Fun with Reductions

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Outline

Show

- Vertex 3-Coloring
- Hamiltonian Cycle
- Super Mario
- Minesweeper

are NP-Hard.
Recap

• P vs NP

• (Polynomial-time) Reductions

• 3-Satisfiability (3SAT)
“Easy to verify” problems: NP

• All decision problems such that we can verify the correctness of a solution in polynomial time.
Polynomial time reductions

• Reduce A to B: a polynomial time algorithm that maps instances of A to instances of problem B, such that the answers are the same.

• $A \leq_p B$: B is at least as hard as A.

  If you can solve B (in poly time) then you can solve A.
3-Satisfiability (3SAT)

given \((x_5 \lor x_3 \lor \neg x_1)\)

\(\land\) \((x_2 \lor \neg x_3) \lor x_5\)

\(\land\) ... variable literals

Q: \exists x_i s.t. formula true?
Gadget-Based Reductions

\[ A \leq_p B: \]

Given instances of \( A \), output instances of \( B \).
Build gadgets for pieces of \( A \).
Put the pieces together.

\[ 3\text{SAT} \leq_p X \]
Fun with Hardness Proofs

Algorithmic Lower Bounds: Fun with Hardness Proofs

Erik Demaine

http://courses.csail.mit.edu/6.890/fall14/lectures/
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Vertex 3-Coloring

Input: a graph

Output: color each vertex using 1 of the 3 colors, so that adjacent vertices do not get the same color.
3-Coloring:
3-Coloring: Yes instance
3-Coloring: No instance
3SAT \leq_p 3\text{-Coloring}

Satisfiable formula \iff \quad 

Unsatisfiable formula \iff
Vertex 3-Coloring

[Garey, Johnson, Stockmeyer 1976]
Vertex 3-Coloring

[Garey, Johnson, Stockmeyer 1976]
Vertex 3-Coloring

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Vertex 3-Coloring
[Garey, Johnson, Stockmeyer 1976]

clause gadget

variable gadget

colors gadget
3SAT $\leq_p$ 3-Coloring

• Consequence:

  3-Coloring is NP-Complete.
  (Because 3-Coloring is also in NP.)
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Hamiltonian Cycle
Hamiltonian Cycle

• Input: a (directed) graph.

• Solution: a cycle visiting every vertex exactly once.
Variable Gadget

$x_i$

Direction we travel along this chain represents whether to set the variable to **true** or **false**.
Clause Gadget

Add a new node for each clause:

Direction we travel along this chain represents whether to set the variable to true or false.

true

false
$3\text{SAT} \leq_p \text{Hamiltonian Cycle}$
3SAT $\leq_p$ Hamiltonian Cycle

3SAT $\leq_p$ Hamiltonian Path
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are NP-Hard.
Super Mario Bros. is NP-Hard
[Aloupis, Demaine, Guo 2012]

clause

variable
Super Mario Bros. is NP-Hard
[Aloupis, Demaine, Guo, Viglietta 2014]
Super Mario Bros. is NP-Hard
[Aloupis, Demaine, Guo, Viglietta 2014]
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Gadgets

Variables:

NOT gate:
Gadgets

splitter

turn

or gate
Gadgets