Substitutions, Bratelli Diagrams, and Dynamics

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Introduction

- We study generalized minimal dynamical systems on Cantor sets.
- An Ordered Bratteli Diagram is a combinatorial graph model for the dynamics of a topological dynamical system.
- We will discuss OBDs in the context of substitution dynamical systems, a type of Cantor minimal system.
- Goal of research is to develop combinatorial invariants for generalized dynamical systems.

General Setting

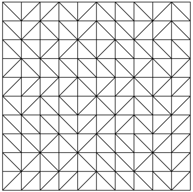
- Let X be a Cantor Set. i.e., any set homeomorphic to the middle thirds cantor set, or any set that is compact, totally disconnected, and perfect.

- Let Γ be a finitely generated discrete group.
- An action $\varphi \colon \Gamma \times X \to X$ defines a generalized dynamical system.
- For $\Gamma = \mathbb{Z}$, the action is generated by one homeomorphism, $\phi(1) \colon X \to X$, so corresponds to traditional dynamical system.
- The action is *minimal* if the orbit of every $x \in X$ is dense.

Examples

Cantor minimal systems arise in many contexts.

- A tiling of the line $\mathbb{R}$ or the plane $\mathbb{R}^2$ yields a generalized dynamical system on the space of tilings Ω defined by the closure of its translates.



T_1

- A tiling which is *repetitive*, *aperiodic*, and has *finite local complexity* implies that Ω is locally homeomorphic to a disk in $\mathbb{R}^n$ times a Cantor set, and is modeled by a minimal Z^n action on a Cantor set.
- Williams solenoids (invariant sets for expanding Axiom A attractors) are modeled by Cantor minimal systems.

Substitution Systems

A substitution system is a particular example of a minimal Cantor system. A *substitution dynamical system* is constructed as follows:

- Let $\mathcal{A}$ be an *alphabet*; a finite set of letters.
- A *substitution* σ is a map from $\mathcal{A}$ to the set of finite words on $\mathcal{A}$.
- Example: $\mathcal{A} = \{a, b\}, \sigma(a) = a, \sigma(b) = ab$

Substitution Systems

- We consider infinite sequences of the form $\{x_k\} = ...x_{-2}x_{-1}.x_0x_1x_2... \in \mathcal{A}^{\mathbb{Z}}$.
- Let $X_\sigma \subset \mathcal{A}^{\mathbb{Z}}$ be the closure of the set $\{\{x_k\}|$ every finite subword of $\{x_k\}$ is contained in $\sigma^n(a)$ for some $a \in A$ and some n. $\}$
- Let T_σ be the shift map on X_σ: $(Tx)_n = x_{n+1}$
- Then (X_σ, T_σ) is a substitution dynamical system.
- Let $\mathcal{A} = \{a, b\}, \sigma(a) = a, \sigma(b) = ab$.
- Iterating $\sigma(b)$ yields $ab, aab, aaab, aaaab, ...$
- Then X_σ consists of all sequences that have no repeated b's.

Proper Substitutions

- A substitution is *proper* if there exists $n \in \mathbb{N}$ and letters $l, r \in \mathcal{A}$ such that $\sigma^n(a)$ starts with l and ends with r for each $a \in A$.

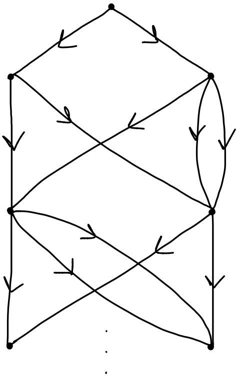
E.g.: $\mathcal{A} = \{a, b\}$, $\sigma(a) = abb, \sigma(b) = abab$.

- For example, as we iterate the above σ we see that $\sigma^2(a) = abbababababab$ and $\sigma^2(b) = abbababbabbabab$. We see that each one starts with a and ends with b.

- As we iterate we continue to see this behavior, so this example is a proper substitution.

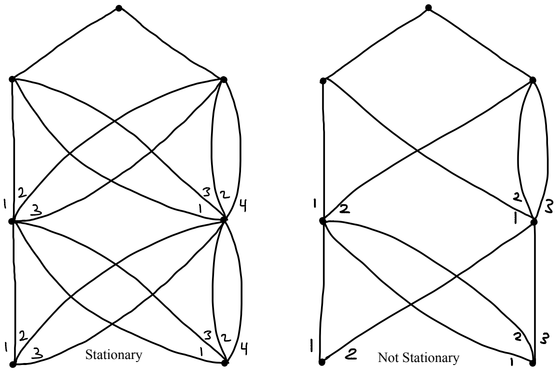
Bratteli Diagrams

- A *Bratteli Diagram* $\mathcal{B}$ is an infinite directed graph with one vertex at the top level, a finite number of vertices at each subsequent level, and edges all directed downwards.



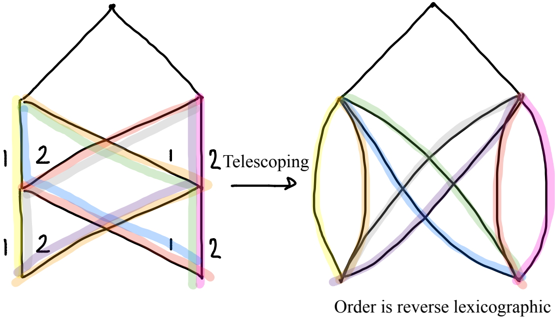
Ordered Bratteli Diagrams

- An *Ordered Bratteli Diagram* (OBD) is a Bratteli Diagram that comes equipped with an order on the edges ranging at each vertex. This induces a (lexicographic) order on the entire space X of infinite paths in $\mathcal{B}$.
- Here we will consider only *stationary* ordered Bratteli diagrams, which are those that repeat after the first level.



Contraction or Telescoping

- *Contraction* or *telescoping* induces an equivalence relation on Bratteli diagrams:



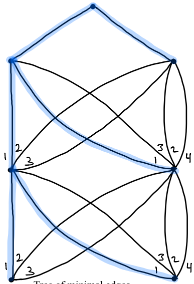
- These two diagrams yield equivalent systems.
- Telescoping an OBD that yields a substitution system is equivalent to replacing the substitution σ with σ^n (in this picture n=2).

Properly ordered OBD

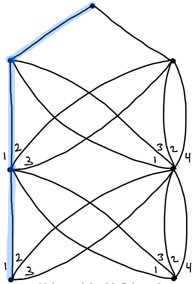
- An OBD is *properly ordered* if there is a unique maximal path and a unique minimal path.

- We can identify whether a diagram is properly ordered by highlighting the tree of all edges that are minimal (resp maximal) and checking that it has a unique infinite path.

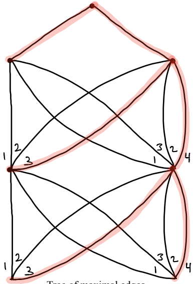
Proper ordering



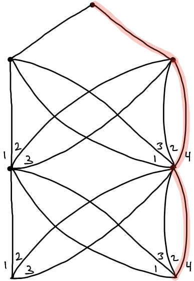
Tree of minimal edges



Unique minimal infinite path



Tree of maximal edges



Unique maximal infinite path

- An OBD is *simple* if, possibly via telescoping, there is a path from any vertex to any other vertex below it.
- We will consider only simple, properly ordered OBDs here.

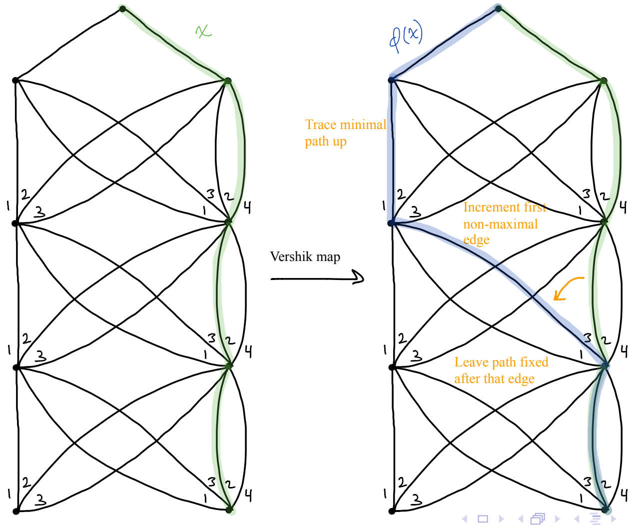
Vershik map

- Given an OBD B, let X be the space of all infinite paths in B
- The topology on X is that given by agreement up to some level n.

  We define the *Vershik map* ϕ on X in the following way:

Vershik map

- The unique maximal path in X is mapped to the unique minimal path.
- For $x \in X$ not maximal:

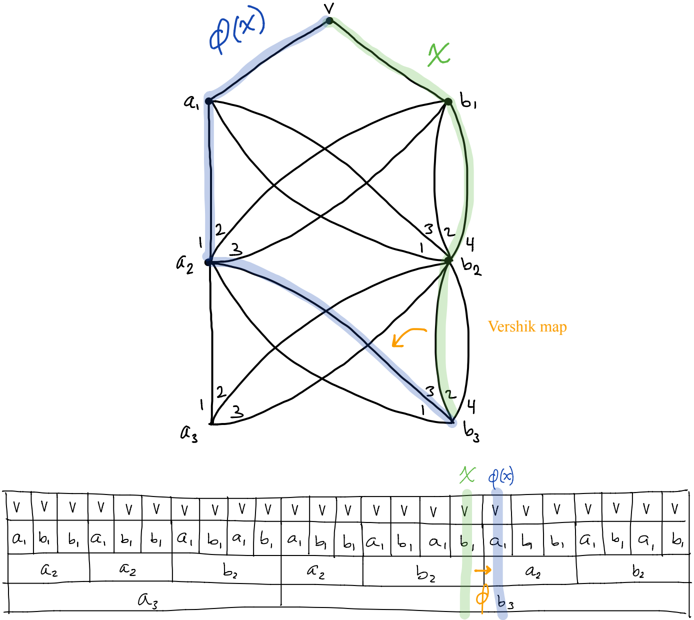


- X under this topology is a Cantor set
- ϕ is a homeomorphism on X
- (X, ϕ) is a dynamical system.

Vershik transform is a shift map

- We can also represent the diagram as an array of boxes (Downarowicz and Maass).
- On this picture the Vershik transform is actually a shift map on the space of infinite paths, here represented as columns.

Vershik transform is a shift map

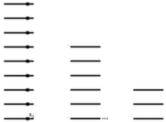


Dynamics of the OBD

Theorem (Herman, Putnam, Skau)

Equivalence classes (under telescoping) of minimal, simple, properly ordered OBDs are in bijective correspondence with pointed topological conjugacy classes of minimal compact zero-dimensional systems.

- Given any minimal dynamical system (X, ϕ, y), we can build an OBD that is conjugate to it.
- The idea of the proof is to construct the OBD using Kakutani-Rokhlin partitions.



- For a substitution system, the K-R partitions needed are

$$\mathcal{P}_n = \{T^k(\sigma^n([a]))|a \in A, 0 \leq k \leq |\sigma^n(a)|\}$$

(Where $[a]$ is the set of sequences that start with a)

Stationary OBDs and Substitution Systems

Theorem (Durand, Host, Skau)

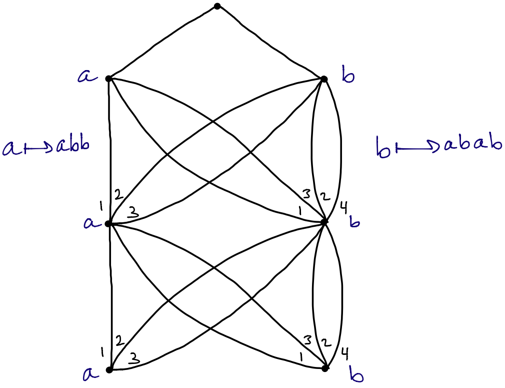
A stationary, properly ordered Bratteli diagram yields a system that is conjugate either to a substitution dynamical system associated to some proper substitution, or to an odometer (the case where there is only one vertex at each level).

Given a substitution system, we can build the corresponding OBD in the following manner:

- For $n \geq 1$, place $|\mathcal{A}|$ number of vertices, labelled by the letters in $\mathcal{A}$.
- Let the edges ranging at each $a \in \mathcal{A}$ be given by $\sigma(a)$, numbered by the order in which those letters appear in $\sigma(a)$.

Stationary OBDs and Substitution Systems

- The substitution system given by $A = \{a, b\}$, $\sigma(a) = abb, \sigma(b) = abab$ corresponds to the diagram:



- This process can be reversed: given a stationary OBD we can read off the substitution.

Further investigations

- *A Bratteli diagram for commuting homeomorphisms of the Cantor set*, by Alan Forrest, Internat. J. Math., 2000, shows how to build an OBD for a minimal action by $\mathbb{Z}^n$ instead of $\mathbb{Z}$, giving one higher dimension extension of this idea.
- *Shape of matchbox manifolds*, by Alex Clark, Steve Hurder and Olga Lukina, preprint, 2013, constructs Markov-Rokhlin towers for a minimal action of an arbitrary fintely-generated group Γ acting on a Cantor set, and thus provides OBD models for general actions.
- We wish to understand properties of the OBDs associated to higher dimensional systems, and other types of groups, and when they admit POBD's. These diagrams are a useful tool for studying dynamical systems, and there are more questions to pursue.

References

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