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Two Theorems on Higher Dimensional Singularities

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§ 1. Introduction

Rational singularities of surfaces have been studied by M. Artin [2], Brieskorn [4], DuVal [6], Laufer [11], Lipman [12], and Tyurina [14]. Recently Burns [5] had a natural generalization of Artin's ration surface singularities to higher dimension. Let us recall the definition.

Definition 1.1. Let p be a normal isolated singularity in the n-dimensional analytic space V and let $\pi : M \to V$ be a resolution of V. p is a rational singularity if $R^i\pi_*(\mathcal{O})$, the i-th direct image sheaf of the structure sheaf $\mathcal{O}$ on M, vanishes near p for $i>0$.

Equivalently, since $R^i\pi_*(\mathcal{O})=0$ at the regular points of V, p is rational if $\operatorname{dir\,lim} H^i(\pi^{-1}(U), \mathcal{O})=0$ for $i>0$, where U runs over a fundamental set of neighborhoods of p in V. Several examples were found of such singularities by Burns [5], especially the Arnold singularities of [1], and all quotient singularities.

The condition for p to be rational is in fact independent of the choice of the resolution by [9], Corollary 2, p. 153. In this paper the following two theorems are proved.

Theorem A. *Let p be a normal n-dimensional isolated singularity of V. Let $N \subset V$ have an admissible representation $\varrho : N \to \Delta$ with $\varrho(p)=0$. Let $U=\varrho^{-1}(|z_1|^2+\ldots+|z_n|^2<r)$ with r chosen sufficiently small so that ϱ is the only singularity of V in U. Then* $\dim R^{n-1}\pi_*(\mathcal{O})_p = \dim \Gamma(U-p, \Omega^n)/L^2(U-p)$. *(Here $L^2(U-p)$ denotes the set of all square integrable holomorphic n forms on $U-p$, see* p. 601 *of* [11].*) In particular, if Γ is a Gorenstein singularity, i.e. there exists a holomorphic n-form ω defined on a deleted neighborhood of p in V, which is nowhere vanishing on this neighborhood, then p is rational if and only if $R^{n-1}\pi_*(\mathcal{O})=0$, i.e. $R^{n-1}\pi_*(\mathcal{O})=0$ if and only if $R^i\pi_*(\mathcal{O})=0$ for all $i>0$.*

The last statement of Theorem A is a trivial consequence of the work of D. Burns [5] and the first part of Theorem A.

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Theorem B. *Let* $f(z_1, \ldots, z_n, z_{n+1}) = z_{n+1}^m + a_1(z_1, \ldots, z_n) z_{n+1}^{m-1} + \ldots + a_m(z_1, \ldots, z_n)$ *be holomorphic near* $(0, \ldots, 0, 0)$. *Let* d_i *be the order of the zero of* $a_i(z_1, \ldots, z_n)$ *at* $(0, \ldots, 0)$, $d_i \geqq 1$. *Let* $d = \min(d_i/i)$, $1 \leqq i \leqq m$. *Suppose that*

$$V = \{z_1, \ldots, z_n, z_{n+1}) : f(z_1, \ldots, z_n, z_{n+1}) = 0\}$$

defined in a suitably small polydisc, has $p = (0, \ldots, 0, 0)$ *as its only singularity. Let* $\pi : M \to V$ *be resolution of* V. *Then* $\dim H^{n-1}(M, \mathcal{O}) > (m-1)d - n$.

For $n = 2$, Theorem A was proved by Laufer. As a consequence of this, he gave a necessary and sufficient condition for a surface singularity to be rational that does not involve a priori knowledge of what a resolution of p looks like. The proof used there cannot be generalized to higher dimension because $C^\infty(0, n-2)$ from θ such that $\bar{\partial}\theta = 0$ does not imply that θ is a holomorphic $(0, n-2)$ form unless $n = 2$. Instead, the proof of Theorem A depends heavily on the properties of analytic cover. Theorem B and its proof are generalizations of Laufer's theorem [10] in the two-dimensional case.

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§ 2. Proof of Theorem A and Theorem B

2.1. Proof of Theorem A. By Artin's algebraization theorem [3], we may assume for a local question near $p \in V$, that V is a normal, affine algebraic variety. As such, by [9], we may assume that we have a resolution $\pi : M \to V$ so that M is Zariski open in a smooth projective variety. In this situation, we can apply the result of Grauert and Riemenschneider [7], that $R^i \pi_* \Omega^n = 0$ for $i > 0$. Let $R = \pi^{-1}(U)$ and $A = \pi^{-1}(p)$. If T is a strictly Levi pseudoconvex neighborhood of A (which may be obtained by letting $T = \pi^{-1}(B \cap N)$ where $B = \{|z|^2 + \ldots + |z_n|^2 < \varepsilon\}$, assuming N is embedded locally in $\mathbb{C}^n$), $H^{n-1}(T, \mathcal{O})$ is finite dimensional by [8, Theorem IX, B. 6, pp. 268—270]. The restriction map $H^{n-1}(R, \mathcal{O}) \to H^{n-1}(T, \mathcal{O})$ is an isomorphism by Lemma 3.1 of [11]. Hence $H^{n-1}(R, \mathcal{O})$ is finite dimensional. So by Serre duality $\dim H^{n-1}(R, \mathcal{O}) = \dim H_*^1(R, \Omega^n)$. (2.2) of [11] gives the exact sequence

$$0 \to \Gamma_*(R, \Omega^n) \to \Gamma(R, \Omega^n) \to \Gamma_\infty(R, \Omega^n) \to H_*^1(R, \Omega^n) \to H^1(R, \Omega^n) \to \ldots .$$

$\Gamma_*(R, \Omega^n) = 0$ and $H^1(R, \Omega^n) = 0$ by Lemma 3.1 of [11]. Thus $H_*^1(R, \Omega^n) \approx \Gamma_\infty(R, \Omega^n)/\Gamma(R, \Omega^n)$. $\Gamma(R, \Omega^n) = L^2(U-p)$ by Theorem 3.1 of [11], so to finish the proof we need that $\Gamma_\infty(R, \Omega^n) \approx \Gamma(R-A, \Omega^n) \approx \Gamma(U-p, \Omega^n)$. By Proposition 2.1 of [11], we must show that every holomorphic n-form defined near the boundary of U has an analytic continuation to $U-p$. It suffices to show that a form defined on $\varrho^{-1}(r-\varepsilon < |z_1|^2 + \ldots + |z_n|^2 < r)$ extends to a form defined on $\varrho^{-1}(r-\varepsilon' < |z_1|^2 + \ldots + |z_n|^2 < r)$ for some $\varepsilon' > \varepsilon$. Then, choosing the greatest such ε' we can extend the form to $\varrho^{-1}(0 < |z_1|^2 + \ldots + |z_n|^2 < r) = U-p$.

Let $s\in\{r-\varepsilon=|z_1|^2+\ldots+|z_n|^2\}$. Let L be a real hyperplane through s which is tangent to the sphere $r-\varepsilon=|z_1|^2+\ldots+|z_n|^2$. For suitable ζ a linear combination of $z_1,\ldots,z_n$, $L=\{|e^\zeta|=\tau\}$ for some $\tau>1$. Let

$$D=\varrho^{-1}\{(z_1,z_2,\ldots,z_n):r-\varepsilon<|z_1|^2+\ldots+|z_n|^2<r-\varepsilon+\eta,|e^\zeta|>\tau-\eta\}$$

and let

$$E=\varrho^{-1}\{(z_1,z_2,\ldots,z_n):|z_1|^2+\ldots+|z_n|^2<r-\varepsilon+\eta,|e^\zeta|>\tau-\eta\}$$

where η is chosen small enough so that each component of E may be covered by a single coordinate system. To finish the proof of the theorem it now suffices to show that every holomorphic function on D can be extended to E. For since E has a single coordinate system $\mathcal{O}\approx\Omega^n$ so that holomorphic forms must extend from D to E. Using the compactness of $r-\varepsilon=|z_1|^2+\ldots+|z_n|^2$, we see that we may extend forms to $\varrho^{-1}(r-\varepsilon'<|z_1|^2+\ldots+|z_n|^2<r)$ for some $\varepsilon'>\varepsilon$ provided that extensions using different E's agree. But if E and E' correspond to s and s', $\varrho(E\cap E')$ is connected and meets $r-\varepsilon<|z_1|^2+\ldots+|z_n|^2<r$ so that each component of $E\cap E'$ meets $\varrho^{-1}(r-\varepsilon<|z_1|^2+\ldots+|z_n|^2<r)$. Then by the identity theorem the extensions using E and E' agree.

Let h be a holomorphic function on D. We want to prove that h can be extended to E. By shrinking D a little bit if necessary, we may assume that h is bounded on D. Since ϱ is proper, E is a Stein manifold. Let E_1 be a connected component of E. Since $\varrho(E)$ is connected, $\varrho(E_1)=\varrho(E)$. It follows that $\varrho:E_1\to\varrho(E)$ is proper and onto. By [8, Theorem III, B. 7, p. 103], $\varrho:E_1\to\varrho(E)$ is an analytic cover. As $\varrho(D)$ is connected, $\varrho(E_1\cap D)=\varrho(D)$. So $\varrho:E_1\cap D\to\varrho(D)$ is also an analytic cover. By [8, Theorem III, B. 14, p. 105], $h/E_1\cap D$ satisfies a polynomial equation

$$h^\lambda+\sum(a_i\circ\varrho)h^i=0 \quad \text{on} \quad E_1\cap D$$

with $a_i\in\mathcal{O}_{\varrho(D)}$. By Hartog's theorem, the a_i have unique holomorphic extensions $a_i\in\mathcal{O}_{\varrho(E)}$. Let V be the subvariety of $E_1\times\mathbb{C}$ defined as follows:

$$V=\{(p,z):p\in E_1,\ z^\lambda+\sum(a_i\circ\varrho(p)z^i=0\}.$$

By shrinking $\varrho(E)$ a little bit, we may assume that the roots of this polynomial are bounded, so the natural projection $\pi_1:V\to E_1$ is proper. Let $\varphi(p)=(p,h(p))$ for $p\in E_1\cap D$. Then $\varphi:E_1\cap D\to V$.

We claim that $E_1\cap D$ is connected. We may write $E_1\cap D=\bigcup\limits_{\alpha\in\Lambda}\cup_\alpha$ where $\cup_\alpha$ are connected components on $E_1\cap D$. If $E_1\cap D$ is disconnected, then Λ has at least two distinct indices. Since $\varrho(D)$ is connected and $\varrho:E_1\cap D\to\varrho(D)$ is proper, it follows that $\varrho(\cup_\alpha)=\varrho(D)$. Hence $\varrho:\cup_\alpha\to\varrho(D)$ is also an analytic cover. Let α,β be two distinct elements in Λ. Let $x_0\in\varrho(D)$, so that $\varrho^{-1}(x_0)\cap\cup_\alpha$ has λ_α distinct points $p_1,\ldots,p_{\lambda_\alpha}$ and $\varrho^{-1}(x_0)\cap\cup_\beta$ has λ_β distinct points $p_{\lambda_\alpha+1},\ldots,p_{\lambda_\alpha+\lambda_\beta}$. Let $f\in\mathcal{O}_{E_1}$ be such that $f(p_i)=i$. By [8, Theorem III, B. 14, p. 105], there is a polynomial.

$$P(X)=X^{\lambda_\alpha}+\sum a_iX^i, \qquad a_i\in\mathcal{O}_{\varrho(D)}$$

such that $P(f)\equiv0$ in $\cup_\alpha$. Now by Hartog's theorem the a_i have unique holomorphic extensions $a_i\in\mathcal{O}_{\varrho(E)}$ and then $P(f)=f^{\lambda_\alpha}+\sum(a_i\circ\varrho)f^i\in\mathcal{O}_{E_1}$. But $P(f)$ is zero on an open set of E_1, so since E_1 is connected, $P(f)\equiv0$ on all of E_1. But then X^{λ_α}

$+\sum a_i(x_0)X^i$ has at least $\lambda_\alpha+\lambda_\beta$ roots, namely the values of f at all points in $p_{\lambda_1},\dots,p_{\lambda_\alpha+\lambda_\beta}$, a contradiction of the fact that $\lambda_\beta>0$.

Let V' be the irreducible branch of V which contains $\varphi(E_1\cap D)$. Since $\varrho\circ\pi_1: V\to\varrho(E)$ is proper, so $\varrho\circ\pi_1: V'\to\varrho(E)$ is proper. Observe that V' is an irreducible Stein space. Thus as in the previous argument $(\varrho\circ\pi)^{-1}(\varrho(D))\cap V'=\pi_1^{-1}(D\cap E_1)\cap V'$ is connected. But $\varphi(D\cap E_1)$ is both open and closed in $\pi^{-1}(D\cap E_1)\cap V'$, so $\varphi(D\cap E_1)=\pi_1^{-1}(D\cap E_1)\cap V'$. Since $\pi_1: V'\to E_1$ is proper, the set $S=\pi(S(V'))\cup\pi\{x\in R(V'):\operatorname{rank}_x\pi<n\}$ is negligible in E_1. [Here we use the usual convention: $s(V')$ denotes the set of singular points of V' and $R(V')$ denotes the set of regular points of V'], and $\pi: V'-\pi^{-1}(S)\to E_1-S$ is a λ-sheeted covering map for some λ. Since $\lambda=1$ over $E_1\cap D$, $\lambda=1$ over E_1-S. Thus every $x\in E_1-S$ has only one inverse image point in V'. For such x, define $H(x)=z$ where $\pi^{-1}(x)\cap V'=\{x,z\}$. H is locally bounded and holomorphic on E_1-S; and hence in E_1 by the Riemann extension theorem). Further, $H=h$ on E_1, so H is an extension of h to all of E_1. Q.E.D.

2.2. Proof of Theorem B. On V, $f(z_1,\dots,z_n,z_{n+1})\equiv 0$ so

$$\frac{\partial f}{\partial z_1}dz_1+\frac{\partial f}{\partial z_2}dz_2+\dots+\frac{\partial f}{\partial z_n}dz_n+\frac{\partial f}{\partial z_{n+1}}dz_{n+1}\equiv 0.$$

Thus

$$\omega\equiv\frac{dz_2\wedge\dots\wedge dz_{n+1}}{\frac{\partial f}{\partial z_1}}=\sigma_2\frac{dz_3\wedge\dots\wedge dz_{n+1}\wedge dz_1}{\frac{\partial f}{\partial z_2}}=\dots=\sigma_{n+1}\frac{dz_1\wedge dz_2\wedge\dots\wedge dz_n}{\frac{\partial f}{\partial z_{n+1}}}$$

where σ_i are either $+1$ or -1. Since p is an isolated singularity, $\frac{\partial f}{\partial z_1}=\frac{\partial f}{\partial z_2}=\dots=\frac{\partial f}{\partial z_n}=\frac{\partial f}{\partial z_{n+1}}=0$ occurs only at p. Hence on $V-p$, ω is a nowhere vanishing holomorphic n form. Holomorphic n-forms λ on $V-p$ extend to M if and only if $\int\lambda\wedge\bar{\lambda}$ is finite on $V-p$ near p [11, p. 603, Theorem 3.1].

Let $D=\dim H^{n-1}(M,\mathcal{O})=\dim H^0(M-A,\Omega^n)/H^0(M,\Omega^n)$. Let $a_1,\dots,a_n$ be integers such that $0\leqq a_i\leqq D$. Then some nonzero linear combination λ with complex coefficients of

$$z_1^{a_1}z_2^{a_2}\dots z_n^{a_n}\omega\qquad\sum_{i=1}^{n}a_i\leqq D\tag{2.1}$$

lies on $H^0(M,\Omega^n)$. So for some polynomial $g(z_1,\dots,z_n)$, $g(0,\dots,0)\neq 0$, $\lambda=z_1^{b_1}z_2^{b_2}\dots z_n^{b_n}g(z_1,\dots,z_n)\omega\in H^0(M,\Omega^n)$ with $z_1^{b_1}z_2^{b_2}\dots z_n^{b_n}\omega$ in (2.1). Also, $z_1^{a_1}\dots z_n^{a_n}\omega\in H^0(M,\Omega^n)$ where $\sum_{i=1}^{n}a_i=D$.

$\varrho:(z_1,\dots,z_n,z_{n+1})\to(z_1,\dots,z_n)$ express V as m-sheeted branched cover over the $(z_1,\dots,z_n)$-plane. The branch locus has zero Lebesgue measure. Let dA be Lebesgue measure on the $(z_1,\dots,z_n)$-plane and let $\int$ denote integration near $(0,\dots,0)$. Let

$\varrho^{-1}(z_1, \ldots, z_n) = \{(z_1, \ldots, z_n, z_{n+1,i}), 1 \leqq i \leqq m\}$. Since $z_1^{a_1} \ldots z_n^{a_n} \omega \in H^0(M, \Omega^n)$ where $\sum_{i=1}^{n} a_i = D$.

$$\int \sum_{i=1}^{m} \frac{|z_1|^{2a_1} \ldots |z_n|^{2a_n}}{\left|\frac{\partial f}{\partial f_{n+1}}(z_1, \ldots, z_n, z_{n+1,i})\right|^2} dA < \infty. \tag{2.2}$$

Let $r^2 = |z_1|^2 + \ldots + |z_n|^2$. For some constant C and for some suitable small neighborhood of $(0, \ldots, 0)$, $|z_{n+1,i}(z_1, \ldots, z_n)| < Cr^d \, 1 \leqq i \leqq m$. Thus for some constant K

$$\left|\frac{\partial f}{\partial z_{n+1}}(z_1, \ldots, z_{n+1,i})\right| < Kr^{(n-1)d}. \tag{2.3}$$

Since $|r|^{2D} = \sum_{\alpha_1 + \ldots + \alpha_n = D} \frac{D!}{\alpha_1! \ldots \alpha_n!} |z_1|^{2\alpha_1} \ldots |z_n|^{2\alpha_n}, 0 \leqq \alpha_i \leqq D, 1 \leqq i \leqq n$, from (2.2) and (2.3),

$$\int \frac{r^{2D}}{r^{2(n-1)d}} dA < \infty.$$

Hence $2(n-1) - d - 2D < 2n$. Q.E.D.

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