MCS 401: Computer Algorithms I (Fall 2020)
Homework 5

Due at 2:00pm CST, Wednesday, Dec 2

1. (2 points) Suppose you are a freelance programmer who needs to decide which job to take in each week. The set of possible jobs is divided into low-stress and high-stress ones. The basic question, each week, is whether to take on a low-stress job or a high-stress job.

   If you select a low-stress job in week i, you get a revenue of $\ell_i > 0$; if you select a high-stress job, you get a revenue of $h_i > 0$. The catch, however, is that in order to take on a high-stress job in week i, it is required that you take no job (of either type) in week $i-1$; you need a full week of prep time to get ready for the crushing stress level. On the other hand, you can take a low-stress job in week i even if you have done a job (of either type) in week $i-1$.

   Given a sequence of n weeks, a (valid) *plan* is specified by a choice of "low-stress," "high-stress," or "none" for each of the n weeks, with the property that if "high-stress" is chosen for week $i > 1$, then "none" has to be chosen for week $i-1$. (It is okay to choose a high-stress job in week 1.) The *value* of the plan is the sum of the revenue you get in each week: ℓ_i if you choose "low-stress" in week i, h_i if you choose "high-stress", and 0 if you choose "none".

   The problem. Given $n > 0$ and $\ell_1, \ldots, \ell_n$, $h_1, \ldots, h_n > 0$, find a plan of maximum value.

   Example. Suppose $n = 4$, and the values of ℓ_i and h_i are given by the following table.

week	1	2	3	4
ℓ	10	1	10	10
h	5	50	5	1

   Then the plan of maximum value would be to choose no job in week 1, a high-stress job in week 2, and low-stress jobs in weeks 3 and 4. The value of this plan would be $0 + 50 + 10 + 10 = 70$.

   (a) Does the following algorithm correctly solve this problem? Justify your answer.

   ```
   for i = 1 to n do
       if h_{i+1} > ℓ_i + ℓ_{i+1} then
           Output "choose no job in week i" ;
           Output "choose a high-stress job in week i + 1" ;
           Continue with iteration i + 2 ;
       else
           Output "choose a low-stress job in week i" ;
           Continue with iteration i + 1 ;
   ```

   (To avoid problems with overflowing array bounds, we define $h_i = \ell_i = 0$ when $i > n$.)

   (b) Give an algorithm that outputs the value of the optimal plan. Your algorithm should run in time $O(n)$. Prove the correctness of your algorithm and analyze its running time.

2. (2 points) In a word processor, the goal of "pretty-printing" is to take text with a ragged right margin, like this,

```
Call me Ishmael.
Some years ago,
never mind how long precisely,
having little or no money in my purse,
and nothing particular to interest me on shore,
I thought I would sail about a little
and see the watery part of the world.
```

and turn it into text whose right margin is as "even" as possible, like this.

```
Call me Ishmael.  Some years ago, never
mind how long precisely, having little
or no money in my purse, and nothing
particular to interest me on shore, I
thought I would sail about a little
and see the watery part of the world.
```

To make this precise enough, we need to define what it means for the right margins to be "even." So suppose our text consists of a sequence of words, $W = \{w_1, w_2, \ldots, w_n\}$, where w_i consists of c_i characters. We have a maximum line length of L. We will assume we have a fixed-width font and ignore issues of punctuation or hyphenation.

A formatting of W consists of a partition of the words in W into lines. In the words assigned to a single line, there should be a space after each word except the last; and so if $w_j, w_{j+1}, \ldots, w_k$ are assigned to one line, then we should have

$$\sum_{i=j}^{k-1} (c_i + 1) + c_k \leq L \ .$$

We will call an assignment of words to a line *valid* if it satisfies this inequality. The difference between the left-hand side and the right-hand side will be called the *slack* of the line – that is, the number of remaining spaces at the right margin.

Give an algorithm to find a partition of a set of words W into valid lines, so that the sum of the squares of the slacks of all lines (including the last line) is minimized. Your algorithm should take as input $n > 0$ and $c_1, \ldots, c_n$, and output the minimum-possible sum of the squares of the slacks. Your algorithm should run in time $O(n^2)$. Prove the correctness of your algorithm and analyze its running time.

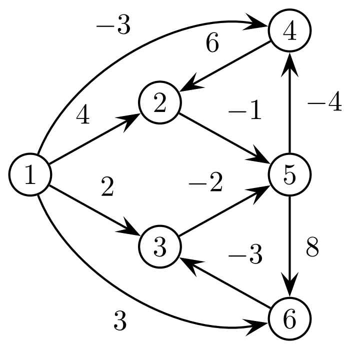
3. (2 points) Suppose it is near the end of the semester and you are taking n courses, each with a final project that still has to be done. Each project will be graded on the following scale: it will receive an integer score on a scale of 0 to 100, higher numbers being better grades. Your goal is to maximize your average grade on the n projects. You have a total of H hours in which to work on the n projects cumulatively, and you want to decide how to divide up this

time. For simplicity, assume $H > 0$ is an integer, and you will spend an integer number of hours on each project.

To figure out how best to divide up your time, you have come up with a set of functions $(f_i)_{i=1}^n$ for each of your n courses; if you spend $h \le H$ hours on the project for course i, you will get a grade of $f_i(h)$. (You may assume that each function f_i are nondecreasing: if $h < h'$, then $f_i(h) \le f_i(h')$. You can also assume $f_i(0) = 0$ for all i.)

The problem. Given these functions $(f_i)_{i=1}^n$, decide how many hours to spend on each project (in integer values only) so that your average grade, as computed according to the f_i, is as large as possible. Your algorithm should run in time $O(n^3H^3)$. Prove the correctness of your algorithm and analyze its running time.

4. Use the Bellman-Ford algorithm to decide the length of the shortest path from $s = 1$ to every node in the graph. Show your work.



An implementation of the Bellman-Ford algorithm is given below. (You can work with other implementations if you prefer.)

Algorithm 1: Bellman-Ford Algorithm

Input : An n-node directed graph G with edge weights w and a source node s.
Output: The length of the shortest path from s to every node in G.
for $u = 1$ **to** n **do**
  $d[u] \leftarrow +\infty$;
$d[s] \leftarrow 0$;
for $i = 1$ **to** n **do**
  $change \leftarrow 0$;
  for *each edge* $e = (u, v) \in E$ *with weight* w_e **do**
    if $d[u] + w_e < d[v]$ **then**
      $d[v] \leftarrow d[u] + w_e$;
      $change \leftarrow 1$;
  if $change = 0$ **then break**;
  if $change = 1$ *and* $i = n$ **then**
    return error "G contains a negative cycle" ;
return d
