Describing groups

Julia F. Knight

University of Notre Dame

Elementary first order formulas

The language of groups has symbols $*$ for the group operation, $^{-1}$ for inverse, and e for the identity. For Abelian groups, we may use $+$ for the group operation, $-$ for inverse, and 0 for the identity.

Elementary first order formulas are the usual formulas, finitely long, and with quantifiers over elements, not sets.

Sentences are formulas that describe the whole structure. For example, $(\forall x)(\forall y)x * y = y * x$ says of a group that it is Abelian. Other formulas describe properties or relations on elements. For example, $(\exists y)y + y + y = x$ says that x is divisible by 3.

The elementary first order theory of G, $Th(G)$, is the set of elementary first order sentences true in G.

Free groups

Tarski. Do the non-Abelian free groups have the same elementary first order theory?

Sela. Yes.

Kharlampovich-Myasnikov

Infinitary formulas

The formulas of $L_{\omega_1\omega}$ are built up from the basic formulas (equalities) using countably infinite disjunctions and conjunctions, in addition to the usual logical connectives and quantifiers.

We classify formulas formulas of $L_{\omega_1\omega}$ according to the number of alternations of $\bigvee\!\!\!\!\bigvee/\exists$ and $\bigwedge\!\!\!\!\bigwedge/\forall$.

Scott Isomorphism Theorem. If $\mathcal{A}$ is a countable structure (for a countable language L), there is a sentence of $L_{\omega_1\omega}$ whose countable models are exactly the isomorphic copies of $\mathcal{A}$.

Such a sentence is called a *Scott sentence*.

Computable infinitary formulas

Computable infinitary formulas are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and conjunctions are restricted to computably enumerable sets. These formulas are comprehensible, even though they are infinitely long.

Examples

1. To say of a group that it is finitely generated, we write the computable Σ_3 sentence

   $$\bigvee\!\!\!\!\bigvee_n (\exists x_1, \ldots, x_n)(\forall y) \bigvee\!\!\!\!\bigvee_w w(x_1, \ldots, x_n) = y$$

2. To describe p-groups, we write the computable Π_2 sentence

   $$(\forall x) \bigvee\!\!\!\!\bigvee_n \underbrace{x * \ldots * x}_{p^n} = e$$

Computable structures

A structure is *computable* if we can effectively determine the truth of the basic statements (equalities).

Not all computable structures have computable infinitary Scott sentences, but many do.

Scott sentences for free groups

Carson-Harizanov-K-Lange-McCoy-Morozov-Quinn-Safranski-Wallbaum. For $\mathbb{F}_n$, there is a Scott sentence of the form φ & ψ, where φ is Σ_2 and a ψ is computable Π_2 sentence. We say that the Scott sentence is *computable d*-Σ_2.

Moreover, this is best possible.

McCoy-Wallbaum. For $\mathbb{F}_\infty$, there is a Scott sentence that is computable Π_4.

Moreover, this is best possible.

Results of Nielsen

Let $a_1, a_2, \ldots, a_n$ be a basis for $\mathbb{F}_n$.

Theorem (Nielsen). Any basis is obtained by finitely many steps of the following kinds:

1. replace a_i by a_i^{-1}
2. permute the a_i
3. replace a_i by $a_i a_j$, for some $j \neq i$

Definition. An n-tuple of words $\overline{w}(\overline{x})$ is *primitive* if $\overline{w}(\overline{a})$ is obtained from $\overline{a}$ by a finite sequence of Nielsen transformations; otherwise, it is *imprimitive*.

Theorem (Nielsen). We can effectively determine whether a given tuple of words is primitive.

Scott sentence for $\mathbb{F}_n$

In Carson, *et al, the Scott sentence for* $\mathbb{F}_n$ *is the conjunction of the following:*

1. *a computable* Π_2 *sentence saying that for* $m > n$*, every* m*-tuple is generated by an* n*-tuple,*
2. *a computable* Σ_2 *sentence saying that there is an* n*-tuple* $\overline{x}$ *s.t.*
   (a) $\overline{x}$ *satisfies no non-trivial relations,*
   (b) *for any* n*-tuple* $\overline{y}$ *and any imprimitive* n*-tuple of words* $\overline{w}$*,* $\overline{w}(\overline{y}) \neq \overline{x}$*.*

How to show that a computable infinitary Scott sentence is optimal

Let $I(\mathcal{A})$ be the set of computable indices (programs for computing) isomorphic copies of $\mathcal{A}$.

Thesis. For a computable structure $\mathcal{A}$, the complexity of the Scott sentence matches that of the "index set" $I(\mathcal{A})$, where this is a set of numbers representing programs that compute isomorphic copies of $\mathcal{A}$.

If there is a computable Π_4 Scott sentence, then the index set is "Π^0_4". If the index set is "on top" among the Π^0_4 sets, then there is no simpler Scott sentence.

Other finitely generated groups

Observation (K-Saraph). If G is a computable group with generators $\overline{a}$, then G has a computable Σ_3 Scott sentence saying that there exists $\overline{x}$ s.t.

1. $\overline{x}$ satisfies the equalitites and inequalities true of $\overline{a}$,
2. $(\forall y) \bigvee\!\!\!\bigvee_w w(\overline{x}) = y$.

Groups G that resemble $\mathbb{F}_n$

Proposition (K-Saraph). Suppose G is a computable group with presentation $\langle \overline{a}, R \rangle$, where $\overline{a}$ has size n, minimal. Suppose

1. all n-tuples of generators are Nielsen equivalent,
2. $Imp(R)$ is c.e., where $Imp(R)$ consists of the n-tuples of words $\overline{w}(\overline{y})$ *not* equivalent modulo $R(\overline{w}(\overline{y}))$ to any primitive tuple $\overline{v}(\overline{y})$.

Then G has a Scott sentence that is computable d-Σ_2.

Scott sentence for G

Our Scott sentence is the conjunction of the following:

1. a computable Π_2 sentence saying that every m-tuple is generated by an n-tuple,
2. a computable Σ_2 sentence saying that there is an n-tuple $\overline{x}$ s.t.
   (a) $\overline{x}$ satisfies the equalities and inequalities true of $\overline{a}$,
   (b) for $\overline{w}(\overline{y}) \in Imp(R)$, $\overline{w}(\overline{y}) \neq \overline{x}$.

Finitely generated Abelian and dihedral groups

Saraph. For any finitely generated Abelian group, there is a computable d-Σ_2 Scott sentence.

Saraph. The di-hedral group D_∞ has a computable d-Σ_2 Scott sentence.

Torsion-free Abelian groups of finite rank

Definition. A *torsion-free Abelian group* is a subgroup G of a $\mathbb{Q}$-vector space. The *rank* is the least dimension of the vector space in which G can be embedded.

Observation (K-Saraph). Every torsion-free Abelian group of finite rank has a computable Σ_3 Scott sentence.

Rank 1

Let G be a torsion-free Abelian group of rank 1—a subgroup of $\mathbb{Q}$. Fix an element to play the role of 1.

$$\text{Let}\quad \begin{array}{l} P^0 = \{p_n : G \models p_n \nmid 1\} \\ P^{fin} = \{p_n : (\exists k) G \models p_n^k \mid 1 \ \& \ p_n^{k+1} \nmid 1\} \\ P^\infty = \{p_n : (\forall k) G \models p_n^k \mid 1\} \end{array}$$

Proposition (K-Saraph). If P^{fin} has an infinite computable subset, then the computable Σ_3 Scott sentence is best possible.

Proposition (K-Saraph). If P^0 consists of a single element p, and P^∞ consists of all other primes, then there is a d-Σ_2 Scott sentence, and this is best possible.

Questions

Question: For a computable finitely generated group G, is there always a computable d-Σ_2 Scott sentence? Could the computable Σ^0_3 Scott sentence ever be optimal?

Question: We have a computable subgroup G of $\mathbb{Q}$ such that the "index set" is d-Σ^0_2, but we have not been able to find a computable d-Σ_2 Scott sentence. Is this a counter-example to the thesis that for a computable group, the complexity of the index set should match that of an optimal Scott sentence?

For this G, $P^{fin} = \varnothing$, and P^∞ is "low" c.e. but not computable.

Limiting density

Let $N(n,t)$ be the number of groups on n generators, with 1 relator of length t. Let $P(n,t)$ be the number of these groups in which all n-tuples of generators are Nielsen equivalent.

Kapovich-Schupp. $\lim_{n\to\infty} \frac{P(n,t)}{N(n,t)} = 1$

For a sentence φ, let $N_\varphi(n,t)$ be the number of groups on n generators with 1 relator of length t in which the sentence φ is true.

Conjecture. For all φ, and for all $n \geq 2$, $\lim_{t\to\infty} \frac{N_\varphi(n,t)}{N(n,t)}$ has value 0 or 1. Moreover, the value is 1 iff $\varphi \in Th(\mathbb{F}_2)$.