MCS 401: Computer Algorithms I (Fall 2020)

Homework 1

Due at 2:00pm CDT, Monday, Sep 21

1. Consider a stable matching instance with $n = 4$ and the following preferences:

$$
\begin{aligned}
m_1 &: w_1 > w_2 > w_3 > w_4 \\
m_2 &: w_2 > w_3 > w_1 > w_4 \\
m_3 &: w_3 > w_1 > w_2 > w_4 \\
m_4 &: w_1 > w_2 > w_3 > w_4 \\
w_1 &: m_2 > m_3 > m_4 > m_1 \\
w_2 &: m_3 > m_4 > m_1 > m_2 \\
w_3 &: m_4 > m_1 > m_2 > m_3 \\
w_4 &: m_1 > m_2 > m_3 > m_4
\end{aligned}
$$

   What is the output of the Gale-Shapley algorithm (G-S algorithm) on this instance?

   (You can solve this question by hand or by code. However, you should know how to simulate the G-S algorithm by hand on small inputs.)

2. (2 points) In this exercise, we will prove that all executions of the G-S algorithm yield the same matching.

   We say a woman w is a *valid partner* of a man m if there is a stable matching that contains the pair (m, w). We say w is the *best valid partner* of m if w ranks the highest (in m's preference) among all valid partners of m. We write $best(m)$ for the best valid partner of m.

   (a) Consider the following stable matching instance with $n = 2$:

$$
\begin{aligned}
m_1 &: w_1 > w_2 \\
m_2 &: w_2 > w_1 \\
w_1 &: m_2 > m_1 \\
w_2 &: m_1 > m_2
\end{aligned}
$$

   Follow the above definitions and compute $best(m_1)$ and $best(m_2)$.

   We will show that in any execution of the G-S algorithm, a man is never rejected by one of his valid partners. Because men propose in decreasing order of preference, the G-S algorithm must output the matching $S^\star = \{(m, best(m))\}$.

Suppose by contradiction that this may happen. Consider the first moment when some man m is rejected by one of his valid partners w.

The rejection happened either because m proposed to w and was turned down, or because w broke her engagement to m in favor of a better proposal. Either way, at this moment w forms or continues an engagement with a man m' whom she likes more than m.

Because w is a valid partner of m, there exists a stable matching S containing the pair (m, w). Suppose in this matching S, m' is paired with $w' \neq w$.

(b) Prove that the above assumptions would lead to the contradiction that S is not stable.

3. Order the following functions in ascending order of asymptotic growth rate. That is, if $g(n)$ immediately follows $f(n)$ in your list, then it should be the case that $f(n) = O(g(n))$.

$$
\begin{aligned}
f_1(n) &= n^2 \ln^3 n \ , \\
f_2(n) &= 2^n \ , \\
f_3(n) &= \ln n \ , \\
f_4(n) &= n^{\ln n} \ , \\
f_5(n) &= \ln \ln n \ , \\
f_6(n) &= n^{4/3} \ , \\
f_7(n) &= 2^{\sqrt{\log_2 n}} \ , \\
f_8(n) &= n! \ .
\end{aligned}
$$

4. Suppose we have two positive functions f and g such that $f(n) = O(g(n))$. Prove or disprove each of the following statement.

   (a) $\ln f(n) = O(\log g(n))$.

   (b) $(f(n))^3 = O\left((g(n))^3\right)$.

   (c) $2^{f(n)} = O\left(2^{g(n)}\right)$.

5. The Floyd-Warshall algorithm is an algorithm for finding all-pair shortest paths in a weighted directed graph (with possibly negative edge weights but no negative cycles).

   Give an asymptotically tight bound on the running time of the Floyd-Warshall algorithm on an input graph with n nodes.

   (Assume that accessing an entry $w[u][v]$ takes $O(1)$ time. The function $\min(x, y)$ returns the minimum of x and y which runs in $O(1)$ time given x and y.)

Algorithm 1: The Floyd-Warshall algorithm.

```
Input  : An n-node graph with edge weights w.
Output : An n × n array d where d[u][v] is the minimum distance between u and v.
for u = 1 to n do
    for v = 1 to n do
        if the edge (u,v) exists then
            d[u][v] ← w[u][v] ;
        else
            d[u][v] ← +∞ ;
for u = 1 to n do
    d[u][u] = 0 ;
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            d[i][j] ← min(d[i][j], d[i][k] + d[k][j]) ;
```