$s^{(n-1)}$ INVARIANT FOR ISOLATED n-DIMENSIONAL SINGULARITIES AND ITS APPLICATION TO MODULI PROBLEM

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0. Introduction. Let A be a complex analytic variety of codimension at least two in a complex manifold M of dimension n. It is well known that any holomorphic p-form defined on M-A can be extended across A. In this note, we shall discuss the case when A is a complex codimension 1 subvariety in M. Throughout the whole paper, we shall assume A is a connected exceptional divisor in a strongly pseudo convex manifold M. In [21], the invariant $s^{(p)}$ was introduced to measure how many holomorphic p-forms cannot be extended across A. Among these numbers, $s^{(n)}$ and $s^{(n-1)}$ are the most interesting ones. $s^{(n)}$ gives $h^{n-1}(\mathcal{O}_M)$ $(:=\dim H^{n-1}(M, \mathcal{O}_M))$. The following is our main theorem.

THEOREM A. *Let (V, q) be a n-dimensional normal singularity with $\mathbf{C}^*$-action, $\pi: M \to V$ an equivariant resolution whose exceptional set is denoted by A. Then*

(a) $s^{(n-1)} \geq h^{n-1}(M, \mathcal{O}_M) - h^{n-1}(A, \mathcal{O}_A)$.

(b) *If V is Gorenstein and $h^{n-1}(M, \mathcal{O}_M) \geq 2$, then $s^{(n-1)} > 0$.*

As an easy consequence of Theorem A, we can classify Gorenstein surface singularities with $\mathbf{C}^*$-action which has $s^{(1)} = 0$.

COROLLARY B. *(V, q) is a Gorenstein surface singularity with $\mathbf{C}^*$-action and $s^{(1)} = 0$ if and only if (V, q) is either a rational double point or a simple elliptic singularity.*

An immediate consequence of the above Corollary B is the following.

COROLLARY C. *Let (V, q) be a Gorenstein surface singularity with $\mathbf{C}^*$-action. Then (V, q) is not rigid.*

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Our presentation goes as follows. In section 1, we formulate a conjecture about the nonvanishing of $s^{(n-1)}$. This conjecture is very interesting because of its immediate application to the deformation of singularities. Example 1.1 and Example 1.4 show that the conditions in the conjecture are important. Here we also recall some formulas which enable us to calculate $s^{(1)}$ in case of surface singularities. These will be used in Example 2.6 and Example 2.7. In section 2 we prove our conjecture in case (V, q) admits a $\mathbf{C}^*$-action (cf. Theorem 2.1). Then we restrict ourself to a general class of weakly elliptic singularities (cf. [15]) which satisfy a maximality condition [20]. In this case $s^{(2)}$ is the length of the elliptic sequence and $s^{(1)}$ is either the length of the elliptic sequence l or $l - 1$ (cf. Theorem 2.5). Recall that the elliptic sequence (cf. Definition 2.2) is defined in a purely topological way, i.e., it can be computed explicitly via the intersection matrix, however $s^{(1)}$ is an analytic invariant. In case (V, q) does not admit $\mathbf{C}^*$-action, we prove our conjecture is still true so long as (V, q) is a special class of simple hyperbolic singularity (cf. Example 2.8). In section 3 we give a classification of regular Gorenstein surface singularities with $\mathbf{C}^*$-action. We also prove that $\dim T^1_V \geq s^{(1)}$ and that Gorenstein surface singularities with $\mathbf{C}^*$-action are not rigid (cf. Theorem 3.2). In the case of maximally elliptic singularity, we show that $\dim T^1_V$ is bounded below by the length of elliptic sequence.

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1. Preliminaries. Let M be a strongly pseudoconvex manifold of dimension $n \geq 2$ with connected exceptional set A. Let $\pi: M \to V$ be the blow-down of A in M, $q = \pi(A)$. π is then a resolution of the Stein normal n-dimension space V with q as its only singularity. Recall that in [21] we define $s^{(p)}$ of the singularity q to be $\dim_{\mathbf{C}} \Gamma(M\text{-}A), \Omega^p)/\Gamma(M, \Omega^p)$. The fact that $s^{(p)}$ so defined depend actually only on the singularity q can be seen as follows. Let $\bar{\Omega}^p_V$ be the sheaf of germs of holomorphic p-forms of $V - \{q\}$ which are locally L^2-integrable in the sense of Griffiths [3]. Then actually $\bar{\Omega}^p_V$ is equal to 0-th direct image sheaf $\pi_*\Omega^p_M$. Let $\theta: V - \{q\} \hookrightarrow V$ be the inclusion map. Then the 0-th direct image sheaf $\theta_*\Omega^p_{V-\{q\}} = \bar{\bar{\Omega}}^p_V$ is coherent by Siu's Theorem [13]. Clearly we have an inclusion $\bar{\Omega}^p_V \hookrightarrow \bar{\bar{\Omega}}^p_V$. It follows easily that $s^{(p)} = \dim(\bar{\bar{\Omega}}^p_{V,q}/\bar{\Omega}^p_{V,q})$. Among these numbers, $s^{(n)}$ and

$s^{(n-1)}$ are the most interesting ones. The following example is due to Steenbrink [14].

Example 1.1. Let Z be a complex manifold of dimension n, and Γ a properly discontinuous group of automorphisms of X. Then the $s^{(p)}$, $0 \leq p \leq n$, for the singularities in $V = Z/\Gamma$ are equal to zero.

The following theorem was proved in our previous paper [19] ($n = 2$ case is due to Laufer [5]).

THEOREM 1.2. $s^{(n)} = \dim H^{n-1}(M, \mathcal{O})$.

Notice that $\dim H^{n-1}(M, \mathcal{O})$ is an important invariant in the theory of isolated singularities. One can classify singularities in terms of this invariant. For $s^{(n-1)}$, we have the following conjecture.

Conjecture. If M is Gorenstein and $\dim H^{n-1}(M, \mathcal{O}) \geq 2$, then $s^{(n-1)} > 0$.

Definition 1.3. M is Gorenstein if there exists a meromorphic n-form ω such that its divisor (ω) is supported on the exceptional set A.

The following example can be found in [21] or [9].

Example 1.4. Let M be a negative line bundle over a nonsingular compact Riemann surface A. Then

$$s^{(1)} = \dim \Gamma(M\text{-}A, \Omega^1)/\Gamma(M, \Omega^1) = \sum_{n=1}^{\infty} h^0(K_A \otimes M^n)$$

where K_A is the canonical line bundle of A.

In particular if $g \geq 2$ and $A \cdot A$ is very negative, then $s^{(1)} = 0$. Therefore in the above conjecture, the Gorenstein property is important.

The following theorem which expresses $s^{(1)}$ in terms of $h^1(M, \Omega^1)$ was first proved in [21].

THEOREM 1.5. *Let M be a two dimensional strongly pseudoconvex manifold such that $V = \{f(x, y, z) = 0\}$ has origin as its only singularity.*

Let $\mu = \dim \mathbf{C}[x, y, z]/(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ and $\tau = \dim \mathbf{C}[x, y, z]/(f, \partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$. Then

$$s^{(1)} = -\frac{1}{6}(K^2 - 5\chi_T(A)) + \tau - \frac{5}{6}(1 + \mu) - \dim H^1(M, \Omega^1)$$

where $\chi_T(A)$ is the topological Euler characteristic of A and K is the canonical divisor on M.

In practice, the following Theorem (cf. [21]) gives a very explicit way to compute $s^{(1)}$, $s^{(2)}$ in case the singularity admits a $\mathbf{C}^*$-action.

THEOREM 1.6. *Suppose $V \subseteq \mathbf{C}^m$ is an analytic variety of dimension two with the origin as its only isolated singularity. Suppose σ is a $\mathbf{C}^*$-action leaving V invariant, defined by*

$$\sigma(t, (z_1, \ldots, z_m)) = (t^{q_1} z_1, \ldots, t^{q_m} z_m) \qquad q_i\text{'s are positive integers.}$$

Let $\varphi: \mathbf{C}^m \to \mathbf{C}^m$ be defined by $\varphi(z_1, \ldots, z_m) = (z_1^{q_1}, \ldots, z_m^{q_m})$ and let $V^1 = \varphi^{-1}(V)$ be the cone above V. Then V^1 has a natural $\mathbf{C}^*$-action defined by $\sigma^1(t, (z_1, \ldots, z_m)) = (tz_1, \ldots, tz_m)$ and the induced map $\varphi: V^1 \to V$ commutes with the $\mathbf{C}^*$-action. Let $A' = (V^1 - \{0\})/\mathbf{C}^* \subseteq P^{m-1}$. Let N' be the universal subbundle (i.e., dual of the hyperplane bundle) of P^{m-1} restricted to A'. Identify Z_{q_i} with the group of q_i^{th} roots of 1. $G = Z_{q_1} \oplus \cdots \oplus Z_{q_m}$ acts on V^1 by coordinate-wise multiplication. G also acts on A' and N'. Let $\pi: A'' \to A'$ be the normalization and $N'' = \pi^*(N')$, the pull back of N' by π. Then $s^{(1)}$ and $s^{(2)}$ can be computed by the following formulas

$$s^{(1)} = \begin{cases} 0 & \text{if } g'' \leq 1 \\ \sum_{-\infty}^{-1} \dim \Gamma(A'', K_{A''} N''^{-n})^G & \text{if } g'' \geq 1 \end{cases}$$

$$s^{(2)} = \begin{cases} 0 & \text{if } g'' = 0 \\ \dim \Gamma(A'', K_{A''})^G & \text{if } g'' = 1 \\ \sum_{-\infty}^{-1} \dim \Gamma(A'', K_{A''} N''^{-n-1})^G & \text{if } g'' \geq 2. \end{cases}$$

where g'' is the genus of A'', $K_{A''}$ is the canonical line bundle of A'' and $\Gamma(A'', K_{A''} N''^{-n})^G$ denotes the G-invariant sections.

COROLLARY 1.7. *Let (V, q) be a normal surface singularity with $\mathbf{C}^*$-action. Then $s^{(1)} + \dim \Gamma(A'', K_{A''})^G = \dim H^1(M, \mathcal{O}_M)$. In particular $s^{(1)} \leq h^1(M, \mathcal{O}_M)$.*

2. Nonvanishing of $s^{(n-1)}$. Let V be a n-dimensional Stein analytic space with $q \in V$ as an isolated singularity. We say that $q \in V$ admits a $\mathbf{C}^*$-action if there exists an embedding $j: (V, q) \to (\mathbf{C}^m, 0)$ for some m

such that $j(V)$ is closed in $\mathbf{C}^m$ and is invariant under the $\mathbf{C}^*$-action $\tilde{\sigma}$ where $\tilde{\sigma}: \mathbf{C}^* \times \mathbf{C}^m$ is defined by

$$\tilde{\sigma}(t, (z_1, \ldots, z_m)) = (t^{q_1} z_1, \ldots, t^{q_m} z_m), \qquad q_i \text{ an integer.}$$

THEOREM 2.1. *Let (V, q) be a n-dimensional normal singularity with $\mathbf{C}^*$-action, $\pi: M \to V$ an equivariant resolution whose exceptional set is denoted by A. Then*

(a) $s^{(n-1)} \geq h^{n-1}(M, \mathcal{O}_M) - h^{n-1}(A, \mathcal{O}_A)$.
(b) *If V is Gorenstein and $h^{n-1}(M, \mathcal{O}_M) \geq 2$, then $s^{(n-1)} > 0$.*

Proof. Since $q \in V$ admits a $\mathbf{C}^*$-action, there is a natural holomorphic vector field X on V of the form

$$X = q_1 z_1 \frac{\partial}{\partial z_1} + \cdots + q_m z_m \frac{\partial}{\partial z_m}$$

By Hironaka, we can choose an equivariant resolution M to which the Euler vector field X lifts; i.e., there exists a holomorphic vector field θ on M such that $\pi_*(\theta) = X$.

For any irreducible component A_i of A, pick a point $x \in A_i$ such that x is a smooth point of A. Choose local coordinate centered at x such that $u_n = 0$ is a local defining equation for the divisor A. Then locally around x, θ has the following representation:

$$\theta = a_1 \frac{\partial}{\partial u_1} + \cdots + a_n \frac{\partial}{\partial u_n}$$

where a_i are holomorphic functions in $u_1, u_2, \ldots, u_n$. Since

$$\left(a_1 \frac{\partial}{\partial u_1} + \cdots + a_n \frac{\partial}{\partial u_n}\right)(\pi^* z_j) = q_j \pi^*(z_j) \qquad 1 \leq j \leq m$$

there exists at least one a_k such that the vanishing order of a_k along A_i is at most one. Let $\iota(\theta)$ be the inner multiplication by θ. Let ω be a holomorphic n-form on M-A. Suppose that ω does not belong to $\Gamma(M, \Omega^n(A))$. Then there exists an irreducible component A_i of A along which ω has pole of order at least two. In $(u_1, \ldots, u_n)$ coordinate as above, let $\omega = \alpha(u_1, \ldots, u_n) du_1 \wedge \cdots \wedge du_n$. Then

$$\iota(\theta) = \sum_{i=1}^{n} (-1)^{i+1} \alpha a_i du_1 \wedge \cdots \wedge d\hat{u}_i \wedge \cdots \wedge du_n$$

where $d\hat{u}_i$ means du_i is deleted from the expression. By looking at the coefficient of $du_1 \wedge \cdots \wedge d\hat{u}_k \wedge \cdots \wedge du_n$, we conclude that $\iota(\theta)$ is not in $\iota(\theta)H^0(M, \Omega_M^n(A)) + H^0(M, \Omega_M^{n-1})$. Therefore we have the following

LEMMA. *$\iota(\theta)$ induces an injection*

$$H^0(M\text{-}A, \Omega_M^n)/H^0(M, \Omega_M^n(A)) \hookrightarrow H^0(M\text{-}A, \Omega_M^{n-1})/[\iota(\theta)H^0(M, \Omega_M^n(A)) + H^0(M, \Omega_M^{n-1})].$$

Now, the second space has dimension less than or equal to $s^{(n-1)}$, and the first has dimension

$$s^{(n)} - \dim H^0(M, \Omega_M^n(A))/H^0(M, \Omega_M^n) = s^{(n)} - \dim H^0(M, \Omega_M^n \otimes \mathcal{O}_A(A))$$

by the Grauert-Riemenschneider vanishing theorem. Observe that $\Omega_M^n \otimes \mathcal{O}_A(A)$ is the dualizing sheaf ω_A, and $H^0(\omega_A)$ is dual to $H^{n-1}(A, \mathcal{O}_A)$. This proves (a).

From the argument above, it is clear that if there exists a holomorphic n-form on M-A with poles along some irreducible component A_i of A of order at least two, then $s^{(n-1)} > 0$. To prove (b), it suffices to prove the existence of such n-forms under the assumption that $h^{n-1}(M, \mathcal{O}_M) \geq 2$. Since (V, q) is a Gorenstein singularity, there exists a nowhere vanishing holomorphic n-form on M-A. Let ω_1 be an element in $\Gamma(M\text{-}A, \Omega^n)$ such that the image of ω and ω_1 in $\Gamma(M\text{-}A, \Omega^n)/\Gamma(M, \Omega^n)$ are linearly independent. Because $\Gamma(V, \mathcal{O}_V) \to \Gamma(M, \mathcal{O}_M) \to \Gamma(M\text{-}A, \mathcal{O}_M) = \Gamma(V - \{q\}, \mathcal{O}_V)$ is an isomorphism, holomorphic functions on M-A extend over A. So $f := \omega_1/\omega$ is a holomorphic on M. By maximum modulus principle, f is constant on the compact analytic set A. Replacing f by $f - f(A)$ if necessary, we may assume without loss of generality that $f(A) = 0$. Since $\omega_1 = f\omega$ is a nonzero element in $\Gamma(M\text{-}A), \Omega^n)/\Gamma(M, \Omega^n)$, we conclude that ω is the required n-form. Q.E.D.

In [20], we developed a theory for a general class of weakly elliptic singularities which satisfy a maximality condition. These are the so-called maximally elliptic singularities which have minimally elliptic singularities in the sense of Laufer as a special case. The following proposition was proved in ([20], p. 305).

PROPOSITION 2.3. *Let $\pi: M \to V$ be the minimal good resolution of* *a normal two-dimensional Stein space with q as its only maximally elliptic singularity. Let*

$$Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_\ell}, Z_E = Z_{B_{\ell+1}}$$

be the elliptic sequence. Suppose $\dim H^1(M, \mathcal{O}) \geq 2$ *i.e.* $\ell \geq 0$. *Then for any* $0 \leq i \leq \ell$, *there exists* $f_i \in \Gamma(M, \mathcal{O}(-G_i))$ *but* $f_i \notin \Gamma(M, \mathcal{O}(-G_{i+1}))$ *where* $G_i = \Sigma_j Z_{B_j}$, $0 \leq j \leq i$. *In fact the vanishing order of* f_i *on* A_k *is precisely* $\Sigma_r z_k^r$, $0 \leq r \leq i$, *where* $A_k \subseteq B_{i+1}$ *and* $Z_{B_r} = \Sigma_j z_j^r A_j$.

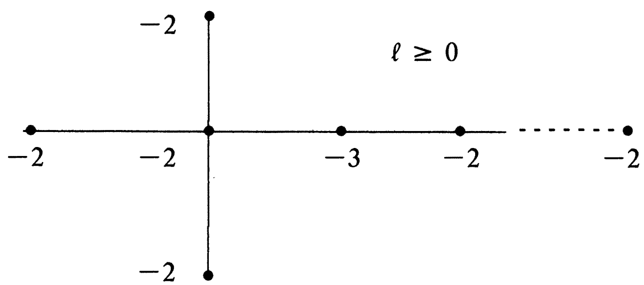
COROLLARY 2.4. *Let V be a normal two-dimensional Stein space with q as its only maximally elliptic singular point. Let $f_0, f_1, \ldots, f_\ell$ be the functions on M as in the above theorem. Then there exists $\omega \in \Gamma(M\text{-}A, \Omega^2)$ such that $\omega, f_0\omega, f_1\omega, \ldots, f_\ell\omega$ form a basis of* $\Gamma(M\text{-}A, \Omega^2)/\Gamma(M, \Omega^2)$.

Proof. This is an easy consequence of the above theorem and Theorem 3.11 of [20].

THEOREM 2.5. *Let (V, q) be a maximally elliptic singularity with* $\mathbf{C}^*$*-action. Then,* $s^{(1)} = \dim H^1(M, \mathcal{O}) =$ *the length of the elliptic sequence if* $\dim H^1(A, \mathbf{R}) = 0$. *If* $\dim H^1(A, \mathbf{R}) \neq 0$, *then* $s^{(1)}$ *is either* $\dim H^1(M, \mathcal{O})$ *or* $\dim H^1(M, \mathcal{O}) - 1$.

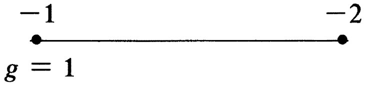
Proof. This follows from the above corollary, the proof of Theorem 2.1 and the fact that if $\dim H^1(A, \mathbf{R}) = 0$, then the minimally elliptic cycle E is not reduced.

Example 2.6. Let $V = \{(x, y, z) \in \mathbf{C}^3 : z^2 = y^3 + x^{9+6\ell}\}$. Then the dual weighted graph 0 of the exceptional set is



As we computed in P. 292 of [20] this is a maximally elliptic singularity and $\dim H^1(M, \mathcal{O}) = \ell + 1$. Since $\dim H^1(A, \mathbf{R}) = 0$ we conclude that $s^{(1)} = \dim H^1(M, \mathcal{O}) = \ell + 1$.

Example 2.7. Let $V = \{(x, y, z) \in \mathbf{C}^3 : x^2 + y^3 + z^{12} = 0\}$. Then the dual weighted graph for the exceptional set is



This is a maximally elliptic singularity and $\dim H^1(M, \mathcal{O}) = 2$. Using the Theorem 1.6, we can compute explicitly that $s^{(1)} = 1 = \dim H^1(M, \mathcal{O}) - 1$. Notice that in this case $H^1(A, \mathbf{R}) \neq 0$.

The following result is the first attempt to solve our conjecture in the case (V, q) does not admit $\mathbf{C}^*$-action.

Example 2.8. Let (V, q) be a two-dimensional Gorenstein singularity. Suppose that the exceptional set A of the minimal resolution $\pi : M \to V$ is a nonsingular compact Riemann surface of genus $g \geq 2$. Suppose $A \cdot A = 2 - 2g$. Then $s^{(1)} > 0$.

Proof. Let $K = -kA$ be the canonical divisor on M. By the adjunction formula, we have

$$-(k - 1)A \cdot A = 2g - 2 \tag{2.1}$$

Since $A \cdot A = 2 - 2g$, it follows from (2.1) that $K = -2A$. Let S be the sheaf of germs of holomorphic vector fields which are tangential to the exceptional set. Then we have

$$0 \to S \to \Theta \to N_A \to 0 \tag{2.2}$$

where Θ is the tangent sheaf of M and N_A is the normal bundle of A. Tensor (2.2) with $\mathcal{O}_A$, we have

$$0 \to \mathcal{O}_A \to S \otimes \mathcal{O}_A \to \Theta \otimes \mathcal{O}_A \to N_A \to 0 \tag{2.3}$$

because $\mathrm{Tor}^1(N_A, \mathcal{O}_A) = \mathcal{O}_A$. Clearly the kernel of the last map is Θ_A where Θ_A is the tangent sheaf of A. Therefore we have

$$0 \to \mathcal{O}_A \to S \otimes \mathcal{O}_A \to \Theta_A \to 0 \tag{2.4}$$

We claim that $\dim H^0(S(-A) \otimes \mathcal{O}_A) - \dim H^1(S(-2A)) \geq g - 1$. We use the exact sequence from (2.4):

$$0 \to \mathcal{O}_A(-A) \to S(-A) \otimes \mathcal{O}_A \to \Theta_A(-A) \to 0 \tag{2.5}$$

Therefore we have

$$\dim H^0(S(-A) \otimes \mathcal{O}_A) \geq \dim H^0(\mathcal{O}_A(-A)) = \begin{cases} g & \text{if } N_A^* = K_A \\ g-1 & \text{if } N_A^* \neq K_A \end{cases} \tag{2.6}$$

where N_A^* and K_A are canonical bundle and canonical bundle of A respectively. Consider the exact sequence

$$0 \to S(-3A) \to S(-2A) \to S(-2A) \otimes \mathcal{O}_A \to 0 \tag{2.7}$$

Let us first recall a theorem of Wahl [16] which says that $\dim H_A^1(S) = 0$. By Serre duality, this simply means that $\dim H^1(S(2K + A)) = \dim H^1(S(-3A)) = 0$ because $S \cong S^* \otimes \Lambda^2 S \cong S^* \otimes K^{-1}(-A)$. Hence we have

$$\dim H^1(S(-2A)) = \dim H^1(S(-2A) \otimes \mathcal{O}_A) \tag{2.8}$$

We again use the exact sequence from (2.4):

$$0 \to \mathcal{O}_A(-2A) \to S(-2A) \otimes \mathcal{O}_A \to \Theta_A(-2A) \to 0$$

Since $2g - 2 + 2A \cdot A = 2 - 2g < 0$, we have $H^1(\mathcal{O}_A(-2A)) = 0$ by Serre duality. Therefore we have

$$\dim H^1(S(-2A)) = \dim H^1(\Theta_A(-2A)) = \begin{cases} 1 & \text{if } N_A^* = K_A \\ 0 & \text{if } N_A^* \neq K_A \end{cases} \tag{2.9}$$

Our claim follows from (2.6) and (2.9). Now from the exact sequence

$$0 \to S(-2A) \to S(-A) \to S(-A) \otimes \mathcal{O}_A \to 0.$$

It follows that there exists a global vector field on M with vanishing order along A is exactly one because the map $H^0(S(-A)) \to H^0(S(-A) \otimes \mathcal{O}_A)$ is not a zero map. The similar argument as in Theorem 2.1 will finish the proof. Q.E.D.

3. Classification of regular Gorenstein surface singularity with C*-action. It is well-known that rational + Gorenstein implies rational double point. In the proof of Theorem 3.1 a simple proof of such fact is included for the sake of completeness. A normal surface singularity is regular if its $s^{(1)}$ is equal to zero.

THEOREM 3.1. *(V, q) is a regular Gorenstein surface singularity with* $\mathbf{C}^*$*-action if and only if (V, q) is either a rational double point or a simple elliptic singularity.*

Proof. Suppose that (V, q) is a regular Gorenstein singularity with $\mathbf{C}^*$-action. Then $\dim H^1(M, \mathcal{O}) \leq 1$ by Theorem 2.1. For the rest of the proof, we shall assume that M is the minimal good resolution of the singularity of V.

Case 1. $\dim H^1(M, \mathcal{O}) = 0$. Then (V, q) is a rational singularity and M is actually the minimal resolution. Let K be a canonical divisor in M. By adjunction formula, we have

$$A_i \cdot K \geq 0 \quad \text{for all} \quad A_i \subseteq A \tag{3.1}$$

Since (V, q) is a Gorenstein rational singularity, we can choose K to be an effective divisor with support on A, i.e. $K = \Sigma\, n_i A_i$, $n_i \geq 0$

$$K^2 = \Sigma\, n_i(A_i K) \geq 0 \tag{3.2}$$

On the other hand, the intersection matrix is negative definite. Therefore $K^2 \leq 0$. It follows that $K^2 = 0$ and consequently $K \cdot A_i = 0$ for all A_i by (3.1) and (3.2). The adjunction formula tells us that $A_i^2 = -2$ for all A_i. Then as an easy exercise, one can show that the weighted dual graph of the exceptional set is one of those from rational double points. By the tautness of rational double points [17], we conclude that (V, q) is a rational double point.

Case 2. $\dim H^1(M, \mathcal{O}) = 1$. Then (V, q) is a minimal elliptic singularity (Theorem 3.10 of [6]). We claim that A is a nonsingular elliptic curve. Suppose on the contrary that A is not a nonsingular elliptic curve. Then all the irreducible components of A are rational curves. Since (V, q) admits a $\mathbf{C}^*$-action, $H^1(A, \mathbf{R}) = 0$. There exists an irreducible component A_i of A such that A_i intersects with three other three distinct components of A. Let ω be a nowhere vanishing holomorphic 2-form on M-A. Then the meromorphic 2-form ω on M has pole along A_i of order at least 2 by adjunction formula. This is because if ω has pole of order one along A_i, then

$A_i \cdot K \leq -A_i^2 - 3$. On the other hand, $A_i \cdot K = -A_i^2 - 2$ by adjunction formula. Now the proof of Theorem 2.1 applies and we conclude that $s^{(1)} > 0$. This is a contradiction. Q.E.D.

Following Schlessinger [11], we define an $\mathcal{O}_V$-module T_V^1 by the exact sequence

$$0 \to \Theta_V \to \Theta_{\mathbf{C}^m}/_V \to N_V \to T_V^1 \to 0.$$

Then T_V^1 is the set of isomorphism classes of first order infinitesimal deformations of V, analogous to $H^1(Y, \Theta_Y)$ for a manifold Y. In [18], Tyurina shows that the T_V^1 may be replaced by $\operatorname{Ext}^1(\Omega_V^1, \mathcal{O}_V)$ (Ω_V^1 denoting Kahler differentials) when V has positive depth along singular locus, e.g. when V is reduced of positive dimension. In [1] Grauert constructs a versal deformation $X \to S$ of V from which every other deformation $W \to T$ may be induced, up to isomorphism, by a map $\phi : T \to S$, with $\phi^*(X) \cong W$. Moreover, the map $t_\phi : t_T \to t_S$ between Zariski tangent spaces is uniquely determined by the isomorphism class of W. As Grauert shows, the Zariski tangent space of S is isomorphic to T_V^1.

V is rigid when every deformation is trivial, or S is reduced to a point. Thus, $T_V^1 = 0$ is the necessary and sufficient condition for rigidity.

In [11] Schlessinger proves that quotient singularities of dimension ≥ 3 are rigid. It is a long standing conjecture that there is no rigid normal surface singularity. The normality condition is important because the singularity obtained by taking two planes in $\mathbf{C}^4$ which meet at a point is rigid.

THEOREM 3.2. *Let (V, q) be a Gorenstein surface singularity with $\mathbf{C}^*$-action. Then (V, q) is not rigid.*

Proof. $\dim T_V^1 = \dim \operatorname{Ext}^1_{\mathcal{O}_V}(\Omega_V^1, \mathcal{O}_V) = \dim H^1_{\{q\}}(V, \Omega_V^1)$ (by local duality). By Grothendieck's local cohomology exact sequence

$$0 \to H^0_{\{x\}}(V, \Omega_V^1) \to H^0(V, \Omega_V^1) \to H^0(V - \{q\}, \Omega_V^1) \to H^1_{\{q\}}(V, \Omega_V^1) \to 0$$

we conclude that $\dim T_V^1 = \dim H^0(V - \{q\}, \Omega_V^1)/H^0(V, \Omega_V^1) \geq s^{(1)}$. Hence we only need to consider the case $s^{(1)} = 0$. In this case (V, q) is a rational double point or a simple elliptic singularity by Theorem 3.1. For either of these cases, $T_V^1 \neq 0$ as deformation theory for these singularities are well developed. Q.E.D.

Added to the proof. We have proved that

$$\dim T_V^1 \geq 1 + \dim H^1(M, \mathcal{O}) \tag{*}$$

for Gorenstein surface singularities with $\mathbf{C}^*$-action (cf. the forthcoming paper [23]). This improves the above Corollary 3.3 by a great deal.

Recently J. Wahl has informed us that he has proved that normal surface singularities with $\mathbf{C}^*$-action must have $\dim T_V^1 > 0$, which is a consequence of (*).

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