Two Theorems on Higher Dimensional Singularities

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§ 1. Introduction

Rational singularities of surfaces have been studied by M. Artin [2], Brieskorn [4], DuVal [6], Laufer [11], Lipman [12], and Tyurina [14]. Recently Burns [5] had a natural generalization of Artin's rational surface singularities to higher dimension. Let us recall the definition.

Definition 1.1. Let \( p \) be a normal isolated singularity in the \( n \)-dimensional analytic space \( V \) and let \( \pi : M \to V \) be a resolution of \( V \). \( p \) is a rational singularity if \( R^n\pi_*(\mathcal{O}) \), the \( i \)-th direct image sheaf of the structure sheaf \( \mathcal{O} \) on \( M \), vanishes near \( p \) for \( i > 0 \).

Equivalently, since \( W_{\pi}(\mathcal{O}) = 0 \) at the regular points of \( V \), \( p \) is rational if \( \text{dir lim} H^i(\pi^{-1}(U), \mathcal{O}) = 0 \) for \( i > 0 \), where \( U \) runs over a fundamental set of neighborhoods of \( p \) in \( V \). Several examples were found of such singularities by Burns [5], especially the Arnold singularities of [1], and all quotient singularities.

The condition for \( p \) to be rational is in fact independent of the choice of the resolution by [9], Corollary 2, p. 153. In this paper the following two theorems are proved.

Theorem A. Let \( p \) be a normal \( n \)-dimensional isolated singularity of \( V \). Let \( N \subset V \) have an admissible representation \( \varphi : N \to \Delta \) with \( \varphi(p) = 0 \). Let \( U = \varphi^{-1}(|z_1|^2 + \ldots + |z_n|^2 < r) \) with \( r \) chosen sufficiently small so that \( \varphi \) is the only singularity of \( V \) in \( U \). Then \( \dim R^{n-1}\pi_*(\mathcal{O})_p = \dim \Gamma(U - p, \mathcal{O}^n)/L^2(U - p) \). (Here \( L^2(U - p) \) denotes the set of all \( n \) square integrable holomorphic \( n \)-forms on \( U - p \), see p. 601 of [11].) In particular, if \( \Gamma \) is a Gorenstein singularity, i.e. there exists a holomorphic \( n \)-form \( \omega \) defined on a deleted neighborhood of \( p \) in \( V \), which is nowhere vanishing on this neighborhood, then \( p \) is rational if and only if \( R^{n-1}\pi_*(\mathcal{O}) = 0 \), i.e. \( R^n\pi_*(\mathcal{O}) = 0 \) if and only if \( R^i\pi_*(\mathcal{O}) = 0 \) for all \( i > 0 \).

The last statement of Theorem A is a trivial consequence of the work of D. Burns [5] and the first part of Theorem A.

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Theorem B. Let \( f(z_1, \ldots, z_m, z_{n+1}) = z_{n+1}^{m+1} + a_1(z_1, \ldots, z_m)z_{n+1}^{m-1} + \ldots + a_m(z_1, \ldots, z_m) \) be holomorphic near \((0, \ldots, 0, 0)\). Let \( d_i \) be the order of the zero of \( a_i(z_1, \ldots, z_m) \) at \((0, \ldots, 0)\), \( d_i \geq 1 \). Let \( d = \min(d_i/j), 1 \leq i \leq m \). Suppose that

\[
V = \{z_1, \ldots, z_m, z_{n+1} : f(z_1, \ldots, z_m, z_{n+1}) = 0\}
\]
defined in a suitably small polydisc, has \( p = (0, \ldots, 0, 0) \) as its only singularity. Let \( \pi : M \rightarrow V \) be resolution of \( V \). Then \( \dim H^{n-1}(M, O) > (m-1)d - n \).

For \( n = 2 \), Theorem A was proved by Laufer. As a consequence of this, he gave a necessary and sufficient condition for a surface singularity to be rational that does not involve a priori knowledge of what a resolution of \( p \) looks like. The proof used there cannot be generalized to higher dimension because \( C^\infty(0, n-2) \) from \( \theta \) such that \( \bar{\partial}\theta = 0 \) does not imply that \( \theta \) is a holomorphic \((0, n-2)\) form unless \( n = 2 \). Instead, the proof of Theorem A depends heavily on the properties of analytic cover. Theorem B and its proof are generalizations of Laufer's theorem [10] in the two-dimensional case.

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§ 2. Proof of Theorem A and Theorem B

2.1. Proof of Theorem A. By Artin's algebraization theorem [3], we may assume for a local question near \( p \in V \), that \( V \) is a normal, affine algebraic variety. As such, by [9], we may assume that we have a resolution \( \pi : M \rightarrow V \) so that \( M \) is Zariski open in a smooth projective variety. In this situation, we can apply the result of Grauert and Riemenschneider [7], that \( R^i\pi_*O^n = 0 \) for \( i > 0 \). Let \( R = \pi^{-1}(U) \) and \( A = \pi^{-1}(p) \). If \( T \) is a strictly Levi pseudoconvex neighborhood of \( A \) (which may be obtained by letting \( T = \pi^{-1}(B \cap N) \) where \( B = \{z_1^2 + \ldots + z_j^2 < \epsilon\} \), assuming \( N \) is embedded locally in \( \mathbb{C}^n \)), \( H^{n-1}(T, \mathcal{O}) \) is finite dimensional by [8, Theorem IX, B. 6, pp. 268–270]. The restriction map \( H^{n-1}(R, \mathcal{O}) \rightarrow H^{n-1}(T, \mathcal{O}) \) is an isomorphism by Lemma 3.1 of [11]. Hence \( H^{n-1}(R, \mathcal{O}) \) is finite dimensional. So by Serre duality \( \dim H^{n-1}(R, \mathcal{O}) = \dim H^1(R, \mathcal{O}^n) \). (2.2) of [11] gives the exact sequence

\[
0 \rightarrow \Gamma_*(R, \mathcal{O}^n) \rightarrow \Gamma(R, \mathcal{O}^n) \rightarrow \Gamma_*(R, \mathcal{O}^n) \rightarrow \cdots
\]

Thus \( \Gamma_*(R, \mathcal{O}^n) = 0 \) and \( H^1(R, \mathcal{O}^n) = 0 \) by Lemma 3.1 of [11]. Thus \( H^1_{\mathcal{O}}(R, \mathcal{O}^n) \approx H_{\mathcal{O}}(R, \mathcal{O}^n) \). \( \Gamma(R, \mathcal{O}^n) = L^2(U - p) \) by Theorem 3.1 of [11], so to finish the proof we need that \( \Gamma_{\mathcal{O}}(R, \mathcal{O}^n) \approx \Gamma(R - A, \mathcal{O}^n) \approx L^2(U - p, \mathcal{O}^n) \). By Proposition 2.1 of [11], we must show that every holomorphic \( n \)-form defined near the boundary of \( U \) has an analytic continuation to \( U - p \). It suffices to show that a form defined on \( q^{-1}(r - \epsilon' < |z_1|^2 + \ldots + |z_n|^2 < r) \) extends to a form defined on \( q^{-1}(r - \epsilon' < |z_1|^2 + \ldots + |z_n|^2 < r) \) for some \( \epsilon' > \epsilon \). Then, choosing the greatest such \( \epsilon' \) we can extend the form to \( q^{-1}(0 < |z_1|^2 + \ldots + |z_n|^2 < r) = U - p \).
Let \( s \in \{ r - \varepsilon = |z_1|^2 + \ldots + |z_n|^2 \} \). Let \( L \) be a real hyperplane through \( s \) which is tangent to the sphere \( r - \varepsilon = |z_1|^2 + \ldots + |z_n|^2 \). For suitable \( \zeta \) a linear combination of \( z_1, \ldots, z_n \), \( L = \{ |e^\zeta| = \tau \} \) for some \( \tau > 1 \). Let

\[
D = q^{-1} \{ (z_1, z_2, \ldots, z_n) : r - \varepsilon < |z_1|^2 + \ldots + |z_n|^2 < r - \varepsilon + \eta, |e^\zeta| > \tau - \eta \}
\]

and let

\[
E = q^{-1} \{ (z_1, z_2, \ldots, z_n) : |z_1|^2 + \ldots + |z_n|^2 < r - \varepsilon + \eta, |e^\zeta| > \tau - \eta \}
\]

where \( \eta \) is chosen small enough so that each component of \( E \) may be covered by a single coordinate system. To finish the proof of the theorem it now suffices to show that every holomorphic function on \( D \) can be extended to \( E \). For since \( E \) has a single coordinate system \( \mathcal{O} \approx \mathcal{O}^* \) so that holomorphic forms must extend from \( D \) to \( E \). Using the compactness of \( r - \varepsilon = |z_1|^2 + \ldots + |z_n|^2 \), we see that we may extend forms to \( q^{-1} (r - \varepsilon' < |z_1|^2 + \ldots + |z_n|^2 < r) \) for some \( \varepsilon' > \varepsilon \) provided that extensions using different \( E \)'s agree. But if \( E \) and \( E' \) correspond to \( s \) and \( s' \), \( q(E \cap E') \) is connected and meets \( r - \varepsilon < |z_1|^2 + \ldots + |z_n|^2 < r \) so that each component of \( E \cap E' \) meets \( q^{-1} (r - \varepsilon < |z_1|^2 + \ldots + |z_n|^2 < r) \). Then by the identity theorem the extensions using \( E \) and \( E' \) agree.

Let \( h \) be a holomorphic function on \( D \). We want to prove that \( h \) can be extended to \( E \). By shrinking \( D \) a little bit if necessary, we may assume that \( h \) is bounded on \( D \). Since \( q \) is proper, \( E \) is a Stein manifold. Let \( E_1 \) be a connected component of \( E \). Since \( q(E) \) is connected, \( q(E_1) = q(E) \). It follows that \( q : E_1 \to q(E) \) proper and onto. By [8, Theorem III, B. 7, p. 103], \( q : E_1 \to q(E) \) is an analytic cover. As \( q(D) \) is connected, \( q(E_1 \cap D) = q(D) \). So \( q : E_1 \cap D \to q(D) \) is also an analytic cover. By [8, Theorem III, B. 14, p. 105], \( h/E_1 \cap D \) satisfies a polynomial equation

\[
h^\alpha + \sum (a_i \circ q) h^i = 0 \quad \text{on} \quad E_1 \cap D
\]

with \( a_i \in \mathcal{O}_q(D) \). By Hartog's theorem, the \( a_i \) have unique holomorphic extensions \( a_i \in \mathcal{O}_q(E) \). Let \( V \) be the subvariety of \( E_1 \times \mathbb{C} \) defined as follows:

\[
V = \{ (p, z) : p \in E_1, \quad z^\lambda + \sum (a_i \circ q(p) z^i = 0) \}.
\]

By shrinking \( q(E) \) a little bit, we may assume that the roots of this polynomial are bounded, so that the natural projection \( \pi_1 : V \to E_1 \) is proper. Let \( \varphi(p) = (p, h(p)) \) for \( p \in E_1 \cap D \). Then \( \varphi : E_1 \cap D \to V \).

We claim that \( E_1 \cap D \) is connected. We may write \( E_1 \cap D = \bigcup_{a \in A} \cup \_a \) where \( \_a \) are connected components on \( E_1 \cap D \). If \( E_1 \cap D \) is disconnected, then \( A \) has at least two distinct indices. Since \( q(D) \) is connected and \( q : E_1 \cap D \to q(D) \) proper, it follows that \( q(\cup_a \_a) = q(D) \). Hence \( q : \cup_a \_a \to q(D) \) is also an analytic cover. Let \( \alpha, \beta \) be two different elements in \( A \). Let \( x_0 \in q(D) \), so that \( q^{-1}(x_0) \cap \cup_a \_a \) has \( \lambda_a \) distinct points \( p_1, \ldots, p_{\lambda_a} \) and \( q^{-1}(x_0) \cap \cup_\beta \_\beta \) has \( \lambda_\beta \) distinct points \( p_{\lambda_a + 1}, \ldots, p_{\lambda_a + \lambda_\beta} \). Let \( f \in \mathcal{O}_{E_1} \) be such that \( f(p_i) = i \). By [8, Theorem III, B. 14, p. 105], there is a polynomial

\[
P(X) = X^{\lambda_a} + \sum a_i X^i, \quad a_i \in \mathcal{O}_q(D)
\]

such that \( P(f) \equiv 0 \) in \( \_a \). Now by Hartog's theorem the \( a_i \) have unique holomorphic extensions \( a_i \in \mathcal{O}_q(E) \) and then \( P(f) = f^{\lambda_a} + \sum (a_i \circ q) f^i \in \mathcal{O}_{E_1} \). But \( P(f) \) is zero on an open set of \( E_1 \), so since \( E_1 \) is connected, \( P(f) \equiv 0 \) on all of \( E_1 \). But then \( X^{\lambda_a} = \ldots \)
+ \sum a_i(x_0)X^i \text{ has at least } \lambda_\alpha + \lambda_\beta \text{ roots, namely the values of } f \text{ at all points in } p_{11}, \ldots, p_{14} + \alpha, \text{ a contradiction of the fact that } \lambda_\beta > 0.

Let \( V' \) be the irreducible branch of \( V \) which contains \( \varphi(E_1 \cap D) \). Since \( \varphi \circ \pi_1 : V \to g(E) \) is proper, so \( \varphi \circ \pi_1 : V' \to g(E) \) is proper. Observe that \( V' \) is an irreducible Stein space. Thus as in the previous argument \( \varphi^{-1}(\varphi(D)) \cap V' = \pi^{-1}_1(D \cap E_1) \cap V' \) is connected. But \( \varphi(D \cap E_1) \) is both open and closed in \( \pi^{-1}(D \cap E_1) \cap V' \), so \( \varphi(D \cap E_1) = \pi^{-1}_1(D \cap E_1) \cap V' \). Since \( \pi_1 : V' \to E_1 \) is proper, the set \( S = \pi(S(V')) \cap \{ x \in R(V') : \text{rank}_x \pi < n \} \) is negligible in \( E_1 \). [Here we use the usual convention: \( s(V') \) denotes the set of singular points of \( V' \) and \( R(V') \) denotes the set of regular points of \( V' \)], and \( \pi : V' \to \pi^{-1}(S) \to E_1 - S \) is a \( \lambda \)-sheeted covering map for some \( \lambda \). Since \( \lambda = 1 \) over \( E_1 \cap D, \lambda = 1 \) over \( E_1 - S \). Thus every \( x \in E_1 - S \) has only one inverse image point in \( V' \). For such \( x \), define \( H(x) = z \) where \( \pi^{-1}(x) \cap V' = \{ x, z \} \). \( H \) is locally bounded and holomorphic on \( E_1 - S \); and hence in \( E_1 \) by the Riemann extension theorem). Further, \( H = h \) on \( E_1 \), so \( H \) is an extension of \( h \) to all of \( E_1 \). Q.E.D.

2.2. Proof of Theorem B. On \( V \), \( f(z_1, \ldots, z_n, z_{n+1}) \equiv 0 \) so
\[
\frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial z_2} dz_2 + \ldots + \frac{\partial f}{\partial z_n} dz_n + \frac{\partial f}{\partial z_{n+1}} dz_{n+1} = 0.
\]
Thus
\[
\omega = \frac{dz_2 \wedge \ldots \wedge dz_{n+1}}{\frac{\partial f}{\partial z_1}} = \frac{dz_3 \wedge \ldots \wedge dz_{n+1} + dz_1}{\frac{\partial f}{\partial z_2}} = \ldots = \frac{dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n}{\frac{\partial f}{\partial z_{n+1}}}
\]
where \( \sigma_i \) are either \(+1\) or \(-1\). Since \( p \) is an isolated singularity, \( \frac{\partial f}{\partial z_1} = \ldots = \frac{\partial f}{\partial z_n} = \frac{\partial f}{\partial z_{n+1}} = 0 \) occurs only at \( p \). Hence on \( V - p \), \( \omega \) is a nowhere vanishing holomorphic \( n \)-form. Holomorphic \( n \)-forms \( \lambda \) on \( V - p \) extend to \( M \) if and only if \( \int \lambda \wedge \bar{\lambda} \) is finite on \( V - p \) near \( p \) [11, p. 603, Theorem 3.1].

Let \( D = \dim H^{n-1}(M, \mathcal{O}) = \dim H^0(M - A, \Omega^n)/H^0(M, \Omega^n) \). Let \( a_1, \ldots, a_n \) be integers such that \( 0 \leq a_i \leq D \). Then some nonzero linear combination \( \lambda \) with complex coefficients of
\[
z_1^{a_1}z_2^{a_2}\ldots z_n^{a_n} \sum_{i=1}^{n} a_i \leq D
\]
lies on \( H^0(M, \Omega^n) \). So for some polynomial \( g(z_1, \ldots, z_n), g(0, \ldots, 0) \neq 0, \lambda = z_1^{b_1}z_2^{b_2}\ldots z_n^{b_n}g(z_1, \ldots, z_n) \omega \in H^0(M, \Omega^n) \) with \( z_1^{b_1}z_2^{b_2}\ldots z_n^{b_n} \omega \) in (2.1). Also, \( z_1^{a_1}\ldots z_n^{a_n} \omega \in H^0(M, \Omega^n) \) where \( \sum_{i=1}^{n} a_i = D \).

\( \varphi : (z_1, \ldots, z_n, z_{n+1}) \to (z_1, \ldots, z_n) \) express \( V \) as \( m \)-sheeted branched cover over the \( (z_1, \ldots, z_n) \)-plane. The branch locus has zero Lebesgue measure. Let \( da \) be Lebesgue measure on the \( (z_1, \ldots, z_n) \)-plane and let \( \int \) denote integration near \( (0, \ldots, 0) \). Let
\( \varphi^{-1}(z_1, \ldots, z_n) = \{(z_1, \ldots, z_m, z_{n+1,i}) \mid 1 \leq i \leq m\} \). Since \( z_1^{a_1} \cdots z_n^{a_n} \omega \in H^0(M, \Omega^n) \) where \( \sum_{i=1}^n a_i = D \).

\[
\int \sum_{i=1}^m \left| \frac{\partial f}{\partial z_{n+1}}(z_1, \ldots, z_m, z_{n+1,i}) \right|^2 dA < \infty. \tag{2.2}
\]

Let \( r^2 = |z_1|^2 + \cdots + |z_n|^2 \). For some constant \( C \) and for some suitable small neighborhood of \((0, \ldots, 0)\), \( |z_{n+1,i}(z_1, \ldots, z_n)| < Cr^d 1 \leq i \leq m \). Thus for some constant \( K \)

\[
\left| \frac{\partial f}{\partial z_{n+1}}(z_1, \ldots, z_{n+1}, i) \right| < Kr^{(n-1)d}. \tag{2.3}
\]

Since \( r^{2D} = \sum_{\alpha_1 + \cdots + \alpha_n = D} D! \alpha_1! \cdots \alpha_n! |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n}, 0 \leq \alpha_i \leq D, 1 \leq i \leq n \), from (2.2) and (2.3),

\[
\int \frac{r^{2D}}{r^{2(n-1)d}} dA < \infty.
\]

Hence \( 2(n-1) - d - 2D < 2n \). Q.E.D.

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