**Subset Sum and Knapsack**

**Subset Sum:** Input: $n$ integers $w_1, \ldots, w_n > 0$, and an integer $W$  
Output: A subset $S \subseteq \{1, 2, \ldots, n\}$  
\[ \text{st. } \sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} w_i \text{ is as large as possible.} \]

Failed attempt 1: large numbers first. \(\times\)  
\[ \{w-1, \frac{w}{2}, \frac{w}{2}\} \]

Failed attempt 2: small numbers first \(\times\)  
\[ \{1, \frac{w}{2}, \frac{w}{2}\} \]

**A dynamic-programming solution.**

\[ \text{OPT}(i, w) = \text{the value of the optimal solution} \]
\[ \text{for } 0 \leq i \leq n \text{ for the first } i \text{ intervals with } 0 \leq w \leq W \text{ maximum-allowed sum } w. \]

\[ \text{OPT}(i, w) = \max \sum_{j \in S} w_j \text{ st. } \sum_{j \in S} w_j \leq w. \]
\[ S \subseteq \{1, \ldots, i\} \]

Final answer = \(\text{OPT}(n, W).\)

If $w_i \leq w$,  
\[ \text{OPT}(i, w) = \max(\text{OPT}(i-1, w) , \text{OPT}(i-1, w-w_i) + w_i) \]

If $w_i > w$  
\[ \text{OPT}(i, w) = \text{OPT}(i-1, w) \]

**Example:** \(W = 6\)  
\(w_1 = 2, \ w_2 = 2, \ w_3 = 3\)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
w & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
i=0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\
\hline
i=1 & 0 & 0 & 2 & 2 & 4 & 4 & 4 \\
\hline
i=2 & 0 & 0 & 2 & 3 & 4 & 4 & 5 \\
\hline
i=3 & 0 & 0 & 2 & 3 & 4 & 5 & 5 \\
\hline
\end{array}
\]

Blue for index  

Final answer = 5.  
we can recover which $w_i$'s are chosen  
by following the red arrows \(\rightarrow \{w_2, w_3\}\)

Runtime = \(O(nW)\)
Knapsack

Input: n items, W > 0
Each item has a weight \( W_i > 0 \) and a value \( V_i > 0 \).

Goal: pick a subset \( S \) of items s.t.
\[
\sum_{j \in S} W_j \leq W \quad \text{and} \quad \sum_{j \in S} V_j \text{ is maximized.}
\]

A similar DP solution.
\[
\text{OPT}(i, w) = \text{the maximum value one can get from}
\text{the first } i \text{ items with a weight limit of } w.
\]
If \( w_i \leq w \)
\[
\text{OPT}(i, w) = \max \left( \text{OPT}(i-1, w), \text{OPT}(i-1, w-w_i) + V_i \right)
\]
If \( w_i > w \)
\[
\text{OPT}(i, w) = \text{OPT}(i-1, w).
\]