Complex Dynamics: Existence of attracting domains

Sara Lapan
University of Michigan

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Introduction

Definition

Let $f : U \to \mathbb{C}^m$ be a holomorphic map such that:

- *$U \subseteq \mathbb{C}^m$ is open, and*
- *f fixes a point $p \in U$ (i.e. $f(p) = p$).*

*This is a **(discrete) holomorphic local dynamical system**.*

$\text{End}(\mathbb{C}^m, p)$ denotes the set of all (discrete) holomorphic local dynamical systems at p.

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Set-Up

Let $f \in \text{End}(\mathbb{C}^m, O)$ be defined on U. Near the origin:

$$f(z) = P_1(z) + P_k(z) + P_{k+1}(z) + \ldots,$$

where:

- P_j is a homogeneous polynomial of degree j and,
- if f is non-linear, $P_k \not\equiv 0$ and k is the **order** of f at the origin.

The n-th iterate of f is f^n and the n-th iterate of $z \in U$ is z_n;

$$f^n := f \circ f^{n-1} = \underbrace{f \circ \cdots \circ f}_{n-times} \quad \text{and} \quad z_n := f^n(z).$$

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The set of all points on which f and its iterates are defined is the ***stable set*** K_f ***of*** f*:*

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Guiding Questions

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   In particular, is f holomorphically conjugate to a simpler looking map?

2. *What happens to points near the origin upon iteration by f?*
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First we address these questions in dimension one.

Dimension One: Types of Fixed Points

For $f \in \text{End}(\mathbb{C}, 0)$, near the origin:

$$f(z) = a_1 z + a_k z^k + \dots.$$

- If $|a_1| \neq 0, 1$, then the origin is **hyperbolic** and:
  - If $0 < |a_1| < 1$, then the origin is **attracting**.
    Ex. $f(z) = \frac{1}{2} z \quad \Rightarrow \quad f^n(z) = 2^{-n} z.$
  - If $|a_1| > 1$, then the origin is **repelling**.
    Ex. $f(z) = 2z \quad \Rightarrow \quad f^n(z) = 2^n z.$
- If $a_1 = 0$, then the origin is **superattracting**.
  Ex. $f(z) = z^2 \quad \Rightarrow \quad f^n(z) = z^{2^n}.$
- If $|a_1| = 1$ and:
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The origin is parabolic, so:

$$f(z) = a_1 z + a_k z^k + \dots,$$

where $q \in \mathbb{N}$ such that $a_1^q = 1$.

To learn about the dynamics of f, we study the dynamics of f^q.

If $a_1 = 1$ and $f \not\equiv \text{Id}$, then f is **tangent to the identity.**

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Dimension One: f tangent to the identity - Example

What happens to points near the origin under iteration?

Consider the map:

$$f(z) = z - z^3 = z(1 - z^2).$$

If $z \in \mathbb{R}$ small, z_n will approach 0:

$$\frac{1}{2} \xrightarrow{f} \frac{3}{8} \xrightarrow{f} \frac{165}{512} \xrightarrow{f} \sim 0.3 \longrightarrow \dots$$

If $z \in i\,\mathbb{R}$ small, z_n will tend towards ∞ along the imaginary axis:

$$\frac{i}{2} \xrightarrow{f} \frac{5}{8}\,i \xrightarrow{f} \frac{445}{512}\,i \xrightarrow{f} \sim 1.5\,i \longrightarrow \dots$$

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Leau-Fatou flower for:

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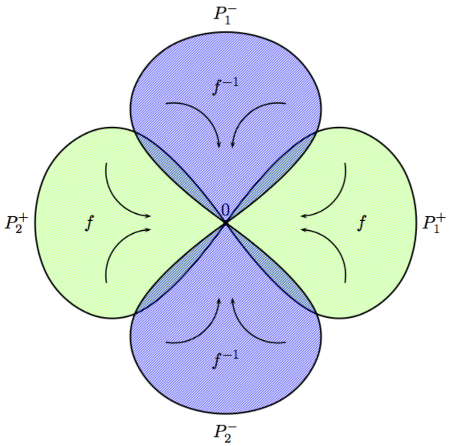

Dimension One: f tangent to the identity

Let $f \in \text{End}(\mathbb{C}, 0)$ be tangent to the identity of order k. Near 0:

$$f(z) = z(1 + a_k z^{k-1} + \ldots),$$

where $a_k \neq 0$. Then f has $k - 1$ attracting (repelling) directions.

Theorem (Leau-Fatou flower theorem)

1. *Each attracting (repelling) direction has an attracting (repelling) petal centered along it. The union of these petals with 0 forms a neighborhood of 0.*
2. $K_f \setminus \{0\} = \bigcap_{k \in \mathbb{N}} f^{-k}(U) \setminus \{0\}$ *is the disjoint union of the basins centered at the attracting directions.*
3. *If P is an attracting petal as in (1), then there exists an injective holomorphic map $\varphi : P \to \mathbb{C}$ such that its image contains a right half-plane and $\varphi \circ f \circ \varphi^{-1}(z) = z + 1$.*

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near the origin, where $P_k \not\equiv 0$.

We want to generalize the concepts of the Leau-Fatou flower theorem to higher dimensions. In particular, we focus on dimension two.

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Characteristic Directions

Definition

*If $v \in \mathbb{C}^2 \setminus \{O\}$ such that $P_k(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, then $[v] \in \mathbb{P}^1$ is a **characteristic direction** of f.*
*$[v]$ is **non-degenerate** if $\lambda \neq 0$, otherwise it is **degenerate.***

Motivation: If $f = \mathrm{Id} + P_k$ and $[v]$ is a characteristic direction of f, then $f([v]) = [v]$.

To every non-degenerate characteristic direction $[v]$, we associate a constant $A(v) \in \mathbb{C}$ called the **director** of f at $[v]$.

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If $v \in \mathbb{C}^2 \setminus \{O\}$ such that $P_k(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, then $[v] \in \mathbb{P}^1$ is a ***characteristic direction*** *of f.*
$[v]$ is ***non-degenerate*** *if $\lambda \neq 0$, otherwise it is* ***degenerate.***

Motivation: If $f = \mathrm{Id} + P_k$ and $[v]$ is a characteristic direction of f, then $f([v]) = [v]$.

To every non-degenerate characteristic direction $[v]$, we associate a constant $A(v) \in \mathbb{C}$ called the **director** of f at $[v]$.

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Theorem (Hakim).

Let f have characteristic direction $[v]$. Suppose that

- $[v]$ is non-degenerate and
- the real part of the director is positive (i.e., $\text{Re}(A(v)) > 0$).

Then f satisfies the **Local Property (LP)**:

1. $\exists$ a domain Ω invariant under f whose points are attracted to O tangentially to $[v]$
2. $f|_\Omega$ is conjugate to translation via a holomorphic map $\varphi : \Omega \to \mathbb{C}^2$. In particular,

$$\varphi \circ f = \varphi + (1, 0).$$

Furthermore, if f is an automorphism of $\mathbb{C}^2$, then f satisfies the **Global Property (GP)**: $\exists\Omega$ as in (LP) biholomorphic to $\mathbb{C}^2$ via φ.

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Suppose f has only one characteristic direction $[v]$ and $[v]$ is non-degenerate. It turns out that the director $A(v) = 0$, so Hakim's theorem does not apply.

Theorem (Hakim).

Let f have non-degenerate characteristic direction $[v]$. If there exists a domain where all orbits converge to O along $[v]$, then the director of $[v]$ has non-negative real part.

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Thank You!

Results on attracting domains in dimension 2

Let:

α be $A(v)$, the director of f corresponding to direction $[v]$.

$\hat{\Omega}$ be a domain of attraction along $[v]$.

Ω be $\hat{\Omega}$ on which f is conjugate to translation.

Non-Degenerate	Degenerate

• $\text{Re}(\alpha) > 0 \Rightarrow \exists\Omega$ (Hakim)
• $\text{Re}(\alpha) < 0 \Rightarrow \nexists\hat{\Omega}$ (Hakim)
• $\text{Re}(\alpha) = 0$:
  •$\alpha = 0 \Rightarrow \exists\Omega$ (L.,Vivas)
  •$\alpha \neq 0 \Rightarrow$?
	• Irregular $\Rightarrow \alpha = 0, \exists\Omega$ (Vivas)
	• Fuchsian $\Rightarrow$ sometimes $\exists\hat{\Omega}$ (Vivas)
	• Apparent $\Rightarrow$?