Tan Jiang
Stephen S.-T. Yau

Intersection lattices and topological structures of
complements of arrangements in $\mathbb{CP}^2$


<http://www.numdam.org/item?id=ASNSP_1998_4_26_2_357_0>
1. – Introduction

An arrangement of hyperplanes $A$ is a finite collection of $\mathbb{C}$-linear subspaces of dimension $(d - 1)$ in $\mathbb{C}^d$. Associated with the arrangement $A$ is an open real $2d$-manifold the complement $M(A) = \mathbb{C}^d - \bigcup\{H : H \in A\}$. The central problem in this area is to decide to what extent the topology or differentiable structure of $M(A)$ is determined by the combinatorial geometry of $A$ and vice versa.

The theory was first initiated in 1969 by V. I. Arnol’d [1], who calculated the Poincaré polynomial of the pure braid space $M_1$ and the cohomology ring structure of $H^*(M_1)$, where $M_1$ is the complement of the complexified braid arrangement $A_1$ defined by

$$\prod_{1 \leq i < j \leq \ell} (z_i - z_j).$$

In general, for an arbitrary arrangement $A$, define holomorphic differential forms $\omega_H = \frac{1}{2\pi i} \frac{d\alpha_H}{\alpha_H}$, where $\alpha_H$ is the linear form defining the hyperplane $H$ for $h \in A$, and let $[\omega_H]$ denote the corresponding cohomology class. Let

$$R(A) = \bigoplus_{p=0}^{\ell} R_p$$

be the graded $\mathbb{C}$-algebra of holomorphic differential forms on $M(A)$ generated by the $\omega_H$ and 1. Arnol’d conjectured that the natural map $\eta : [\eta]$ of $R(A) \rightarrow H^*(M(A), \mathbb{C})$ is an isomorphism of graded algebras. This was proved by Brieskorn [2] in 1971, who showed that the $\mathbb{Z}$-subalgebra of $R(A)$ generated by the forms $\omega_H$ and 1 is isomorphic to the singular cohomology
Although Brieskorn proved the Arnol'd conjecture that $R(A)$ is isomorphic to $H^*(M(A), \mathbb{C})$ as a graded algebra for the arbitrary arrangement $\mathcal{A}$, it was not known whether the algebra $R$ is determined by the combinatorial data of $\mathcal{A}$, since the linear forms enter the definition of $R(A)$. In 1980, Orlik and Solomon [16] introduced a graded algebra $A(\mathcal{A})$ to an arbitrary arrangement $\mathcal{A}$. $A(\mathcal{A})$ is a combinatorial invariant of $\mathcal{A}$. The beautiful result of Orlik and Solomon asserts that there is an isomorphism of algebra $A(\mathcal{A}) \simeq R(\mathcal{A})$. This together with the Brieskorn's solution to the Arnol'd conjecture implies that the cohomological ring $H^*(M(\mathcal{A}), \mathbb{C})$ is a combinatorial invariant of $\mathcal{A}$.

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^3$ and let $\mathcal{A}^*$ be the corresponding arrangement of lines in $\mathbb{CP}^2$. Then we have $M(\mathcal{A}) = M(\mathcal{A}^*) \times \mathbb{C}^*$ (cf. [18]), where $M(\mathcal{A}^*) = \mathbb{CP}^2 - \cup \mathcal{A}^*$. Topology and differentiable structure of $M(\mathcal{A}^*)$ are important in the theory of hypergeometric functions (see the work of Gel'fand [8] and his subsequent papers, the work of Deligne and Mostow [3], and subsequent papers by Mostow). Moreover, they play a role in some interesting problems in algebraic geometry (see especially the works of Hirzebruch [9] and Moishezon [13]). Although the conjecture that the homotopic type of $M(\mathcal{A}^*)$ is a combinatorial invariant of the projective arrangement of $\mathcal{A}^*$ seems disproved by G. Rybnikov [20] in 1994, we have shown [11] that for a very large class of projective arrangements in $\mathbb{CP}^2$, the diffeomorphic type of $M(\mathcal{A}^*)$ is indeed a combinatorial invariant of $\mathcal{A}^*$.

**Definition.** Let $\mathcal{A}^*$ be a projective arrangement of lines in $\mathbb{CP}^2$, The set of all intersections of elements of $\mathcal{A}^*$ partially ordered by reverse inclusion is denoted as $L(\mathcal{A}^*)$.

It is natural to ask whether the combinatorial data $L(\mathcal{A}^*)$ of the projective arrangement are determined by the homotopic type, topological type, or diffeomorphic type of $M(\mathcal{A}^*)$. For the first question: Falk has written a series of papers [5], [6], and [7] on whether there are combinatorially distinct arrangements that have homotopic equivalent complements. In [6], Falk constructed two projective arrangements in $\mathbb{CP}^2$, each of which has two triple points and nine double points. The homotopic equivalence of their complements was shown in [7]. In view of this example, one would like to know whether $L(\mathcal{A}^*)$ is determined by the topological type of $M(\mathcal{A}^*)$. The following theorem answers this question affirmatively.

**Main Theorem.** Let $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ be two projective arrangements in $\mathbb{CP}^2$. If $M(\mathcal{A}_1^*)$ is homeomorphic to $M(\mathcal{A}_2^*)$, then $L(\mathcal{A}_1^*)$ is isomorphic to $L(\mathcal{A}_2^*)$.

In view of Falk's example mentioned above, we have the following corollary.

**Corollary.** There exist two projective arrangements $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ in $\mathbb{CP}^2$ such that $M(\mathcal{A}_1^*)$ and $M(\mathcal{A}_2^*)$ have the same homotopic type, but they do not have the same topological type.

In Section 2 we recall some necessary definitions and results in three-manifolds that are due to Waldhausen [22]. In Section 3 we study the boundary
of a regular neighborhood of an arrangement $A^*$ in $\mathbb{C}P^2$, using Waldhausen's theory on graphed manifolds [21]. By restricting ourselves to nonexceptional projective arrangements in $\mathbb{C}P^2$, we show in Section 4 that if two such arrangements have the same topological types, then they have the same graph structures (again in the sense of Waldhausen). In Section 5 we prove the main theorem for nonexceptional arrangements. In Section 6 and Section 7 we finish the proof of the remaining part of our main theorem for the exceptional arrangements.

The second author learned this important open problem during P. Orlik's interesting lectures at CBMS conference on arrangements at Flagstaff in 1988. The main theorem of this paper was announced in [12].

Acknowledgment. We gratefully acknowledge both referees for their careful reading of this paper and especially for providing us many useful comments.

2. – Definitions and preliminaries

In this section we recall some necessary definitions and important results on three-manifolds due to Waldhausen [22].

Throughout this section, by a manifold, we mean an orientable compact three-dimensional manifold with or without boundary.

A surface is a connected two-manifold. It is compact and orientable, unless the contrary is stated explicitly. A surface $F$ in the manifold $M$ is properly embedded (i.e., $F \cap \partial M = \partial F$, where $\partial$ denotes boundary). A surface in $\partial M$ is a submanifold of $\partial M$. A system of surfaces in $M$ or $\partial M$ consists of finitely many, mutually disjoint components of the above two types.

Let $F$ be a subspace of $M$. $U(F)$ denotes a regular neighborhood of $F$. A regular neighborhood is always compact and sufficiently small. A typical construction is as follows. Choose a finite triangulation in which $F$ is a subcomplex. The closed star of $F$ in the second barycenter subdivision of this triangulation is then a regular neighborhood of $F$.

An isotopy deformation of $M$ is a level preserving map $h : M \times I \rightarrow M \times I$, $I = [0,1]$, such that from each level $h_t : M \times I \rightarrow M \times I$, $t \in I$, such that $h|_{M \times \{t\}} = h_t$, $h_0$ is a homeomorphism from $M$ onto itself and $h_0 = \text{Identity}$. We often abbreviate “isotopy deformation” as “deformation.”

Subspaces $N_1$ and $N_2$ in $M$ are called isotopic in $M$ if there is an isotopy deformation of $M : h_t, t \in I$, such that $h_1(N_1) = N_2$.

Definition 2.1. Let $M$ be a manifold. Let $F$ be a system of surfaces in $M$ or $\partial M$. $F$ is compressible in $M$ if either one of the following two cases hold.

(a) There is a noncontractible simple closed curve $k$ in $\text{Int}(F)$, and a disk $D$ in $M$, $\text{Int}(D) \subseteq \text{Int}(M)$, such that $D \cap F = \partial D = k$.
(b) There is a ball $E$ in $M$ such that $E \cap F = \partial E$.

$F$ is incompressible in $M$ if and only if it is not compressible in $M$. 

INTERSECTION LATTICES AND TOPOLOGICAL STRUCTURES
DEFINITION 2.2. A manifold $M$ is called irreducible if every two-sphere in $M$ is compressible.

Thus $M$ is irreducible if and only if each two-sphere in $M$ bounds a three-cell in $M$. (Remember: If $M$ is irreducible, and $\partial M \neq \emptyset$, then either $M$ is a ball, or else genus($\partial M$) > 0, and hence $H_1(M)$ is infinite.)

DEFINITION 2.3. A manifold $M$ is called boundary-irreducible if $\partial M$ is incompressible.

The following lemma is a well-known corollary of the sphere theorem.

LEMMA 2.1. Suppose $M$ is irreducible and $\pi_1(M)$ is not finite. Then $M$ is aspherical, that is, $\pi_j(M) = 0$, for $j \geq 2$.

Lemma 2.2 below seems to be widely known. A proof is given in [23].

LEMMA 2.2. Let $M$ be an irreducible manifold.

(a) If $\partial M \neq \emptyset$, and $M$ is not a ball, then there exists in $M$ an incompressible surface $F$ such that $0 \neq [\partial F] \in H_1(\partial M)$.

(b) If $\partial M = \emptyset$, then there exists in $M$ an incompressible surface if and only if either $H_1(M)$ is not finite or $\pi_1(M)$ is a nontrivial free product with amalgamation (or both).

If $F$ is a separating incompressible surface in $M$, $\partial M = \emptyset$, then $\pi_1(M)$ is a nontrivial free product with amalgamation, $\pi_1(M) \simeq A \ast_C B$, where $C \simeq \pi_1(F)$, in a natural way.

DEFINITION 2.4. Let $M$ be an irreducible manifold that is not a ball. $M$ is sufficiently large if and only if there exists an incompressible surface in $M$.

REMARK 2.1. There exist irreducible manifolds with infinite fundamental group, which are not sufficiently large [23].

Let $T = T_1 \cup \cdots \cup T_n (n \geq 0)$ be a system of tori in Int$(M)$, and $U(T)$ be a regular neighborhood of $T$ in $M$.

DEFINITION 2.5. If each component of $M - \text{Int}(U(T))$ is homeomorphic to a fiber bundle with $S^1$ as fiber, then $T$ is called a graph structure of $M$. A manifold with a graph structure is called a graph manifold.

Let $T_1$ be an arbitrary fixed component of $T$. $U(T_1) \supseteq T_1$ is a component of $U(T)$. So $U(T_1)$ is homeomorphic to $T_1 \times I$. Let $T'$ and $T''$ be the boundary surfaces of $U(T_1)$. The component of $M - \text{Int}(U(T))$ which is pasted along $T'$ (respectively, $T''$) is denoted by $M_1$ (respectively, $M_2$). We can compare the homology class of curves in $T'$ and $T''$ by the natural isomorphism

$$H_1(T') \leftrightarrow H_1(U(T_1)) \leftrightarrow H_1(T'').$$

Hence we can talk about intersection of homology classes of curves on $T'$ and $T''$. 

DEFINITION 2.6. A graph structure \( T \) of a manifold \( M \) is called reduced if none of the following ten situations occur. A manifold with a reduced graph structure is called a reduced graph manifold.

(W1) \( M_1 \neq M_2 \) and \( M_1 \) is a \( S^1 \)-bundle over the annulus.
(W2) A fiber of \( M_1 \) is homologous in \( T_1 \) to a fiber of \( M_2 \).
(W3) \( M_1 = S^1 \times D^2 \) \( (D^2 = 2\text{-cell}) \) is a solid torus, and a meridian curve \( \{ p \} \times S^1 \subseteq \partial M_1 \) has intersection number \( \{ \pm 1 \} \) with a fiber of \( M_2 \) in \( T_1 \).
(W4) \( M_1 \simeq S^1 \times D^2 \) \( (D^2 = 2\text{-cell}) \) is a solid torus and a meridian curve \( \{ p \} \times S^1 \subseteq \partial M_1 \) is homologous to a fiber of \( M_2 \) in \( T_1 \).
(W5) \( M_1 \) is a \( S^1 \)-bundle over the Möbius band, and the homology class \( \mu_1 \) in \( \partial M_1 = T' \) of the boundary of a section of \( M_1 \) is homologous to a fiber of \( M_2 \) in \( T_1 \).
(W6) \( M_1 \) and \( M_2 \) are \( S^1 \)-bundles over the Möbius band, and \( \mu_1 \) is homologous to \( \mu_2 \), where \( \mu_4 \) is defined as in (W5).
(W7) \( M - \text{Int}(U(T)) \) has two components. One of them is a graph manifold \( Q \), defined in Section 3 of [21], which is homeomorphic to an \( S^1 \)-bundle with orientable total space over the Möbius band. The other is not a solid torus.
(W8) \( M_1 = M_2 \simeq A \times S^1 \) \( (A = \text{annulus}) \simeq I \times S^1 \times S^1 \) and the pasting map \( S^1 \times S^1 \to S^1 \times S^1 \) is given by a matrix of trace \( \pm 2 \).
(W9) \( M_1 \) and \( M_2 \) are solid tori.
(W10) \( T = \emptyset \), and \( M \) is a \( S^1 \)-bundle over \( S^2 \) or \( \mathbb{R}P^2 \) (real projective plane).

3. - The boundary of a regular neighborhood of an arrangement \( A^* \) in \( \mathbb{C}P^2 \)

Let \( A^* \) be an arrangement in \( \mathbb{C}P^2 \) and \( N(A^*) = \bigcup_{\ell \in A^*} \ell \). Suppose that \( A^* \) has \( x_1, \ldots, x_k (k \geq 0) \) as multiple intersection points (i.e., multiplicity \( t(x_i) \geq 3 \)). We blow up \( \mathbb{C}P^2 \) at \( \{ x_i \} \). We get a set \( \tilde{A}^* \) of lines that includes the proper transforms of the \( x_i \) in a blown-up surface \( \mathbb{C}P^2 \). \( \tilde{A}^* \) is called an associate arrangement in \( \mathbb{C}P^2 \) induced by \( A^* \). Suppose that \( \tilde{A}^* = \{ \ell_1, \ldots, \ell_n \} \). Each pair of lines of \( \tilde{A}^* \) intersects at most at one point. Let \( N(\tilde{A}^*) = \bigcup_{\ell_i \in \tilde{A}^*} \ell_i \), which is connected. Let \( U(\tilde{A}^*) \) be a regular neighborhood of \( N(\tilde{A}^*) \) and \( K(\tilde{A}^*) = \partial(U(\tilde{A}^*)) \). Thus \( K(\tilde{A}^*) \) is a plumbed three-manifold which is homeomorphic to \( K(A^*) \), the boundary of a regular neighborhood of \( N(A^*) \) in \( \mathbb{C}P^2 \).

\( K(\tilde{A}^*) \) can be also obtained by pasting some \( S^1 \)-bundles together. Consider the boundary of regular neighborhood of a line \( \ell_i \in \tilde{A}^* \) as a \( S^1 \)-bundle \( E_i \to \ell_i \). If two lines \( \ell_1 \) and \( \ell_2 \) of \( \tilde{A}^* \) intersect at a point \( x \), let \( D_i \) be a disc in \( \ell_i \) such that \( x \in \text{Int}(D_i) (i = 1, 2) \). Define \( E'_i := E_i|_{\ell_i - \text{Int}(D_i)} \), and glue \( E'_1 \) and \( E'_2 \) along \( E_i|_{\partial D_i} \). In other words, we choose a trivialization of \( E_i|_{\partial D_i} \) which is \( S^1 \times S^1 \). We then glue them together according to the map \( f \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : S^1 \times S^1 \to S^1 \times S^1 \), which switches the base of \( E_i|_{\partial D_i} \) with the fiber of \( E_2|_{\partial D_2} \). More generally, if
362

\[\ell_i\] intersects \(n_i\) number of times in \(\mathcal{A}^* - \{\ell_i\}\), then we consider the restriction of the \(S^1\)-bundle over \(\ell_i\) to \(n_i\)-punctured sphere. The boundary of its total space is a disjoint union of \(n_i\) tori, each of which is pasted along with another \(S^1\)-bundle. Let \(T(\mathcal{A}^*) = T_1 \cup \ldots \cup T_m\) be the disjoint union of all such tori in \(K(\mathcal{A}^*)\). Thus \(K(\mathcal{A}^*)\) is a graph manifold with graph structure \(T(\mathcal{A}^*)\).

From this graph manifold \(K(\mathcal{A}^*)\) and its graph structure \(T(\mathcal{A}^*)\), we define a weighted graph \(G(\mathcal{A}^*)\) as follows. For each \(\ell_i\) of \(\mathcal{A}^*\), one vertex \(v_i\) with weight \((n_i - n_i)\) corresponds to the self-intersection number of \(\ell_i\) in \(\mathbb{C}P^2\). Each torus in the graph structure \(T(\mathcal{A}^*)\) corresponds to an edge of \(G(\mathcal{A}^*)\). If lines \(\ell_i\) and \(\ell_j\) of \(\mathcal{A}^*\) intersect at a point, then an edge of \(G(\mathcal{A}^*)\) is defined to have \(v_i\) and \(v_j\) as its adjacent vertices. Thus \(G(\mathcal{A}^*)\) consists of \(n\) vertices \(v_1, \ldots, v_n\) and \(m\) edges \(e_1, \ldots, e_m\).

Now let us consider the case when the graph manifold \(K(\mathcal{A}^*)\) with graph structure \(T(\mathcal{A}^*)\) is irreducible. First we have the following lemma.

**Lemma 3.1.** If \(\mathcal{A}^*\) is an arrangement such that each \(\ell \in \mathcal{A}^*\) has at least three intersection points with other lines of \(\mathcal{A}^*\), then \(K(\mathcal{A}^*)\) is a reduced graph manifold with reduced graph structure \(T(\mathcal{A}^*)\).

**Proof.** Suppose that the arrangement \(\mathcal{A}^*\) in \(\mathbb{C}P^2\) satisfies the condition in Lemma 2.1. Then \(\tilde{\mathcal{A}}^*\) induced by \(\mathcal{A}^*\) satisfies the same condition, since each added exceptional line \(\mathbb{C}P^1\) from blowing up of \(N(\mathcal{A}^*)\) must intersect at least three original lines in \(\mathcal{A}^*\).

Let \(\tilde{M} = K(\tilde{\mathcal{A}}^*) - \text{Int}(U(T(\tilde{\mathcal{A}}^*)))\), where \(U(T(\tilde{\mathcal{A}}^*))\) is a regular neighborhood of \(T(\tilde{\mathcal{A}}^*)\) in \(K(\tilde{\mathcal{A}}^*)\). We can see that each component \(M_i\) of \(\tilde{M}\) corresponds to a line \(\ell_i\) in \(\tilde{\mathcal{A}}^*\) and \(M_i\) is an \(S^1\)-bundle over an \(n_i\)-punctured sphere \(B_i\) (\(n_i \geq 3\) by our assumption). So \(M_i\) is not homeomorphic to a solid torus \((S^1 \times D^2)\), an \(S^1\)-bundle over the annulus, or an \(S^1\)-bundle over the Möbius band for each \(i = 1, \ldots, n\). Thus the situations from (W1) to (W9) except (W2) are excluded. (W10) is obviously not true here. With regard to the exclusion of (W2) by looking at the glue map \(f\), one can see that fibers of \(M_1\) and \(M_2\) are representative of the two generators of \(H_1(T_1)\), respectively, when \(M_1\) and \(M_2\) are glued together along \(\partial(U(T_1))\).

On the other hand, if there is a line \(\ell \in \mathcal{A}^*\) that contains at most two intersection points, then we have only the following cases.

(Case 2a) \(\ell\) contains no intersection point. It follows that \(\mathcal{A}^* = \{\ell\}\). Then \(K(\mathcal{A}^*) = K(\ell)\) is an \(S^1\)-bundle over the two-sphere, which is precisely case (W10) of Definition 2.6. So \(K(\mathcal{A}^*)\) is not a reduced graph manifold.

(Case 2b) \(\ell\) contains only one intersection point. Thus \(\mathcal{A}^*\) is a pencil. The component of \(\tilde{M} = K(\tilde{\mathcal{A}}^*) - \text{Int}(U(\{A\}))\) that corresponds to \(\ell\) is homeomorphic to \(S^1 \times D^2\) and its meridian curve \(\{p\} \times S^1\) in \(\partial(S^1 \times D^2)\) is homologous to a fiber of an adjacent component of \(\tilde{M}\) by the glue map \(f\). So (W4) of Definition 2.6 is true, and \(K(\tilde{\mathcal{A}}^*)\) is not a reduced graph manifold.
(Case 2c) \( \ell \) contains exactly two intersection points \( y_1 \) and \( y_2 \). So their multiplicities \( t(y_1) \) and \( t(y_2) \) are at least two. There are two further subcases.

(Case 2c-i) \( t(y_1) = t(y_2) = 2 \). Then \( \mathcal{A}^* \) is a triangle; that is, \( \mathcal{A}^* \) has exactly three lines \( \ell_1, \ell_2, \ell_3 \) and three intersection points \( y_1, y_2, y_3 \). Clearly we have \( K(\mathcal{A}^*) = K(\mathcal{A}^*) \) with graph structure \( T(\mathcal{A}^*) = T_1 \cup T_2 \cup T_3 \) and \( M = M_1 \cup M_2 \cup M_3(= K(\mathcal{A}^*) - U(T(\mathcal{A}^*))) \). Their relations are as follows: \( \ell_1 \leftrightarrow M_1, \ell_2 \leftrightarrow M_2, \ell_3 \leftrightarrow M_3; \{y_1\} = \ell_2 \cap \ell_3, \{y_2\} = \ell_1 \cap \ell_3, \{y_3\} = \ell_1 \cap \ell_2, T_i \leftrightarrow y_i \) \( (i = 1, 2, 3) \). So \( M_1, M_2 \) and \( M_3 \) are \( S^1 \)-bundle over the annulus (i.e., \( T \times I \)), which means that (W1), of Definition 2.6 is satisfied here. Hence \( K(\mathcal{A}^*) \) is not a reduced graph manifold.

(Case 2c-ii) \( t(y_1) \) or \( t(y_2) > 2 \). To fix our notation, we shall assume that \( t(y_1) > 2 \). Then the exceptional line \( \ell_1 \) obtained by blowing up \( y_1 \) contains at least three intersection points. Let \( M \) (respectively, \( M_1 \)) be the component of \( \mathcal{M} \) that corresponds to \( \ell \) (respectively, \( \ell_1 \)). Thus \( M \) is an \( S^1 \)-bundle over the annulus, and \( M_1 \) is an \( S^1 \)-bundle over \( n_1 \)-punctured sphere \((n_1 \geq 3) \). So (W1) of Definition 2.6 is valid here.

Thus we have the following proposition.

**Proposition 3.2.** Suppose that \( \mathcal{A}^* \) is an arrangement in \( \mathbb{C}P^2 \). Then \( K(\mathcal{A}^*) \) is a reduced graph manifold with a reduced graph structure \( T(\mathcal{A}^*) \) if and only if each line of \( \mathcal{A}^* \) contains at least three intersection points.

Recall the following theorem and lemma in Section 7 of [21].

**Theorem 3.3.** A reduced graph manifold is irreducible.

**Lemma 3.4.** Let \( M \) be a reduced graph manifold with the graph structure \( T = T_1 \cup \ldots \cup T_n \). Then \( T_1 \) is compressible if and only if one component of \( M - \text{Int}(U(T)) \) which is pasted along \( U(T_1) \) is a solid torus.

From these results we have the following corollary.

**Corollary 3.5.** If \( \mathcal{A}^* \) is an arrangement with at least three intersection points on each of its line or \( \mathcal{A}^* \) is a triangle arrangement, then \( K(\mathcal{A}^*) \) is irreducible and \( T(\mathcal{A}^*) \) is an incompressible surface system in \( K(\mathcal{A}^*) \).

### 4. – Two arrangements \( \mathcal{A}^* \) and \( \mathcal{B}^* \) in \( \mathbb{C}P^2 \) with the same topological type

Throughout this section, let \( \mathcal{A}^* \) and \( \mathcal{B}^* \) be two arrangements in \( \mathbb{C}P^2 \), \( M(\mathcal{A}^*) = \mathbb{C}P^2 - N(\mathcal{A}^*), M(\mathcal{B}^*) = \mathbb{C}P^2 - N(\mathcal{B}^*) \), and let \( \varphi : M(\mathcal{A}^*) \rightarrow M(\mathcal{B}^*) \) be a homeomorphism.
**Proposition 4.1.** Let \( U(A^*) \) and \( U(B^*) \) be two regular neighborhoods of \( A^* \) and \( B^* \) respectively. Then \( U(A^*) - N(A^*) \) is homotopic equivalent to \( U(B^*) - N(B^*) \).

**Proof.** Let \( U_1(B^*) \) be an arbitrary regular neighborhood of \( N(B^*) \). Then \( K := \varphi((U(A^*))^c) \) is a neighborhood of \( N(A^*) \). There is a regular neighborhood \( U(A^*) \) of \( N(A^*) \) such that \( U(A^*) \subseteq K \). So \( \varphi((U(A^*))^c) \cap \varphi(K^c) \), i.e., \( V := \varphi(U(A^*))^c \subseteq \varphi(K^c) = U_1(B^*) \). Since \( V \) is a neighborhood of \( N(B^*) \), we can choose a regular neighborhood \( U_2(B^*) \) of \( N(B^*) \) such that \( U_2(B^*) \subseteq V \). Thus we get

\[
U_2(B^*) - N(B^*) \subseteq V - N(B^*) \subseteq U_1(B^*) - N(B^*).
\]

Observe that \( V - N(B^*) \) is exactly \( \varphi(U(A^*) - N(A^*)) \). So we have

\[
U_2(B^*) - N(B^*) \xrightarrow{i_2} \varphi(U(A^*) - N(A^*)) \xrightarrow{i_1} U_1(B^*) - N(B^*),
\]

where \( i_1 \) and \( i_2 \) are inclusion maps. Since \( U_i(B^*) \) (\( i = 1, 2 \)) are regular neighborhoods of \( N(B^*) \) in \( \mathbb{CP}^2 \) and \( U_1(B^*) \) can be contracted to \( U_2(B^*) \), the inclusion map \( i = i_1 \circ i_2 : U_2(B^*) - N(B^*) \hookrightarrow U_1(B^*) - N(B^*) \) induces an isomorphism \( i_* : \pi_j(U_2(B^*) - N(B^*)) \rightarrow \pi_j(U_1(B^*) - N(B^*)) \). Consider \( (i_1 \circ \varphi) \circ (\varphi^{-1} \circ i_2) = i \). We have the following induced maps

\[
\pi_j(U_2(B^*) - N(B^*)) \xrightarrow{(\varphi^{-1} \circ i_2)_*} \pi_j(U(A^*) - N(A^*)) \xrightarrow{(i_1 \circ \varphi)_*} \pi_j(U_1(B^*) - N(B^*))
\]

for \( j \geq 1 \). Since \( i_* = (i_1 \circ \varphi)_* \circ (\varphi^{-1} \circ i_2)_* \), \( (i_1 \circ \varphi)_* \) is onto and \( (\varphi^{-1} \circ i_2)_* \) is one-to-one.

Similarly, we can show that \( (\varphi^{-1} \circ i_2)_* \) is onto and \( (i_1 \circ \varphi)_* \) is one-to-one. It follows that \( \pi_j(U_1(B^*) - N(B^*)) \simeq \pi_j(U(A^*) - N(A^*)) \). In view of Whitehead theorem, \( U_1(B^*) - N(B^*) \) is homotopic equivalent to \( U(A^*) - N(A^*) \). Since any two regular neighborhoods of \( N(B^*) \) (or \( N(A^*) \)) are homotopic equivalent, the proposition follows immediately. \( \square \)

**Remark 4.1.** More generally, by the same proof, the above proposition is still true for any pairs \( (X, K), (Y, H) \) of complexes, such that \( X - K \) is homeomorphic to \( Y - H \).

Observe that \( K(A^*) \), the boundary of an arbitrary regular neighborhood \( U(A^*) \) of \( N(A^*) \), is homotopic equivalent to \( U(A^*) - N(A^*) \). So we have the following corollary.

**Corollary 4.2.** If \( K(A^*) \) and \( K(B^*) \) are boundaries of regular neighborhoods \( U(A^*) \) and \( U(B^*) \) of \( N(A^*) \) and \( N(B^*) \) respectively, then \( K(A^*) \simeq K(B^*) \) homotopically.

**Corollary 4.3.** Let \( \tilde{A}^* \) (respectively \( \tilde{B}^* \)) be the induced arrangement from \( A^* \) (respectively \( B^* \)) by blowing up. Then \( K(\tilde{A}^*) \simeq K(\tilde{B}^*) \) homotopically.
Before we can proceed, we need to recall a result of Waldhausen.

**Definition 4.1.** Let $M$ and $N$ be compact orientable 3-manifolds. An isomorphism $\psi$ of $\pi_1(N)$ onto $\pi_1(M)$ is said to respect the peripheral structure if for each boundary surface $F$ of $N$ there is a boundary surface $G$ of $M$ such that $\psi(i_*(\pi_1(F))) \subset R$ and $R$ is conjugate in $\pi_1(M)$ to $i_*(\pi_1(G))$, where $i_*$ denotes inclusion homomorphism.

**Theorem 4.4 (6.5 of [22]).** If $M$ and $N$ are irreducible and boundary-irreducible compact orientable three-manifolds and $\psi$ is an isomorphism from $\pi_1(N)$ onto $\pi_1(M)$ which respects the peripheral structure and $M$ is sufficiently large, then there exists a homeomorphism $f : N \to M$ that induces $\psi$.

**Lemma 4.5.** $K(\mathcal{A}^*)$ and $K(\mathcal{B}^*)$ are boundary irreducible, and $\varphi_\ast : \pi_1(K(\mathcal{A}^*)) \to \pi_1(K(\mathcal{B}^*))$ respects the peripheral structure.

**Proof.** The lemma follows immediately from the fact that $\partial K(\mathcal{A}^*) = \phi = \partial K(\mathcal{B}^*)$. \hfill \Box

**Lemma 4.6.** If $\mathcal{A}^*$ is an arrangement with at least three intersection points on each of its line, then $K(\mathcal{A}^*)$ is reduced, irreducible, boundary irreducible and, sufficiently large.

**Proof.** By Proposition 3.2 and Corollary 3.5, we need only to show that $K(\mathcal{A}^*)$ is sufficiently large. In view of a result of D. Mumford [14], we know that the first Betti number of $K(\mathcal{A}^*)$ is at least $p$ if $\mathcal{A}^*$ is $p$-connected (i.e., $p$ is the minimal number such that there exist some points $P_1, \ldots, P_p \in N(\mathcal{A}^*)$ making $N(\mathcal{A}^*) - \{P_1, \ldots, P_p\}$ a tree). By Lemma 2.2 (b), $K(\mathcal{A}^*)$ is sufficient large. \hfill \Box

From Corollary 4.3 and Theorem 4.4 of Waldhausen we have the following proposition.

**Proposition 4.7.** If $\mathcal{A}^*$ and $\mathcal{B}^*$ are two arrangements in $\mathbb{CP}^2$ such that each of their lines contains at least three intersection points and if $M(\mathcal{A}^*)$ and $M(\mathcal{B}^*)$ are homeomorphic, then for $K(\mathcal{A}^*)$ and $K(\mathcal{B}^*)$, the boundaries of arbitrary regular neighborhoods $U(\mathcal{A}^*)$ and $U(\mathcal{B}^*)$ of $N(\mathcal{A}^*)$ and $N(\mathcal{B}^*)$, respectively, there is an isomorphism $\phi$ from $\pi_1(K(\mathcal{A}^*))$ onto $\pi_1(K(\mathcal{B}^*))$ and a homeomorphism $f : K(\mathcal{A}^*) \to K(\mathcal{B}^*)$ that induces $\phi$.

Now we need to review some results of Waldhausen [21] before we can prove our theorem.

**Definition 4.2.** Let $M$ be a reduced graph manifold. We say that $M$ has the Waldhausen property if none of the following three cases occurs.

- **(E1)** $M - \text{Int}(U(T))$ consists of the bundle over the two-sphere with three-punctures and three solid tori,
- **(E2)** $M - \text{Int}(U(T))$ consists of the bundle over the Möbius band and one solid torus,
- **(E3)** $T \neq \phi$ and $M - \text{Int}(U(T))$ is torus × interval.
LEMMA 4.8. If $A^*$ is an arrangement with at least three intersection points on each of its lines, then $K(A^*)$ is a reduced graph manifold with the Waldhausen property.

PROOF. This follows from Lemma 4.6 and Definition 4.2. \qed

In Section 9 of [21], for a reduced graph manifold $M$ that satisfies the Waldhausen property, a weighted graph $G(M)$ was introduced. It can be described axiomatically as follow.

(G1) $G(M)$ has only finitely many weighted vertices $\mu_1, \mu_2, \ldots$ and finitely many directed edges $\tau_1, \tau_2, \ldots$. For each edge, each of its end point is incident with one vertex. $G(M)$ is connected.

(G2) Each vertex $\mu_j$ is assigned a triple of integers $(g_j, r_j, s_j)$. Here $r_j$ is nonnegative. When $r_j = 0$, $s_j$ is arbitrary. When $r_j > 0$, $s_j$ is replaced by a dash (or is omitted). $g_j$ is arbitrary. (As for the graph manifolds discussed in our paper, each vertex $\mu_j$ corresponds to a component $M_j$ of $M - \text{Int}(\mathcal{U}(T))$ with weight given by $(g_j, r_j, s_j)$. Here $g_j$ is the genus of the base of $M_j$, $r_j$ is the number of boundary surfaces that are not connected to any component of $T$, and $s_j$ is the cross-section obstruction when $r_j$ is zero).

(G3) If a vertex of degree one is assigned the triple $(0, 0, s_j)$, it is replaced by a dash.

(G4) (a) If $G(M)$ has only two vertices, both two vertices are not weighted by a dash.

(b) A vertex of degree zero is not assigned the triple $(0, 0, s_j)$ or $(-1, 0, s_j)$.

(c) A vertex of degree two is not assigned the triple $(0, 0, s_j)$.

(d) A vertex of degree one is not assigned the triple $(0, 1, -)$. (e) If $G(M)$ has three vertices that are weighted by dashes, and only three edges, then the fourth vertex is not assigned $(0, 0, s)$.

(f) If $G(M)$ has one vertex that is weighted by a dash, and only one edge, then the second vertex cannot be weighted $(-1, 0, s_j)$.

(G5) If edge $\tau_i$ is incident with a vertex weighted by a dash, then $\tau_i$ is directed to this vertex and is weighted by a pair of integers $\alpha_i, \beta_i$, where $\alpha_i, \beta_i$ are co-prime and $1 \leq \beta_i < \alpha_i$.

(G6) If the vertices that are incident with $\tau_i$ are not weighted by dashes, then the edge $\tau_i$ is weighted by a pair of integers $(\alpha_i, \beta_i)$, where $\alpha_i$ and $\beta_i$ are co-prime and $0 \leq \beta_i < \alpha_i$.

(G7) The following graphs or subgraphs are not considered.
Let $G^*$ (possibly disconnected) be the subgraph of $G(M)$ where all the vertices with $g_j < 0$ and adjacent edges are removed. The homology group $H_1(G^*)$ is weighted by a homomorphism to $\mathbb{Z}_2$.

Suppose that the arrangement $\mathcal{A}^*$ in $\mathbb{C}P^2$ is chosen such that each line of $\mathcal{A}^*$ contains at least three intersection points. Recall that for each $\ell_i$ of $\mathcal{A}^*$ there is a corresponding vertex $\mu_i$ with weight $(0, 0, s_i)$, where $s_i = \ell_i \cdot \ell_i$, the self-intersection number of $\ell_i$ in $\mathbb{C}P^2$. In $G(\mathcal{A}^*)$ each vertex $\mu_i$ has degree at least three. There is no dash for any vertex in $G(\mathcal{A}^*)$. So the definition of $G(\mathcal{A}^*)$ has no conflict with (G3), (G4), and (G7). For (G5) and (G6), first we describe the direction of weight of $\tau_i$.

Let $U(T_i)$ be a component of $U(T)$ containing $T_i$. The boundary surfaces are $T_i^-$ and $T_i^+$ with their orientation induced by inclusion $T_i^- \hookrightarrow M - \text{Int}(U(T))$ and $T_i^+ \hookrightarrow M - \text{Int}(U(T))$. The direction and weight of edge $\tau_i$ are decided by the gluing homeomorphism from $T_i^-$ to $T_i^+$. Let the direction of $\tau_i$ be from $T_i^-$ to $T_i^+$. From the orientation of $T_i^-$ and $T_i^+$ we choose bases $\{a_2, b_2\}$ ($b_2$ is represented by a fiber of $M_j$, $j = 1, 2$, $T_i^- \in M_1$, $T_i^+ \in M_2$) for $\pi_1(T_i^\pm)$. The gluing homeomorphism induces an isomorphism from $\pi_1(T_i^-)$ to $\pi_1(T_i^+)$, which is expressed by

$$
\begin{pmatrix}
a_2 \\
b_2
\end{pmatrix}
= \epsilon
\begin{pmatrix}
\gamma_i & \delta_i \\
\alpha_i & \beta_i
\end{pmatrix}
\begin{pmatrix}
a_1 \\
b_1
\end{pmatrix},
$$

where $\epsilon_i = 1$ or $-1$, $\det\begin{pmatrix}\gamma_i & \delta_i \\\alpha_i & \beta_i\end{pmatrix} = -1$, $0 \leq \beta_i < \alpha_i$, $(\alpha_i, \beta_i) = 1$.

Then $\tau_i$ is weighted by $(\alpha_i, \beta_i)$.

In our situation, we have simply $a_i = \beta_i = 1$, $\beta_i = \gamma_i = 0$.

For (G8) the definition of assigning $H_1(G^*)$ a weight that is a homomorphism to $\mathbb{Z}_2$ is as follows. Let $M^*$ be a (possibly disconnected) submanifold of $M$ obtained by taking away all those $S^1$-bundle over nonorientable surfaces together with those $U(T_i)$ which pasted along on them from $M = U(T) \cup M_1 \cup \ldots \cup M_m$. $H_1(M^*) \to H_1(G^*)$ is surjective. The kernel is generated by $H_1(M_j)$ for $M_j \subseteq M^*$. For a closed path $\ell$ in $M^*$, we define $\rho^*(\ell) = \sum_i \sigma(\ell, T_i) \epsilon_i'$ where $\sigma(\ell, T_i)$ is the intersection number of $\ell$ and $T_i$ modulo 2, $\epsilon_i' \in \mathbb{Z}_2$,

$$
\epsilon_i' = \begin{cases}
1 & \text{if } \epsilon_i = -1 \\
0 & \text{if } \epsilon_i = 1.
\end{cases}
$$

Then $\rho : H_1(G^*) \to \mathbb{Z}_2$ is the desired homomorphism such that the following diagram commutes.

\[ H_1(M^*) \longrightarrow H_1(G^*) \]

\[ \rho^* \downarrow \ \\
\rho \uparrow \]

\[ \mathbb{Z}_2 \]

DEFINITION 4.3. Let $G_1$ and $G_2$ be two graphs with properties (G1) to (G8). $G_1$ and $G_2$ are said to be equivalent if there is $1 \to 1$ incidence preserving map $\phi : G_1 \to G_2$ with the following properties.

(a) $\phi$ carries one vertex to a vertex with same weight
(b) \( \phi \) carries one weighted edge with \((\alpha, \beta)\) to a edge with weight \((\alpha, \beta^\omega)\), where \(\omega = + \) or \(-1\) according to the edge orientation being preserved or not. \(\beta^{-1}\) is the standard representative of the coset \((\text{mod } \alpha)\) of the inverse coset of \(\beta\). We use the convention that \(0^{-1} = 0\).

(c) The induced homeomorphism \(\phi : G_1^* \to G_2^*\) by \(\phi\) induces a commutative diagram.

\[
\begin{array}{ccc}
H_1(G_1^*) & \to & H_1(G_2^*) \\
\downarrow & & \downarrow \\
Z_2 & & Z_2
\end{array}
\]

(d) Each of the following pair of graphs is equivalent.

\[
\begin{array}{ccc}
(2, 1) & \cong & (-1, 1, -) \\
(0, 1, -) & & \\
\end{array}
\]

\[
\begin{array}{ccc}
(2, 1) & \cong & (-2, 0, 0) \\
(0, 0, 2) & & \\
\end{array}
\]

Remark 4.2. Condition (d) above does not occur in our graphs \(G(\tilde{A}^*)\) or \(G(\tilde{B}^*)\), since they are not weighted by dashes.

We are ready to state the main theorem of [21] (cf. (9.4) of [21]), which is essential to the proof of our theorem.

**Theorem 4.9.** An oriented reduced graph manifold that satisfies the Waldhausen property (Definition 4.2) determines and is determined by its weighted graph with the properties (G1)-(G8).

Thus, two oriented reduced graph manifolds are homeomorphic if and only if the corresponding graphs are equivalent.

**Corollary 4.10.** Let \(A^*\) and \(B^*\) be two arrangements in \(\mathbb{C}P^2\) such that each of their lines contains at least three intersection points. If \(M(A^*)\) is homeomorphic to \(M(B^*)\), then \(G(\tilde{A}^*)\) is equivalent to \(G(\tilde{B}^*)\).

**Proof.** This follows from Proposition 4.7, Lemma 4.8, and Theorem 4.9.

---

5. – Proof of the main theorem for non-exceptional arrangements in \(\mathbb{C}P^2\)

In this section we shall prove a weak form (Theorem 5.4) of our main theorem.

**Theorem 5.1.** Let \(A^*\) and \(B^*\) be two arrangements in \(\mathbb{C}P^2\). By blowing up their multiple points (multiplicity \(\geq 3\)), we obtain two associated arrangements \(\tilde{A}^*\) and \(\tilde{B}^*\)
and \( B^* \) in some blownup surfaces \( \mathbb{CP}^2 \). Let \( L(\mathcal{A}^*) \) and \( L(\mathcal{B}^*) \) be the set of all intersections of elements of \( \mathcal{A}^* \) and \( \mathcal{B}^* \), respectively, which are partially ordered by \( X \leq Y \iff Y \subseteq X \). Then \( L(\mathcal{A}^*) \cong L(\mathcal{B}^*) \) if and only if \( L(\mathcal{A}^*) \cong L(\mathcal{B}^*) \), which preserves weights (self-intersection numbers) of lines in \( \mathcal{A}^* \) and \( \mathcal{B}^* \).

**Proof.** Obviously \( L(\mathcal{A}^*) \cong L(\mathcal{B}^*) \Rightarrow L(\mathcal{A}^*) \cong L(\mathcal{B}^*) \).

To prove the converse, for the sake of convenience but without loss of generality, we simply assume that \( \mathcal{A}^* = \mathcal{B}^* \). Let \( A_1 = \mathcal{A}^* \), \( A_2 = \mathcal{B}^* \) and \( \phi_i : \mathcal{A}^* \rightarrow A_i \) be a blowing up map \( (i = 1, 2) \). An element of \( \mathcal{A}^* \) that corresponds to an element of \( A_i \) by \( \phi_i \) is called regular with respect to \( \phi_i \). The set of all regular elements of \( \mathcal{A}^* \) with respect to \( \phi_i \) is denoted by \( R_i \). The remaining elements of \( \mathcal{A}^* \) that are blown down to points by \( \phi_i \) are called exceptional with respect to \( \phi_i \). The set that consists of all such elements is denoted by \( E_i \).

For fixed \( i \) \( (i = 1 \text{ or } 2) \), we list the following three basic properties for elements in \( \mathcal{A}^* \).

- **(P1)** For each \( e \in E_i \), it has the self intersection number \( e^2 = -1 \); for \( r \in R_i \), \( r^2 = 1 - |\{ e \in E_i : e \cap r \neq \emptyset \}| \). Here \( |S| \) denotes the cardinal number of the set \( S \).

- **(P2)** Either two elements in \( R_i \) intersect with exactly one exceptional element in \( E_i \) without intersecting to each other themselves, or they intersect exactly at one double point.

- **(P3)** For each \( e \in E_i \), \( \{ \text{star}(e) \setminus \{ e \} \} \subseteq R_i \) and \( |\text{star}(e) \setminus \{ e \}| \geq 3 \), where \( \text{star}(e) \) is the set consists of all elements in \( \mathcal{A}^* \) that intersect with \( e \).

**Lemma 5.2.** For an \( \ell \in \mathcal{A}^* \), if \( \ell^2 \neq -1 \), then \( \ell \in E_1 \cap E_2 \). If \( \ell^2 = -1 \) and at most one element in \( \text{star}(\ell) \setminus \{ \ell \} \) has the same self intersection, then \( \ell \in E_1 \cap E_2 \) and \( \text{star}(\ell) \setminus \{ \ell \} \subseteq R_1 \cap R_2 \).

**Proof.** By (P1), \( \ell \notin E_1 \cup E_2 \) if \( \ell^2 \neq -1 \). Hence \( \ell \in \mathcal{A}^* \setminus (E_1 \cup E_2) = R_1 \cap R_2 \). If \( \ell^2 = -1 \), then \( \ell \in E_1 \cap E_2 \) by (P1) and \( \text{star}(\ell) \setminus \{ \ell \} \subseteq R_1 \cap R_2 \) by (P3). \( \square \)

Clearly we have \( \mathcal{A}^* = (R_1 \setminus R_2) \cup (R_1 \cap R_2) \cup (E_1 \cap E_2) \) as the union of three disjoint sets. (Recall \( R_1 \setminus R_2 = (R_1 - R_2) \cup (R_2 - R_1) \). We also have \( R_1 - R_2 = E_2 - E_1 \), \( R_2 - R_1 = E_1 - E_2 \), \( R_1 \setminus R_2 = E_1 \setminus E_2 \), etc.

If \( R_1 \setminus R_2 = \emptyset \) (i.e., \( E_1 \setminus E_2 = \emptyset \)), then \( E_1 = E_2 \), \( \phi_1 = \phi_2 \) and \( \mathcal{A} = \mathcal{B} \). So in this case, we have nothing to prove. If \( R_1 \setminus R_2 \neq \emptyset \), then we wish to find out what it looks like.

Define a set \( K(\ell) = (\{ \text{star}(\ell) \setminus \{ \ell \} \}) \cap (R_1 \setminus R_2) \) for each \( \ell \in \mathcal{A}^* \).

**Lemma 5.3.** (i) If \( \ell \in E_1 \cap E_2 \), \( K(\ell) = \emptyset \).

(ii) If \( \ell \in R_1 \cap R_2 \), then \( |K(\ell) \cap E_1| = |K(\ell) \cap E_2| \).

(iii) If \( \ell \in R_1 \setminus R_2 \), then \( |K(\ell)| = 2 \). \( K(\ell) \subseteq R_1 - R_2 \) if \( \ell \in R_2 - R_1 \) and \( K(\ell) \subseteq R_2 - R_1 \) if \( \ell \in R_1 - R_2 \).

(iv) \( K(\ell) \) is discrete for each \( \ell \in \mathcal{A}^* \) in the sense that any two elements of \( K(\ell) \) do not intersect to each other in \( \mathcal{A}^* \).
PROOF. (i) It is clear because $E_1 \cap E_2$ is disjoint from $R_1 \cup R_2$ (P3).

(ii) In view of (P1), we have

$$1 - \ell^2 = \left| \{ e \in E_1 : e \cap \ell \neq \emptyset \} \right|$$

$$= \left| (\text{star}(\ell) - \{ \ell \}) \cap E_1 \right|$$

$$= \left| (\text{star}(\ell) - \{ \ell \}) \cap E_1 \cap E_2 \right| \cup \left| (\text{star}(\ell) - \{ \ell \}) \cap (E_1 - E_2) \right|$$

$$= \left| (\text{star}(\ell) - \{ \ell \}) \cap E_1 \cap E_2 \right| + \left| (K(\ell) \cap E_1) \right|$$

Thus we have $|K(\ell) \cap E_1| = |K(\ell) \cap E_2|$

(iii) When $\ell \in R_1 \cup R_2$, we have $\ell^2 = -1$ by Lemma 5.2. If $\ell \in E_1 - E_2 = R_2 - R_1$, then we have $-1 = \ell^2 = 1 - \left| \{ e \in E_2 : e \cap \ell \neq \emptyset \} \right|$ by (P1); that is, $\left| \{ e \in E_2 : e \cap \ell \neq \emptyset \} \right| = 2$. On the other hand, by (P3) we know that $\ell$ does not intersect with any element in $E_1$. So

$$|K(\ell)| = \left| (\text{star}(\ell) - \{ \ell \}) \cap (E_1 - E_2) \right| + \left| (\text{star}(\ell) - \{ \ell \}) \cap (E_2 - E_1) \right| = 0 + 2 = 2$$

and $K(\ell) \subseteq E_2 - E_1 = R_1 - R_2$. Similarly, one can show that if $\ell \in E_2 - E_1$, then $|K(\ell)| = 2$ and $K(\ell) \subseteq R_2 - R_1$

(iv) Since $E_i$ is discrete by (P3), $K(\ell) = (K(\ell) \cap E_1) \cup (K(\ell) \cap E_2)$ and $K(\ell) \cap E_i = \emptyset$ for all $\ell \in E_i$ by (P3). We need only to show that if $\ell \in R_1 \cap R_2$, then $\ell_1 \cap \ell_2 = \emptyset$. However, since $\ell_1, \ell_2 \in R_2$ intersect and $\ell$ intersects with $\ell_2 \in E_2$, we have $\ell_1 \cap \ell_2 = \emptyset$ in view of (P2). $\square$

Now we shall continue the proof of Theorem 5.1. In view of (iii) of Lemma 5.3, we have $|R_1 \cup R_2| = 2|R_2 - R_1| = 2|R_1 - R_2| \geq 4$, and $R_1 \cup R_2$ must consist of some cycles. In fact, the elements of $R_1 - R_2$ and $R_2 - R_1$ form the edges of each cycle alternately. But by (P2), any two elements in $R_1 - R_2$ must either intersect at exactly one double point or intersect with exactly one element in $R_2 - R_1$ without intersecting each other themselves. Thus $R_1 \cup R_2$ must consist of only one hexagon (fig. 1). Let us label the edges of this hexagon by $a_1 \leftrightarrow e_2 \leftrightarrow a_3 \leftrightarrow e_1 \leftrightarrow a_2 \leftrightarrow e_3(\leftrightarrow a_1)$, where $a_j \in R_1 - R_2$, $e_j \in E_1 - E_2 (j = 1, 2, 3)$.

![Fig. 1. Hexagon](attachment:image.png)
For \( \ell \in R_1 \cap R_2 \), consider \( \ell \) as a line in \( A_1 \). If \( \ell \) passes through the intersection points of \( a_1 \cap a_2 \), \( a_2 \cap a_3 \) or \( a_1 \cap a_3 \) in \( A_1 \subseteq \mathbb{CP}^2 \), then clearly \( \ell \) intersects with \( e_3 \), \( (e_1 \text{ or } e_2) \) in \( \mathbb{CP}^2 \). On the other hand, if \( \ell \) does not pass through the intersection points of \( a_1 \cap a_2 \), \( a_2 \cap a_3 \) or \( a_1 \cap a_3 \) in \( A_1 \subseteq \mathbb{CP}^2 \), then clearly \( \ell \) must intersect with \( a_1 \), \( a_2 \), and \( a_3 \) in \( A_1 \subseteq \mathbb{CP}^2 \). It follows that, if we consider \( \ell \) as a line in \( A_2 \), \( \ell \) passes through \( e_1 \cap e_2 \), \( e_2 \cap e_3 \), and \( e_3 \cap e_1 \) in \( A_2 \subseteq \mathbb{CP}^2 \), which is impossible because these three points are not colinear. So only the first situation is allowed. By (ii) and (iv) of Lemma 5.3, we conclude that \( |K(\ell)| = 2 \). In fact, \( \ell \) must intersect with \( R_1 \cap R_2 \) on exactly two opposite edges namely, \( |\ell \cap a_1| = 1 = |\ell \cap e_1| \) or \( |\ell \cap a_2| = 1 = |\ell \cap e_2| \) or \( |\ell \cap a_3| = 1 = |\ell \cap e_3| \). Thus we can define

\[
\phi : (E_1 \cap E_2) \cup (R_1 \cap R_2) \cup \{a_1, a_2, a_3\} \mapsto (E_1 \cap E_2) \cup (R_1 \cap R_2) \cup \{e_1, e_2, e_3\}
\]

by \( \phi|_{E_1 \cap E_2} = \text{Identity} \), \( \phi|_{R_1 \cap R_2} = \text{Identity} \), \( \phi(a_j) = e_j \) for \( j = 1, 2, 3 \). It is clear that \( \phi \) induces an isomorphism from \( L(A_1) \) to \( L(A_2) \). Thus the theorem is proved. A configuration for illustration of \( \phi_2 \circ \phi_1^{-1} : L(A_1) \rightarrow L(A_2) \) can be shown as in figure 2. \( R_1 \cap R_2 = \{\ell_1, \ell_2, \ldots, \ell_9\} \), \( R_2 - R_1 = \{e_1, e_2, e_3\} \), \( R_1 - R_2 = \{a_1, a_2, a_3\} \), \( E_1 \cap E_2 = \) exceptional sets obtained by blowing up three points \( \ell_1 \cap \ell_6 \cap \ell_9 \), \( \ell_2 \cap \ell_5 \cap \ell_9 \), \( \ell_4 \cap \ell_6 \cap \ell_8 \).

\[\square\]

Fig. 2. Rational map
THEOREM 5.4. Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be two arrangements in $\mathbb{C}P^2$ such that each of their lines contains at least three intersection points. If $M(\mathcal{A}^*)$ is homeomorphic to $M(\mathcal{B}^*)$, then $L(\mathcal{A}^*)$ is isomorphic to $L(\mathcal{B}^*)$.

PROOF. In view of Corollary 4.10 we know that $G(\mathcal{A}^*)$ is equivalent to $G(\mathcal{B}^*)$. It means that each $\ell$ of $\mathcal{A}^*$ with weight $(\ell \cdot \ell)$ (self-intersection number) is one-to-one correspondent to a line of $\mathcal{B}^*$ with the same weight and $L(\mathcal{A}^*)$ is equivalent to $L(\mathcal{B}^*)$. Thus Theorem 5.4 follows immediately from Theorem 5.1.

6. – Exceptional arrangements in $\mathbb{C}P^2$

An arrangement in $\mathbb{C}P^2$ is called exceptional if one of its lines has at most two intersection points. We have shown in Section 4 that the topological type of the complement of a nonexceptional arrangement in $\mathbb{C}P^2$ determines the lattice of this arrangement. In this section we shall study the cohomology rings as well as the fundamental groups of complements of exceptional arrangements.

We shall first study the cohomology algebra of arrangement in $\mathbb{C}^3$. In general, let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$. Recall that the cohomology of its complement $M(\mathcal{A})$ as an algebra is isomorphic to Orlik-Solomon algebra over $\mathbb{C}$ which is defined as follows.

Let $E = E(\mathcal{A}) = \wedge(E_1)$ be the exterior algebra of $E_1 := \bigoplus_{H \in A} \mathbb{C}e_H$. Write $uv = u \wedge v$ for $u, v \in E$. If $|A| = n$, then $E$, as a graded algebra, can be written as $E = \bigoplus_{p=0}^n E_p$, where $E_0 = \mathbb{C}$, $E_p$ is spanned by all $e_{H_1} \ldots e_{H_p}$ with $H_k \in A$. Define a map $\partial : E \to E$ by $\partial(1) = 0$, $\partial(e_H) = 1$ and $\partial(e_{H_1} \ldots e_{H_p}) = \sum_{k=1}^p (-1)^{k-1} e_{H_1} \ldots \hat{e}_{H_k} \ldots e_{H_p}$ for all $H_1, \ldots, H_p \in A$.

Consider a $p$-tuple of hyperplanes $S = (H_1, \ldots, H_p)$. Write $|S| = p$, $e_S = e_{H_1} \ldots e_{H_p} \in E$ and $\cap S = H_1 \cap \ldots \cap H_p \in L(\mathcal{A})$ (Lattice of $\mathcal{A}$).

DEFINITION 5.1. Let $S = (H_1, \ldots, H_p)$, $r(\cap S) =$ codimension of $\cap S$. $S$ is said to be independent if $r(\cap S) = p$ and dependent if $r(\cap S) < p$. Let $S_p$ be the set of all $p$-tuple and $S = \bigcup_{p \geq 0} S_p$. Let $I = I(\mathcal{A})$ be the ideal of $E$ generated by $\partial e_s$ for all dependent $s \in S$. Orlik-Solomon algebra of $\mathcal{A}$ is defined as $A = A(A) = E/I$.

$I$ is also a graded ideal. If we let $I_p = I \cap E_p$, then $I = \bigoplus_{p=0}^n I_p$ and $\partial I_p \subset I_{p-1}$ ($p \geq 1$). Let $\phi : E \to A$ be the natural homomorphism. Let $A_p = \phi(E_p)$, $a_H = \phi(e_H)$ for $H \in A$ and $a_s = \phi(e_s)$ for $s \in S$.

Now let $n = 3$, $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$ corresponding to an exceptional arrangement $\mathcal{A}^*$ in $\mathbb{C}P^2$. Write

$\mathcal{A} = \{H_0, H_1, \ldots, H_m, H_{m+1}, \ldots, H_{m+n}\}$,

$\mathcal{A}^* = \{\ell_0, \ell_1, \ldots, \ell_m, \ell_{m+1}, \ldots, \ell_{m+n}\}$

where $\bigcap_{i=0}^m \ell_i \neq \emptyset$ and $\bigcap_{j=m+1}^{m+n} \ell_j \cap \ell_0 \neq \emptyset$. Thus any three hyperplanes
in \{H_0, \ldots, H_m\} (or \{H_0, H_{m+1}, \ldots, H_{m+n}\}) are dependent. Such \mathcal{A} is an 
exceptional arrangement in \mathbb{C}^3.

**Lemma 6.1.** Let \mathcal{A} be an exceptional arrangement in \mathbb{C}^3 as above. Then the co-

homology algebra of the complement of \mathcal{A} is isomorphic to \mathbb{A}(\mathcal{A}) = \bigoplus_{p=0}^{3} A_p \text{ where}

\[ A_0 = \mathbb{C}, \quad A_1 = \bigoplus_{i=0}^{m+n} \mathbb{C}a_i \quad (\text{where } a_i = a_{H_i}), \quad A_2 = \left(\bigoplus_{k=1}^{m+n} \mathbb{C}a_0a_k\right) \bigoplus \bigoplus_{1 \leq i < j \leq m+n} \mathbb{C}a_ia_j \quad \text{and} \quad A_3 = \bigoplus_{1 \leq i < j \leq m+n} \mathbb{C}a_ia_0a_j.

**Proof.** \( A_0 = \mathbb{C} \) follows from \( I_0 = 0 \) and \( E_0 = \mathbb{C} \).

Since any two distinct hyperplanes are independent, \((H_i, H_j)\) are the only 
dependent elements in \mathbb{S}_2. But \( \partial(e_i, e_j) = 0 \). So \( I_1 = 0 \) and \( A_1 \cong E_1 \), that is, 
\[ A_1 = \bigoplus_{i=0}^{m+n} \mathbb{C}a_i. \]

For \( A_2 \), we know that it is spanned by \( a_ia_j \) for \( 0 \leq i < j \leq m+n \). Let 
\( e_i = e_{H_i}, \quad i = 0, \ldots, m+n \). For \( 1 \leq i < j \leq m \) or \( m < i < j \leq m+n \), we 
have \( e_ie_j + e_je_i = \partial(e_ie_je_0) \in I \). So 
\[ (6.1) \quad a_ia_j = a_ia_0a_j \quad \text{in } \mathbb{A} \quad \text{for} \quad 1 \leq i < j \leq m \quad \text{or} \quad m < i < j \leq m+n. \]

Thus \( A_2 \) is spanned by \( a_0a_0, \quad 0 \leq \ell \leq m+n \) and \( a_ia_j, \quad 1 \leq i \leq m < j \leq m+n \). 
Next we need to show that if there are \( c_{\ell}, \quad 1 \leq \ell \leq m+n \) and \( c_{ij}, \quad 1 \leq i \leq m < j \leq m+n \) in \( \mathbb{C} \) such that 
\[ (6.2) \quad \sum_{\ell=1}^{m+n} c_{\ell}a_0a_\ell + \sum_{1 \leq i < j \leq m+n} c_{ij}a_ia_j = 0, \]

then all \( c_{\ell}, \ c_{ij} \) are zero. From equation 6.2 we have \( \sum_{\ell=1}^{m+n} c_{\ell}e_0e_\ell + \sum_{1 \leq i < j \leq m+n} c_{ij}e_ie_j \in I_2 \). Remember that \( I_2 \) is spanned by \( \{ \partial(e_ie_je_k) : 0 \leq i < j < k \leq m \text{ or } m < i < j < k \leq m+n+1 \} \), where we set \( e_{m+n+1} = e_0 \). There are \( c_{rst} \in \mathbb{C} \) such that 
\[ \sum_{0 \leq r < s < t \leq m \text{ or } m \leq r < s < t \leq m+n+1} c_{rst} \partial(e_re_se_t) = \sum_{\ell=1}^{m+n} c_{\ell} e_0 e_\ell + \sum_{1 \leq i < j \leq m+n} c_{ij}e_ie_j, \]

which is equivalent to 
\[ \sum_{0 \leq r < s < t \leq m \text{ or } m \leq r < s < t \leq m+n+1} c_{rst}(e_se_t - e_te_s) + \sum_{\ell=1}^{m+n} c_{\ell} e_0 e_\ell = \sum_{1 \leq i < j \leq m+n} c_{ij}e_ie_j. \]

Since \( \{ e_ie_j, \quad 0 \leq i < j \leq m+n \} \) is a base of \( E_2 \), the above equation implies 
immediately \( c_{ij} = 0 \) for \( 1 \leq i \leq m < j \leq m+n \). So 
\[ \sum_{\ell=1}^{m+n} c_{\ell} e_0 e_\ell = \partial \left( \sum_{0 \leq r < s < t \leq m \text{ or } m \leq r < s < t \leq m+n+1} c_{rst} e_re_se_t \right). \]
As \( \partial^2 = 0 \), we have

\[
0 = \partial \left( \sum_{\ell=1}^{m+n} c_\ell (e_0 e_\ell) \right) = \sum_{\ell=1}^{m+n} c_\ell (e_\ell - e_0).
\]

It follows that \( c_\ell = 0 \).

We next prove that \( A_3 = \bigoplus_{1 \leq i \leq m} C_{a_0 a_i a_j} \). Since any four hyperplanes in \( \mathbb{C}^3 \) are linearly dependent, we have

\[
e_i e_k e_0 - e_i e_k e_0 + e_i e_j e_0 - e_i e_k e_0 = \partial(e_i e_j e_k e_0) \in I.
\]

So in \( A \), we have

\[
a_i a_j a_k = a_0 a_j a_k + a_i a_0 a_k + a_i a_j a_0.
\]

On the other hand,

\[
e_i \partial(e_i e_j e_k) = e_i (e_j e_k - e_i e_k + e_i e_j) = e_i e_j e_k.
\]

If \((H_i, H_j, H_k)\) is dependent, then \( e_i e_j e_k \in I_3 \), which means \( a_i a_j a_k = 0 \). In view of this statement and equation 6.3, we see that \( A_3 \) is spanned by \( \{a_0 a_i a_j : 1 \leq i \leq m < j \leq m + n\} \).

We have left to check that if there are \( c_{ij} \in \mathbb{C} \) such that

\[
\sum_{1 \leq i \leq m} \sum_{m < j \leq m + n} c_{ij} a_i a_j = 0,
\]

then \( c_{ij} = 0 \).

Let us denote \( e_{\alpha} e_{\beta} e_{\gamma} e_{\delta} \) and \( e_i e_j e_k \) by \( e_{\alpha\beta\gamma\delta} \) and \( e_{ijk} \), respectively. By the definition of \( I \), we have

\[
I_3 = \langle \partial(e_{\alpha\beta\gamma\delta}), \ e_\ell \partial(e_{ijk}) : 0 \leq \alpha < \beta < \gamma < \delta \leq m + n, \ 0 \leq i < j < k \leq m \text{ or } m < i < j < k \leq m + n + 1 \rangle,
\]

where we still use the convention \( e_{m+n+1} = e_0 \). The relation \( \sum_{1 \leq i \leq m} \sum_{m < j \leq m + n} c_{ij} a_i a_j = 0 \) means \( \sum_{m < j \leq m + n} c_{ij} e_{ij} \in I \), which implies there are \( c_{\alpha\beta\gamma\delta} \) and \( c'_{ijk} \in \mathbb{C} \) such that

\[
\sum_{1 \leq i \leq m} \sum_{m < j \leq m + n} c_{ij} e_{ij} = \sum_{0 \leq \gamma < \beta < \delta \leq m + n} c_{\alpha\beta\gamma\delta} \partial(e_{\alpha\beta\gamma\delta}) + \sum_{0 \leq i < j < k \leq m} \sum_{0 \leq i < j < k \leq m + n + 1} c'_{ijk} \partial(e_{ijk}).
\]

Applying \( \partial \) on both sides of the above equation, we have

\[
\sum_{1 \leq i \leq m} \sum_{m < j \leq m + n} c_{ij} (e_{ij} + e_{0i} + e_{0j}) = \sum_{0 \leq i < j < k \leq m} \sum_{0 \leq i < j < k \leq m + n + 1} c''_{ijk} \partial(e_{ijk}),
\]

where \( c''_{ijk} = \sum_{\ell=0}^{m+n} c'_{ijk} \). The above equation can be written as

\[
\sum_{1 \leq i \leq m} \sum_{m < j \leq m + n} c_{ij} e_{ij} = \sum_{1 \leq i \leq m} \sum_{m < j \leq m + n} c_{ij} (e_{i0} + e_{0j}) + \sum_{0 \leq i < j < k \leq m + n + 1} c''_{ijk} (e_{jk} + e_{ki} + e_{ij}).
\]

Since the terms \( e_{ij} \) for \( 1 \leq i \leq m < j \leq m + n \) do not appear in the right hand side of this equation, we conclude that \( c_{ij} = 0 \) for \( 1 \leq i \leq m < j \leq m + n \). Thus \( A_3 = \bigoplus_{1 \leq i \leq m} C_{a_0 a_i a_j} \). \( \square \)
In fact, from the proof of Lemma 6.1, we see that for any given arrangement \( A = \{ H_1, \ldots, H_k \} \) in \( \mathbb{C}^3 \), the ideal of exterior algebra \( E(A) \) is \( I = I_2 \oplus I_3 \) where \( I_2 \) is spanned by the set of \( \partial e_{ijk} \) with dependent \( (H_i, H_j, H_k) \in S \) and \( I_3 \) is spanned by the set of all \( \partial(e_{ijk}) \) and the set of all \( e_3 \partial(e_{a\beta\gamma}) \) with dependent \( (H_a, H_\beta, H_\gamma) \in S \). Thus we have the following proposition.

**PROPOSITION 6.2.** Let \( A = \{ H_1, \ldots, H_k \} \) be a central arrangement in \( \mathbb{C}^3 \). Then the cohomology algebra of the complement of \( A \) in \( \mathbb{C}^3 \) is isomorphic to \( h(A) = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \) where \( A_0 = \mathbb{C} \), \( A_1 = \bigoplus_{i=1}^k \mathbb{C} a_i \), \( A_2 \) is spanned by the set of all \( a_i a_j (i < j) \) with relations \( a_\alpha a_\beta + a_\beta a_\gamma + a_\gamma a_\alpha = 0 \) for dependent \( (H_\alpha, H_\beta, H_\gamma) \in S \) and \( A_3 \) is spanned by \( a_\alpha a_\beta a_\gamma (\alpha < \beta < \gamma) \) subject to the relations \( a_\alpha a_\beta a_\gamma = a_\alpha a_\beta a_\gamma + a_\alpha a_\gamma a_\beta + a_\beta a_\gamma a_\alpha \) and the relations \( a_\chi (a_a a_v + a_v a_w + a_w a_a) = 0 \) with dependent \( (H_a, H_v, H_w) \in S \).

**LEMMA 6.3.** Let \( A \) be a central arrangement in \( \mathbb{C}^3 \). Let \( A^* = \{ \ell_1, \ell_2, \ldots, \ell_n \} \) be the corresponding projective arrangement in \( \mathbb{CP}^2 \). Suppose that \( n = |A^*| \geq 3 \). If \( A^* \) is not a pencil, then \( b_3(M(A)) \), the third Betti number of \( M(A) \), is nonzero.

**PROOF.** We need only to show that \( A_3(A) \) is nonzero, where \( A_3(A) \) is the third graded piece of the Orlik-Solomon algebra. Choose three lines \( \ell_1, \ell_2, \ell_3 \) in general position from \( A^* \). So \( \ell_1, \ell_2, \ell_3 \) do not form a pencil. Then we claim that \( a_{123} \neq 0 \) in \( A_3(A) \). If \( a_{123} = 0 \), that is, \( e_{123} \in I_3 \) where \( I = \bigoplus_{i=0}^3 I_i \) is the ideal generated by \( \partial e_s \) for dependent \( s \), then we would have some \( c_{ijk\ell} \) and \( c_{\alpha\beta\gamma\delta} \in \mathbb{C} \) such that

\[
e_{123} = \sum_{1 \leq i < j < k < \ell \leq n} c_{ijk\ell} \partial(e_{ijk\ell}) + \sum_{\delta, \text{ dependent } (\alpha, \beta, \gamma)} c_{\alpha\beta\gamma\delta} e_3 \partial(e_{\alpha\beta\gamma}).
\]

By acting \( \partial \) on each side of the above equation, we get

\[
e_{12} + e_{23} + e_{31} = \sum_{(\alpha, \beta, \gamma) \text{ dependent}} c_{\alpha\beta\gamma} \partial(e_{\alpha\beta\gamma}),
\]

where \( c_{\alpha\beta\gamma} = \sum_{\delta} c_{\alpha\beta\gamma\delta} \). Observe that each dependent \( (\alpha, \beta, \gamma) \) corresponds to a vertex \( P(\alpha, \beta, \gamma) = \ell_\alpha \cap \ell_\beta \cap \ell_\gamma \) in \( A^* \) and that if \( P(\alpha, \beta, \gamma) \neq P(\alpha', \beta', \gamma') \), then \( \{ \pm e_{\alpha}, \pm e_{\beta}, \pm e_{\gamma} \} \) is disjoint from \( \{ \pm e_{\alpha'}, \pm e_{\beta'}, \pm e_{\gamma'} \} \). Therefore we have

\[
e_{12} = \sum_{\text{dependent } (\alpha, \beta, \gamma), \ell_\alpha \cap \ell_\beta \cap \ell_\gamma = \ell_\gamma \cap \ell_1} c_{\alpha\beta\gamma} \partial(e_{\alpha\beta\gamma}).
\]

By taking \( \partial \) on both sides of the above equation, we get \( e_2 - e_1 = 0 \), which is absurd. \( \square \)
7. – End of the proof of the main theorem

Let \( A_1^* \) and \( A_2^* \) be two arrangements in \( \mathbb{CP}^2 \) and let at least one of them be exceptional. In this section we shall finish the proof of the main theorem. In view of Theorem 5.4, it remains to prove that under the above hypothesis, if \( L(A_1) \) is not isomorphic to \( L(A) \), then \( M(A_1^*) \) is not homeomorphic to \( M(A_2^*) \).

**CASE 7.1.** Both \( A_1^* \) and \( A_2^* \) are exceptional. Write

\[
A_1^* = \{ H_0, H_1, \ldots, H_p, H_{p+1}, \ldots, H_{p+q} \},
\]
\[
A_2^* = \{ G_0, G_1, \ldots, G_s, G_{s+1}, \ldots, G_{s+t} \},
\]

where \( H_0 \) (respectively \( G_0 \)) intersects with \( H_1, \ldots, H_p \) (respectively, \( G_1, \ldots, G_s \)) at one point and intersects with \( H_{p+1}, \ldots, H_{p+q} \) (respectively, \( G_{s+1}, \ldots, G_{s+t} \)) at another point.

If \( M(A_1^*) \) is homeomorphic to \( M(A_2^*) \), then \( M(A_1) \) is homeomorphic to \( M(A_2) \). By Lemma 6.1, we have \( p + q = s + t \) and \( pq = st \) which imply either \( (p, q) = (s, t) \) or \( (p, q) = (t, s) \). Thus \( L(A_1) \) is isomorphic to \( L(A_2) \).

**CASE 7.2.** \( A_1^* \) is exceptional but \( A_2^* \) is not. In this case we shall show that \( M(A_1^*) \) is not homeomorphic to \( M(A_2^*) \) by considering the following subcases.

Case 7.2.a \( A_1^* \) consists of at most three lines.
Case 7.2.b \( A_1^* \) is a pencil, and \( |A_1| \geq 4 \) (fig. 3).

Case 7.2.c \( A_1^* \) consists of a pencil and a line in general position, and \( |A_1| \geq 4 \) (fig. 4).
Case 7.2.d \( A_1^* = \{H_0, H_1, \ldots, H_p, H_{p+1}, \ldots, H_{p+q}\} \) where \( \bigcap_{i=0}^{p} H_i \) and \( H_0 \cap (\bigcap_{i=p+1}^{p+q} H_i) \) are two different nonempty intersections, \( p > 1, q > 1 \) (fig. 5).

From Proposition 6.2 we know that the first Betti number of \( \mathcal{M}(A) \) in \( \mathbb{C}^3 \) is precisely \(|A|\), the number of elements in \( A \). In case of (7.2.a), we have \(|A_1^*| \leq 3 < |A_2^*|\). Thus \( \mathcal{M}(A_1^*) \) is not homeomorphic to \( \mathcal{M}(A_2^*) \).

In case (7.2.b), we have the third Betti number of \( \mathcal{M}(A_1) = 0 \) by Lemma 6.1 while the third Betti number of \( \mathcal{M}(A_2) \) is nonzero by Lemma 6.3. Thus \( \mathcal{M}(A_1^*) \) is not homeomorphic to \( \mathcal{M}(A_2^*) \).

For Case (7.2.c), let \( A_1^* = \{H_0, H_1, \ldots, H_p\} \) \( p \geq 3 \), where \( H_1, \ldots, H_p \) form a pencil and \( H_0 \) is in general position. Let \( G(A_1^*) \) be the weighted graph of \( A_1^* \), and \( K(A_1^*) \) be the manifold that is the boundary of a tubular neighborhood of \( A_1^* \) in \( \mathbb{C}P^2 \). \( G(A_1^*) \) is obtained by blowing up \( \bigcap_{i=1}^{p} H_i \) as an exceptional line \( E \). We denote the vertices by \( G(A_1^*) v, v_0, v_1, \ldots, v_p \) corresponding to \( E, H_0, H_1, \ldots, H_p \), respectively. The pictures of \( A_1^* \) and \( G(A_1^*) \) are shown as in figure 6 and the left-hand side of figure 7, where \((q_i, r_i, s_i)\) is the weight of the vertex \( v_i \) which is defined in Section 4. By the Neuman’s calculus of plumbing in Section 2 of [15], we can reduce \( G(A_1^*) \) to \( G' \) and then to a single vertex graph \( G'' \) with certain weight \((g, 0, 0)\) (fig. 8). Notice that if we apply \( R_4 \) or \( R_5 \) in the \( i \)th step, the weight of \( V + V_0 \) will change from \((g_i, 0, 0)\) to

\[
\begin{array}{c}
E(-1) \\
\hline
H_0(1) \\
\hline
H_1(0) \\
\hline
H_2(0) \\
\vdots \\
\hline
H_p(0)
\end{array}
\]

Fig. 6. Blown-up pencil
(gi \#(-2), 0, 0) (for R4) or (gi \# 1, 0, 0) (for R5), where

\[
g_{i, \#(-2)} = \begin{cases} 
-2 & \text{if } g_i = 0 \\
-2g_i - 2 & \text{if } g_i > 0 \\
g_i - 2 & \text{if } g_i < 0 
\end{cases}
\]

\[
g_{i, \#1} = \begin{cases} 
1 & \text{if } g_i = 0 \\
g_i + 1 & \text{if } g_i > 0 \\
g_i - 2 & \text{if } g_i < 0.
\end{cases}
\]

(R4) \hspace{1cm} (R5)

So if we use only step R5, then we have \( g \geq 2 \), since \( p \geq 3 \). On the other hand, R4 will bring any \( g_i \) down to a number \( \leq -2 \). Once \( g_i \) is negative, both step R4 and R5 will bring \( g_i \) to a smaller number less than \(-2\). Thus we conclude that \( |g| \geq 2 \). The plumbed three-manifold \( M(V + V_0) \) corresponding to \( V + V_0 \) is diffeomorphic to \( K(A^*_1) \) by Proposition 2.1 of [15]. Observe that \( M(V + V_0) \)
is a reduced graph manifold with reduced graph structure $T = \phi$ (one needs to check only the last condition of the definition of reduced graph manifold). and it has the Waldhausen property (cf. Definition 4.2). Now by Theorem 3.12 of Waldhausen (cf. (9.4) of [21]), $K(A_i^+)$ homeomorphic to $K(A_j^+)$ will imply that $G(A_i^+)$ is equivalent to $G''$. Since $G(A_i^+)$ is not equivalent to $G''$, we conclude that $M(A_i^+)$ is not homeomorphic to $M(A_j^+)$.

For the last case (7.2.d), we blow up the points $\bigcap_{i=1}^p H_i$ and $\bigcap_{j=p+1}^{p+q} H_j$ and get exceptional line $E_1$ and $E_2$, respectively (fig. 9). Number in the parentheses is the self-intersection number. We blow down $H_0$ to a point and get figure 10. The graph manifold is then reduced with respect to the graph $G$ (fig. 11), since $p \geq 2$ and $q \geq 2$ in view of Lemma 4.6. Notice that $G$ has weights $(0, 0, 0)$ for all its vertices. If $M(A_i^+)$ were homeomorphic to $M(A_j^+)$, this would imply $K(A_i^+)$ homeomorphic to $K(A_j^+)$ in view of Proposition 4.7, which in turn would
imply $G$ equivalent to $G(A^2_4)$. The last assertion is not possible because $G(A^2_4)$ has nonzero weights. Therefore we conclude that $M(A^2_4)$ is not homeomorphic to $M(A^2_3)$. This finishes the proof of our main theorem.

REFERENCES


2606 Estero Pkwy
Valparaiso, IN
46383
E-mail: tjiang@aoc-resins.com

Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
M/C 249 Chicago, IL 60607-7045, USA
E-mail: u32790@uicvm.bitnet