On the Character Degree Simplicial Complex of a Finite Solvable Group

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Definitions

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- Vertices (0-simplices) are the members of X.
- Edges (1-simplices) between all distinct $x, y \in X$ satisfying $(x, y) > 1$.
- The *dimension* of a simplicial complex $\mathcal{X}$ is one less than the size of the largest simplex.

Example

Consider the set $X = \{2, 3, 5, 11, 10, 22, 15, 33\}$.

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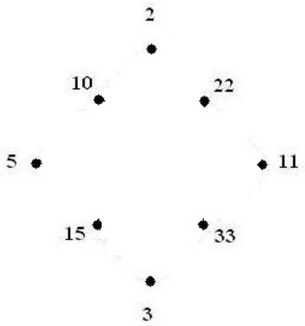

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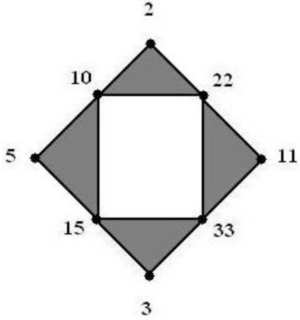

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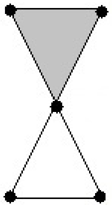

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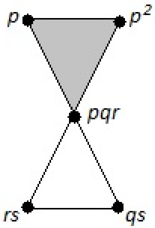

Characters and Representations

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- *A representation of a finite group* G *is a homomorphism* $\rho : G \to \mathrm{GL}_n(\mathbb{C})$.
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NOTE: By including $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{C})$ into $\mathrm{GL}_{n+m}(\mathbb{C})$, we see that we can add representations and therefore characters.

A character χ of G is said to be *irreducible* if it cannot be written as a sum of other characters. For a finite group G, the set of irreducible characters of G is denoted $\mathrm{Irr}(G)$.

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We are particularly interested in the common divisor complex of $\text{cd}(G) \setminus \{1\}$ where G is a finite group. We call this the *character degree simplicial complex*, and denote this by $\mathcal{G}(G)$.

History

In classical character theory, one begins with a group G and derives properties satisfied by $\mathrm{cd}(G)$.

e.g. $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$, or $\chi(1)$ divides $|G|$ for $\chi \in \mathrm{Irr}(G)$.

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Recently ($\sim$ 1970), begin with a set X of positive integers and derive properties of any finite group G satisfying $\text{cd}(G) = X$.

More generally, begin with a graph $\mathcal{X}$ and derive properties of any finite group G having character degree graph $\mathcal{X}$.

Theorem (Manz '85)

Suppose G is a finite solvable group. Then the character degree graph of G has at most two connected components.

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Isaacs pointed out that by only considering the character degree graph of G that we are losing information.

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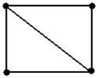

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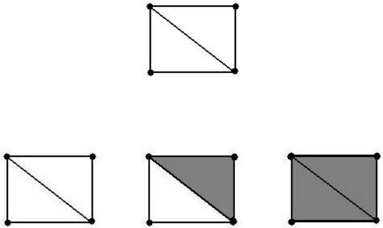

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Theorem (Benjamin '97)

Let G be a nonabelian finite solvable group with $\dim(\mathcal{G}(G)) = n$. *Then*

$$|\mathrm{cd}(G)| \leq \begin{cases} 3 & \text{if } n = 0 \\ 6 & \text{if } n = 1 \\ 9 & \text{if } n = 2 \\ n^2 + n + 2 & \text{if } n \geq 3 \end{cases}$$

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Suppose G is a nonabelian finite solvable group with $\dim(\mathcal{G}(G)) = n$. What other constraints can be placed on the structure of $\mathcal{G}(G)$ in terms of n?

The Fundamental Group

We chose to examine the fundamental group of $\mathcal{G}(G)$, which we denote by $\pi_1(G)$. For this, we assume

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The fundamental group of simplicial complexes is very well understood. Our $\pi_1(G)$ will just be the free product of some number of copies of $\mathbb{Z}$. We call the number of copies of $\mathbb{Z}$ that appear in this free product the *rank* of $\pi_1(G)$.

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Benjamin's result implies that there is a bound on $\text{rk}(\pi_1(G))$ in terms of the dimension of $\mathcal{G}(G)$.

That is, for $n \geq 3$ we have $|\text{cd}(G)| \leq n^2 + n + 2$, which implies a bound on $\text{rk}(\pi_1(G))$ on the order of n^4.

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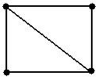

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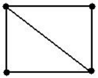

Benjamin's bound does not exclude this simplicial complex, but our result does.

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Theorem

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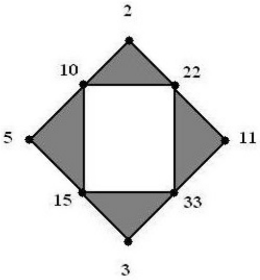
Suppose A is a solvable group with $\mathrm{cd}(A) = \{1, 2, 3\}$ and B a solvable group with $\mathrm{cd}(B) = \{1, 5, 11\}$. The group $A \times B$ has $\mathrm{cd}(A \times B)$ represented by



Main Ideas

Suppose X is a set of integers and $\Omega \subseteq X$ with both $\mathcal{G}(X)$ and $\mathcal{G}(\Omega)$ path connected. There is a natural inclusion map $i : \mathcal{G}(\Omega) \to \mathcal{G}(X)$ inducing a homomorphism $i_\star : \pi_1(\Omega) \to \pi_1(X)$.

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Lemma

Assume Ω and X have the properties above, and assume that $|\Omega| > 1$. Then for all $x \in \Omega$ and $y \in X$ satisfying $\pi(x) \subseteq \pi(y)$, we have that $i'_\star : \pi_1(\Omega') \to \pi_1(X)$ is surjective for $\Omega' = (\Omega \setminus \{x\}) \cup \{y\}$.

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Corollary

Suppose Ω and X have the properties above and that $|\Omega| > 1$. If $x, y \in \Omega$ are distinct, then $\pi(x) \nsubseteq \pi(y)$.

Idea of Proof

Suppose $x \in \Omega$ and $y \in X$ with $\pi(x) \subseteq \pi(y)$. Want to look at the set $(\Omega \setminus \{x\}) \cup \{y\}$.

A *path* is a sequence of vertices $\{x_1, \ldots, x_n\}$ such that there is an edge between x_i and x_{i+1} for all $1 \leq i < n$.

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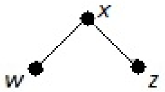

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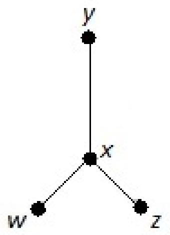

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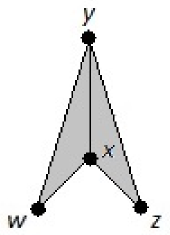

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Thank you!