Chapter 5 Review:

(1) *Base Case.* $n = 1$. The left hand side is $\sum_{i=1}^{1} 3^{i-1} = 3^0 = 1$. The right hand side is $\frac{3^1-1}{2} = 1$. These are equal so it is true for $n = 1$.
Inductive Step. Assume that $\sum_{i=1}^{n} 3^{i=1} = \frac{3^n-1}{2}$. Prove that $\sum_{i=1}^{n+1} 3^{i=1} = \frac{3^{n+1}-1}{2}$. Using the inductive hypothesis, we have

$$\sum_{i=1}^{n+1} 3^{i=1} = \sum_{i=1}^{n} 3^{i=1} + 3^{(n+1)-1} = \frac{3^n - 1}{2} + 3^n = \frac{3^n - 1 + 2 \cdot 3^n}{2} = \frac{3 \cdot 3^n - 1}{2} = \frac{3^{n+1} - 1}{2}.$$

(2) *Base Case.* $n = 1$. The left hand side is $\left(1 - \frac{1}{2}\right)^1 = \frac{1}{2}$. The right hand side is $1 - \frac{1}{2} = \frac{1}{2}$. These are equal so it is true for $n = 1$.
Inductive Step. Assume that $\left(1 - \frac{1}{2}\right)^n \geq 1 - \frac{n}{2}$. Show that $\left(1 - \frac{1}{2}\right)^{n+1} \geq 1 - \frac{n+1}{2}$. Using the inductive hypothesis, we have

$$\begin{aligned}
\left(1 - \frac{1}{2}\right)^{n+1} &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)^n \\
&\geq \left(1 - \frac{1}{2}\right)\left(1 - \frac{n}{2}\right) \\
&= 1 - \frac{1}{2} - \frac{n}{2} + \frac{n}{4} = 1 - \frac{n+1}{2} + \frac{n}{4} \\
&\geq 1 - \frac{n+1}{2}.
\end{aligned}$$

(3) *Base Case.* $n = 1$. $10^{1+2} + 10^1 + 1 = 1011 = 337 \cdot 3$ is divisible by 3 so it is true for $n = 1$.
Inductive Step. Assume that $10^{n+2} + 10^n + 1$ is divisible by 3. That is, assume that $10^{n+2} + 10^n + 1 = 3k$, or $10^{n+2} + 10^n = 3k - 1$, for some integer k. Prove that $10^{n+3} + 10^{n+1} + 1$ is divisible by 3. Using the induction hypothesis we have

$$10^{n+3} + 10^{n+1} + 1 = 10(10^{n+2} + 10^n) + 1 = 10(3k - 1) + 1 = 30k - 10 + 1 = 3(10k - 3)$$

so it is divisible by 3.

(4) (a) $a_1 = 1$ and $a_n = 6a_{n-1} - 1$ for $n \geq 2$.
(b) $a_1 = 3$ and $a_n = \frac{1}{2}(a_{n-1}^2 + 1)$ for $n \geq 2$.
(c) $a_1 = 1$ and $a_n = \left(\sqrt{a_{n-1}} + n\right)^2$ for $n \geq 2$.

(5) *Base Cases.* $n = 1$. $\frac{2}{9}\left(1 - \left(-\frac{1}{2}\right)^{1-1}\right) = \frac{2}{9}(1 - 1) = 0 = a_1$. $n = 2$. $\frac{2}{9}\left(1 - \left(-\frac{1}{2}\right)^{2-1}\right) = \frac{2}{9}(1 + \frac{1}{2}) = \frac{2}{9} \cdot \frac{3}{2} = \frac{1}{3} = a_2$. Thus it is true for $n = 1$ and $n = 2$.
Inductive Step. Assume the formula works for *all* k with $1 \leq k \leq n$. In particular, assume that $a_n = \frac{2}{9}\left(1 - \left(-\frac{1}{2}\right)^{n-1}\right)$ and that $a_{n-1} = \frac{2}{9}\left(1 - \left(-\frac{1}{2}\right)^{n-2}\right)$.

Prove that $a_{n+1} = \frac{2}{9}\left(1 - \left(-\frac{1}{2}\right)^n\right)$. We use the recursive definition of the sequence and the induction hypothesis to get

$$\begin{aligned}
a_{n+1} &= \frac{1}{2}(a_n + a_{n-1}) \\
&= \frac{1}{2}\left[\frac{2}{9}\left(1 - \left(-\frac{1}{2}\right)^{n-1}\right) + \frac{2}{9}\left(1 - \left(-\frac{1}{2}\right)^{n-2}\right)\right] \\
&= \frac{1}{9}\left[1 - \left(-\frac{1}{2}\right)^{n-1} + 1 - \left(-\frac{1}{2}\right)^{n-2}\right] \\
&= \frac{1}{9}\left[2 - \left(-\frac{1}{2}\right)^{n-2}\left(-\frac{1}{2} + 1\right)\right] \\
&= \frac{2}{9}\left[1 - \left(-\frac{1}{2}\right)^{n-2}\left(\frac{1}{4}\right)\right] \\
&= \frac{2}{9}\left[1 - \left(-\frac{1}{2}\right)^{n}\right]
\end{aligned}$$

(6) This is an arithmetic sequence with $a = 5$ and $d = 4$.
 (a) $a_{32} = a + (32 - 1)d = 5 + 31 \cdot 4 = 129$. $a_{100} = 5 + 99 \cdot 4 = 401$.
 (b) Notice that $125 = 129 - 4$, so 125 will be the term just before 129 in the sequence. So $a_{31} = 125$.
 (c) Now $429 = 401 + 28$. $28 = 4 \cdot 7$, so 429 should be in the sequence seven terms along, so $a_{107} = 429$. 1000 is not in the sequence because our sequence consists only of odd numbers because we add an even number to an odd number each time.
 (d) The sum is $S = \frac{18}{2}[2a + (18 - 1)d] = 9[10 + 17 \cdot 4] = 702$.

(7) (a) If it is arithmetic, $a = 6$ and $d = -4$. So $a_{27} = 6 + (27-1)(-4) = -98$. The sum of the first 30 terms is $\frac{30}{2}[2 \cdot 6 + (30 - 1)(-4)] = -1560$.
 (b) If it is geometric, $a = 6$ and $r = \frac{1}{3}$. So $a_{27} = ar^{27-1} = 6\left(\frac{1}{3}\right)^{26}$. The sum of the first 30 terms is $\frac{6\left(1-\frac{1}{3^{30}}\right)}{1-\frac{1}{3}} = 9\left(1 - \frac{1}{3^{30}}\right)$.

(8) (a) $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
 (b) It is not possible for three consecutive terms to be odd. Suppose two terms, f_n and f_{n+1} are both odd. Then the next term is their sum which must be even since they are both odd.
 (c) It is not possible for two consecutive terms to be even. Suppose f_n and f_{n+1} are both even. Since $f_{n+1} = f_n + f_{n-1}$, we must have that f_{n-1} is even. Since $f_n = f_{n-1} + f_{n-2}$, we must have that f_{n-2} is even and so on. So every preceding term must be even. However, we know that there are odd terms in the sequence, so this cannot happen.

(9) The characteristic polynomial is $x^2 - 5x + 4 = (x - 1)(x - 4)$, which has roots 1 and 4. The solution is of the form

$$a_n = c_1 1^n + c_2 4^n = c_1 + c_2 4^n.$$

Use the initial conditions to solve for the constants. $a_0 = -3 = c_1 + c_2 4^0$ so $c_1 + c_2 = -3$. $a_1 = 6 = c_1 + c_2 4^1$ so $c_1 + 4c_2 = 6$. We now have two equations and two unknowns which is easily solved for $c_1 = -6$ and $c_2 = 3$. The solution is

$$a_n = -6 + 3 \cdot 4^n.$$