1 Right Angled Coxeter Groups (RACGs) and the Davis complex

1.1 Definition and Examples

Definition 1. *If G is a simplicial graph on a set of S elements, then* $W = \langle s \in S : st = ts \text{ if } (s,t) \in E(G) \rangle$ *is a* right angled Coxeter group*.*

EXAMPLES

- If G is the complete graph on S, then $W = (\mathbb{Z}/2\mathbb{Z})^{|S|}$.
- If G has no edges, then $W = \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$, the free product of $|S|$ copies of $\mathbb{Z}_2$.
- (Geometric example) Let P be a right angled (normal vectors to faces, after translating to the origin, are orthogonal)polyhedron (the convex hull cut out by k hyperplanes) in $\mathbb{R}^n$ or $\mathbb{H}^n$. Let $W = \langle s_H : H$ hyperplanes through codim 1 facets $\rangle$. Then W is a RACG, where each s_H represents a reflection.

Exercise: The class of RACGs is stable under direct products and free products.

1.2 Geometric Representation

We define a bilinear form on $\mathbb{R}^{|S|}$ by $B(e_s, e_t) = \begin{cases} 1 & s = t \\ 0 & (s,t) \in E(G) \\ -1 & \text{else} \end{cases}$, where e_s, e_t denote the usual basis elements.

Set $\delta_s : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$ by $U \mapsto U - 2B(u, e_s)e_s$. This sends e_S to -1, e_t to e_t if there is an edge between them, and e_t to $e_t + 2e_s$ if there is not [this seems weird...]

If $(s,t) \in E(G)$, then $[\delta_s, \delta_t] = 1$. So $s \mapsto \delta_s$ gives a representation from W to $GL(\mathbb{R}^{|S|})$.

Theorem 1. *(Tits) This representation is faithful.*

From this theorem and the next, we derive the corollary.

Theorem 2. *(Mol'cov) Every finitely generated subgroup of $GL(n;\mathbb{R})$ is residually finite.*

Corollary 1. *Every RACG is residually finite.*

Recall that

Definition 2. *A group G is* residually finite *if these equivalent conditions hold:*

1. *For all $g \neq 1$, there exists a finite quotient $p : G \to \bar{G}$ such that $p(g) \neq 1$.*
2. *For all finite subsets $F \subset G$, there exists a map $p : G \to \bar{G}$, where $\bar{G}$ is finite, such that p is injective on F.*

1.3 The Davis Complex

Definition 3. *The* Cayley graph *of W with respect to S, Cayley(W, S) is built as such: the vertex set is W, and there is one nonoriented edge between (w, ws).*

EXAMPLES

- G is a vertex. Then Cayley(W, S) is a single edge, with one vertex labeled 1 and another labeled s.
- G is a single edge between s, t. Then Cayley(W, S) is a square with corners $1, s, t, st = ts$.
- G is a set of vertices with no edges. Then Cayley(W, S) is an infinite tree with valence $|S|$.

Lemma 1. *Suppose we have a subset $T \subset S$ such that for all $t_1, t_2 \in T$, there is an edge between them in G (that is, T is a clique in G). Let $W_T = \langle t : t \in T\rangle \subset W$. Then W_T is embedded and Cayley$(\mathbb{Z}_2^{|T|}, T) \to$Cayley$(W, S)$ is an embedding.*

Sketch of proof: There's a natural map of $(\mathbb{Z}_2)^{|T|} \to W_T$. There's also a retraction of W to $(\mathbb{Z}_2)^{|T|}$ by $t \mapsto t, u \neq t \mapsto 1$. Now $W_T \subset W$ is easy, while following the retraction and natural map shows that it's an embedding. [thought I understood this but I guess not].

The lemma implies that there's a lot of cubes in any Cayley graph of a RACG.

Finally, we iteratively construct the CW-*Davis complex* of a RACG.

$D^{(1)}$ is Cayley(W, S).

$D^{(2)}$: for each edge $(s, t) \in E(G)$, for each $g \in W$, add a square along the boundary 4-cycle $(g, gs, gst = gts, gt)$.

$D^{(n+1)}$: for each $(k + 1)$ clique in G, add in copies of $[0, 1]^{n+1}$, with one for each $g \in W$.

2 CAT(0) cube complexes

Definition 4. *A* cube complex X *is a CW complex build inductively.*

$X^{(0)}$ *is a discrete set, while* $X^{(1)}$ *is a graph on* $X^{(0)}$*. To be precise,* X^1 *is* X^0 *with attached 1-cells, which are copies of* $[0, 1]$*, attached by* $\partial[0, 1] = \langle 0, 1\rangle \to X^0$.

Now $X^{(k+1)}$ *is* $X^{(k)}$ *with attached* $k+1$*-cells,* $[0, 1]^{k+1}$*, which are attached by combinatorial maps.*

We implicitly defined cubes above

TO BE FINISHED ALMOST THERE there's like 20 % of the lecture left.