M)$$

is bijective for $0\leqq p\leqq n-3$.

If f is a polynomial and the compactification $\bar{V}$ of V in $\mathbf{P}^n$ is irreducible and has no singular points other than the original singular point $0\in V$, then

$$\bar{\pi}^*\colon \Gamma(\bar{V}, \Omega^p_{\bar{V}})\to\Gamma(\bar{M}, \Omega^p_{\bar{M}})$$

is bijective for $0\leqq p\leqq n-3$ where $\bar{\pi}\colon \bar{M}\to\bar{V}$ is a resolution of $\bar{V}$ which has M as an open subset. Moreover, both π^* and $\bar{\pi}^*$ are injective for $p=n-2$.

Proof. By Theorem 2.5, $K^p=0$ for $0\leqq p\leqq n-2$. This means that $\tilde{\Omega}^p_V=\Omega^p_V$ for $0\leqq p\leqq n-2$. Hence π^* and $\bar{\pi}^*$ are injective for $0\leqq p\leqq n-2$.

We claim that π^* is surjective for $0 \leqq p \leqq n-3$. Let $\omega \in \Gamma(M, \Omega_M^p)$. Let ω_1 be the element of $\Gamma(V\setminus\{0\}, \tilde{\Omega}_V^p)$ induced by $\omega/M\setminus\pi^{-1}(0)$ via π. By Theorem 2.7, there exists $\omega_2 \in \Gamma(V, \Omega_V^p)$ such that $\omega_2/V\setminus\{0\} = \omega_1$. Then $\pi^*(\omega_2) = \omega$ and the surjectivity of π^* is proved. Similarly $\bar{\pi}^*$ is surjective.

Now we continue our proof of Theorem B.. We prove that

$$\sum_{i=0}^{2n} (-1)^i \dim H^i(\bar{M}, \Omega^p) = \sum_{i=0}^{2n} (-1)^i \dim H^i(\bar{V}_0, \Omega_{\bar{V}_0}^p) + \sum_{i=1}^{2n-1} (-1)^i \dim H^i(M, \Omega^p)$$

$$\text{i.e.}\quad \chi^p(\bar{M}) = \chi^p(\bar{V}_0) + \chi^p(M), \qquad 0 \leqq p \leqq 2n-2 \tag{2.6}$$

as follows. By the Mayer Vietoris sequence, the rows of the following commutative diagram are exact.

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(\bar{M}, \Omega^p) & \longrightarrow & H^0(M, \Omega^p) \oplus & H^0(\bar{M}\setminus A, \Omega^p) & \longrightarrow & H^0(M\setminus A, \Omega^p) \rightarrow \\
& \uparrow \pi_1 & & \uparrow \pi_2' & \uparrow \pi_2'' & & \uparrow \pi_3' \\
0 \longrightarrow & H^0(\bar{V}_0, \tilde{\Omega}^p) & \longrightarrow & H^0(V_0, \tilde{\Omega}^p) \oplus & H^0(\bar{V}_0\setminus\{0\}, \tilde{\Omega}^p) & \longrightarrow & H^0(V_0\setminus\{0\}, \tilde{\Omega}^p) \rightarrow \\
\longrightarrow & H^1(\bar{M}, \Omega^p) & \longrightarrow & H^1(M, \Omega^p) \oplus & H^1(\bar{M}\setminus A, \Omega^p) & \longrightarrow & H^1(M\setminus A, \Omega^p) \rightarrow \cdots \\
& \uparrow \pi_4 & & \uparrow \pi_5' & \uparrow \pi_5'' & & \uparrow \pi_6 \\
\longrightarrow & H^1(\bar{V}_0, \tilde{\Omega}^p) & \longrightarrow & H^1(V_0, \tilde{\Omega}^p) \oplus & H^1(\bar{V}_0\setminus\{0\}, \tilde{\Omega}^p) & \longrightarrow & H^1(V\setminus\{0\}, \tilde{\Omega}^p) \rightarrow \cdots \\
\longrightarrow & H^{2n-1}(\bar{M}, \Omega^p) & \longrightarrow & H^{2n-1}(M, \Omega^p) \oplus & H^{2n-1}(\bar{M}\setminus A, \Omega^p) & \rightarrow & \\
& \uparrow \pi_{6n-2} & & \uparrow \pi_{6n-1}' & \uparrow \pi_{6n-1}'' & & \\
\longrightarrow & H^{2n-1}(\bar{V}_0, \tilde{\Omega}^p) & \longrightarrow & H^{2n-1}(V_0, \tilde{\Omega}^p) \oplus & H^{2n-1}(\bar{V}_0\setminus\{0\}, \tilde{\Omega}^p) & \rightarrow & \\
\longrightarrow & H^{2n-1}(M\setminus A, \Omega^p) & \longrightarrow & H^{2n}(\bar{M}, \Omega^p) & \longrightarrow 0 & & \\
& \uparrow \pi_{6n} & & \uparrow \pi_{6n+1} & & & \\
\longrightarrow & H^{2n-1}(V_0\setminus\{0\}, \tilde{\Omega}^p) & \longrightarrow & H^{2n}(\bar{V}_0, \tilde{\Omega}^p) & \longrightarrow 0. & &
\end{array} \tag{2.7}$$

The higher terms in (2.7) are 0. Since $\bar{\pi}$ is biholomorphic on $\bar{M}\setminus A$, (2.6) follows from Lemma 2.1 and Corollary 2.8. Put (2.6) into (2.3), we get

$$\sigma = -\sum_{p=q}^{2n-q} \chi^p(M) - 2\sum_{p=0}^{q-1} \chi^p(M) + \sigma(M).$$

This completes the proof of Theorem B. Q.E.D.

Proof of the Theorem A. Let V be a stein space with finitely many isolated hypersurface singularities $\{q_1, \ldots, q_\rho\}$ such that M is a modification of V at $\{q_1, \ldots, q_\rho\}$, i.e. $\pi: M \to V$ is a resolution of singularities of V. Let $U_1, \ldots, U_\rho$ be the Stein neighborhoods of $q_1, \ldots, q_\rho$ respectively such that $\pi^{-1}(U_1), \ldots, \pi^{-1}(U_\rho)$ are holomorphically convex neighborhoods of A_α, $1 \leqq \alpha \leqq \rho$ respectively. Then the restriction mapping

$$\begin{aligned}\gamma\colon\ & H^i(M,\mathfrak{I})\to H^i(\bigcup \pi^{-1}(U_\alpha),\mathfrak{I})\\ &=\oplus H^i(\pi^{-1}(U_\alpha),\mathfrak{I})\end{aligned}$$

is an isomorphism for $i\geqq 1$, by Lemma 3.1 of [13]. We may assume without loss of generality that M is a resolution of the singularity of V where $V=N\cap f^{-1}(0)$ has the origin as its only singular point and $f(z_0, z_1, \dots, z_n)$ is holomorphic in $N\subseteq \mathbf{C}^{n+1}$, $n\geqq 4$, a Stein neighborhood of $(0,0,\dots,0)$ with $f(0,0,\dots,0)=0$. As in the proof of Theorem B, we may further assume that f is a polynomial and $\bar{V}_t$ the compactification of $V_t=f^{-1}(t)$ in $\mathbf{P}^{n+1}$ has the following properties. $\bar{V}_0$ has $(0,0,\dots,0)\in\mathbf{C}^{n+1}$ as its only singularity, $\bar{V}_t$ is non-singular for small $t\neq 0$, and $\bar{V}_t$ is irreducible for all small t. By Serre duality, $\chi^p(\bar{V}_t)=(-1)^n\chi^{n-p}(\bar{V}_t)$ for $0\leqq p\leqq n$ and small $t\neq 0$. Hence $\chi^p(\bar{V}_0)=(-1)^n\chi^{n-p}(\bar{V}_0)$ by the proof of Theorem B. Now (2.6) implies

$$\begin{aligned}&\chi^p(\bar{M})-\chi^p(M)=(-1)^n[\chi^{n-p}(\bar{M})-\chi^{n-p}(M)] \qquad \text{for } 2\leqq p\leqq n-2\\ &\Rightarrow\ \chi^p(M)=(-1)^n\chi^{n-p}(M)\end{aligned}$$

because $\chi^p(\bar{M})=(-1)^n\chi^{n-p}(\bar{M})$ by Serre duality. Q.E.D.

§ 3. Surface Singularities

Let $f(z_0,\dots,z_n)$ be a holomorphic function defined near $0=(0,\dots,0)$ such that $V=\{(z_0,\dots,z_n)\colon f(z_0,\dots,z_n)=0\}$ has an isolated singularity at 0. We choose V to be Stein. Let μ be the Milnor number of the singularity at 0. The Milnor number, originally defined for polynomial f [18] is also defined for holomorphic f [2, 15, 20], and is a topological invariant of the local embedding near 0 of V in $\mathbf{C}^{n+1}$ [17].

For $n=1$, the plane curve case, Milnor [18] showed that

$$\mu=2\delta-\gamma+1 \tag{3.1}$$

where γ is the number of irreducible components of V and δ is the "number" of nodes and cusps at 0. In this section we would like to apply our method developed in Section 2 to study surface singularities.

Theorem 3.1. *Let $f(x,y,z)$ be holomorphic in N, a Stein neighborhood of $(0,0,0)$ with $f(0,0,0)=0$. Let $V=\{(x,y,z)\in N\colon f(x,y,z)=0\}$ have $(0,0,0)$ as its only singular point. Let μ be the Milnor number of $(0,0,0)$ and σ be the signature of V'. Let $\pi\colon M\to V$ be a resolution of V and $A=\pi^{-1}(0,0,0)$. Let $\chi_T(A)$ be the topological Euler characteristic of A and s be the number of irreducible components of A. Then*

$$\sigma\leqq -s+2\dim H^1(M,\mathcal{O})+\dim H^1(M,\Omega^1), \tag{3.2}$$

$$1+\mu\geqq \chi_T(A)+2\dim H^1(M,\mathcal{O})-\dim H^1(M,\Omega^1), \tag{3.3}$$

and

$$1+\mu+\sigma=\chi_T(A)-s+4\dim H^1(M,\mathcal{O}), \tag{3.4}$$

where Ω^1 is the sheaf of germs of holomorphic 1-forms on M.

Remark. (3.3) was obtained in [1]. The ingredient of the proof of Laufer's formula [14] is the Riemann-Roch Theorem. The ingredient of the proof of Durfee's formula (Theorem 1.5 of [4]) is the Hirzebruch index Theorem. We prove Theorem 3.1 without using the Riemann-Roch Theorem or Hirzebruch index Theorem. If we combine (3.4) and Durfee's formula for signature (Theorem 1.5 of [4]), we obtain Laufer's formula for μ in [14]. Since the proof of Durfee's formula does not depend on the Riemann-Roch Theorem, we derive Laufer's formula without using the Riemann-Roch Theorem.

Proof. We will use the notations in the proof of Theorem B. By (2.5), we have

$$\sigma=\chi^1(\bar{V}_0)+2\chi^0(\bar{V}_0)-\chi^1(\bar{M})-2\chi^0(\bar{V}_0)+\sigma(M). \tag{3.5}$$

Because of (2.6), Corollary 2.8 and Lemma 2.1, (3.5) becomes

$$\sigma=-2\chi^0(M)-\chi^1(M)-\dim\operatorname{coker}\pi_1^*-\dim\operatorname{coker}\pi_2'^*+\sigma(M) \tag{3.6}$$

where

$$\pi_1^*:\ \Gamma(V_0,\tilde{\Omega}^1)\to\Gamma(M,\Omega^1)$$

and

$$\pi_2'^*:\ \Gamma(\bar{V}_0,\tilde{\Omega}^1)\to\Gamma(\bar{M},\Omega^1).$$

Let $A=\bigcup A_i$, $1\leqq i\leqq s$ be the decomposition of A into irreducible components. By Grauert-Mumford criterion, the intersection matrix $[A_i\cdot A_j]$ is negative definite. Hence $\sigma(M)=-s$. So the inequality (3.2) follows from (3.6). Recall that in [1], we prove that

$$1+\mu=-2\chi^0(M)+\chi^1(M)+\dim\operatorname{coker}\pi_1^*+\dim\operatorname{coker}\pi_2'^*+\chi_T(A). \tag{3.6$'$}$$

By Combining (3.6) and (3.6)′, we get (3.4). Q.E.D.

Theorem 3.2. *Let the notations be as Theorem* 3.1. *Let K be the canonical divisor on M. Then*

$$\sigma=-K^2-s-8\dim H^1(M,\mathcal{O}). \tag{3.7}$$

Remark. By combining (3.4) and (3.7), we obtain Durfee's formula for signature. Since (3.7) is just a simple application of Noether's formula and Hirzebruch index formula, the Durfee's method of singature defect can be completely avoided.

Proof. We will use the notations in the proof of Theorem B. By (2.1),

$$\sigma=\sigma_{\bar{V}_t}-\sigma_{\bar{M}}-s. \tag{3.8}$$

$\omega=\dfrac{dx\wedge dy}{f_z}=\dfrac{dy\wedge dz}{f_x}=\dfrac{dz\wedge dx}{f_y}$ is a non-zero holomorphic 2-form on V_t, $t\neq 0$, and on $V\setminus\{0\}$. Let $K_{\infty,t}$ be the part of the divisor of ω on $\bar{V}_t$ which is supported on $\bar{V}_t\setminus V_t$, for t small. $K_{\infty,t}\cdot K_{\infty,t}$ is independent of t since the family $\{\bar{V}_t\}$ is differentiably trivial away from $(0,0,0)\in\mathbf{C}^3$. Let $K_\infty\cdot K_\infty$ denote this constant value for $K_{\infty,t}\cdot K_{\infty,t}$. Noether's formula says

$$\chi^0(\bar{M})=\tfrac{1}{12}(K\cdot K+K_\infty\cdot K_\infty+\chi_T(\bar{M})), \tag{3.9}$$

$$\chi^0(\bar{V}_t =\tfrac{1}{12}(K_\infty\cdot K_\infty+\chi_T(\bar{V}_t)), \quad t\neq 0. \tag{3.10}$$

Hirzebruch index theorem says

$$\sigma_{\bar{V}_t}=\tfrac{1}{3}(K_\infty\cdot K_\infty-2\chi_T(\bar{V}_t)), \quad t\neq 0, \tag{3.11}$$

$$\sigma_{\bar{M}}=\tfrac{1}{3}(K\cdot K+K_\infty\cdot K_\infty-2\chi_T(\bar{M})). \tag{3.12}$$

(3.7) follows from (3.8), (3.9), (3.10), (3.11), (3.12) and (2.6). Q.E.D.

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