1. (2 points) Consider the following stable matching instance with \( n = 4 \):

\[
\begin{align*}
  m_1 : & w_1 > w_2 > w_3 > w_4 \\
  m_2 : & w_2 > w_3 > w_1 > w_4 \\
  m_3 : & w_3 > w_1 > w_2 > w_4 \\
  m_4 : & w_1 > w_2 > w_3 > w_4 \\
  w_1 : & m_2 > m_3 > m_4 > m_1 \\
  w_2 : & m_3 > m_4 > m_1 > m_2 \\
  w_3 : & m_4 > m_1 > m_2 > m_3 \\
  w_4 : & m_1 > m_2 > m_3 > m_4
\end{align*}
\]

Simulate the Gale-Shapley algorithm on this input (men propose to women).

For grading purposes, you only need to give the output.

(You can solve this question by code if you prefer. However, you should know how to simulate the G-S algorithm by hand on small input like this.)

2. (2 points) Order the following functions in ascending order of asymptotic growth rate. That is, if \( g(n) \) immediately follows \( f(n) \) in your list, then it should be the case that \( f(n) = O(g(n)) \).

\[
\begin{align*}
  f_1(n) &= \frac{1}{10} n^2 \ln n \\
  f_2(n) &= 2^n \\
  f_3(n) &= \ln^3 n = (\ln n)^3 \\
  f_4(n) &= \ln \ln n = \ln(\ln(n)) \\
  f_5(n) &= 5 \cdot n^{4/3} \\
  f_6(n) &= n^! 
\end{align*}
\]

3. (2 points) The Floyd-Warshall algorithm (Algorithm 1) finds all-pair shortest paths in a weighted directed graph (with possibly negative edge weights but no negative cycles).

Give an asymptotically tight bound on the runtime of Algorithm 1 on a graph with \( n \) nodes.
(Assume that accessing an entry $w[u][v]$ takes $O(1)$ time. The function $\text{min}(x, y)$ returns the minimum of $x$ and $y$ which runs in $O(1)$ time.)

**Algorithm 1:** The Floyd-Warshall algorithm.

| Input : | An $n$-node directed graph with edge weights $w$. |
| Output: | An $n \times n$ array $d$ where $d[u][v]$ is the minimum distance from $u$ to $v$. |
| for $u = 1$ to $n$ do |
| for $v = 1$ to $n$ do |
| if the edge $(u,v)$ exists then |
| $d[u][v] \leftarrow w[u][v]$; |
| else |
| $d[u][v] \leftarrow +\infty$; |
| for $u = 1$ to $n$ do |
| $d[u][u] = 0$; |
| for $k = 1$ to $n$ do |
| for $i = 1$ to $n$ do |
| for $j = 1$ to $n$ do |
| $d[i][j] \leftarrow \text{min}(d[i][j], d[i][k] + d[k][j])$; |

4. (extra credit, 1 point) We will prove the G-S algorithm always outputs the same matching. We start with some definitions.

A woman $w$ is said to be a valid partner of a man $m$ if $(m, w)$ appears in some stable matching. Let $\text{best}(m)$ denote the best valid partner of $m$, which is the highest-ranked valid partner of $m$ (according to $m$’s preference).

Prove that in any execution of the G-S algorithm, a man is never rejected by one of his valid partners. (Consequently, since men propose in decreasing order of preference, the G-S algorithm can only output the matching $S^* = \{(m, \text{best}(m))\}.)

(Hint: Proof by contradiction. Consider the first moment when a man $m$ is rejected by one of his valid partners $w$. The rejection happened because $w$ was/became engaged to another man $m'$ that she likes more than $m$. Because $w$ is a valid partner of $m$, there exists a stable matching $S$ containing the pair $(m, w)$. Suppose $m'$ is paired with $w' \neq w$ in $S$. Prove that $S$ is not a stable matching which leads to a contradiction.)