

CHAPTER 6 REVIEW:

- (1) We know that $|A \cup B| = |A| + |B| - |A \cap B|$. We are given that $|A| + |B| > |A \cup B| = |A| + |B| - |A \cap B|$. Cancel the like terms from both sides to get $0 > -|A \cap B|$ or $|A \cap B| > 0$. Since this is a strict inequality, there must be at least one item in $A \cap B$, so $A \cap B \neq \emptyset$.
- (2) Let $A = \{n \in \mathbf{N} | 1 \leq n \leq 2000, 2|n\}$, $B = \{n \in \mathbf{N} | 1 \leq n \leq 2000, 3|n\}$, $C = \{n \in \mathbf{N} | 1 \leq n \leq 2000, 5|n\}$, and $D = \{n \in \mathbf{N} | 1 \leq n \leq 2000, 7|n\}$. We want $|A \cup B \cup C \cup D|$. To do this we must find the size of the intersections.

$$\begin{aligned}
 |A| &= \lfloor \frac{2000}{2} \rfloor = 1000 \\
 |B| &= \lfloor \frac{2000}{3} \rfloor = 666 \\
 |C| &= \lfloor \frac{2000}{5} \rfloor = 400 \\
 |D| &= \lfloor \frac{2000}{7} \rfloor = 285 \\
 |A \cap B| &= \lfloor \frac{2000}{2 \cdot 3} \rfloor = 333 \\
 |A \cap C| &= \lfloor \frac{2000}{2 \cdot 5} \rfloor = 200 \\
 |A \cap D| &= \lfloor \frac{2000}{2 \cdot 7} \rfloor = 142 \\
 |B \cap C| &= \lfloor \frac{2000}{3 \cdot 5} \rfloor = 133 \\
 |B \cap D| &= \lfloor \frac{2000}{3 \cdot 7} \rfloor = 95 \\
 |C \cap D| &= \lfloor \frac{2000}{5 \cdot 7} \rfloor = 57 \\
 |A \cap B \cap C| &= \lfloor \frac{2000}{2 \cdot 3 \cdot 5} \rfloor = 66 \\
 |A \cap B \cap D| &= \lfloor \frac{2000}{2 \cdot 3 \cdot 7} \rfloor = 47 \\
 |A \cap C \cap D| &= \lfloor \frac{2000}{2 \cdot 5 \cdot 7} \rfloor = 28 \\
 |B \cap C \cap D| &= \lfloor \frac{2000}{3 \cdot 5 \cdot 7} \rfloor = 19 \\
 |A \cap B \cap C \cap D| &= \lfloor \frac{2000}{2 \cdot 3 \cdot 5 \cdot 7} \rfloor = 9
 \end{aligned}$$

We then have $|A \cup B \cup C \cup D| = 1000 + 666 + 400 + 285 - 333 - 200 - 142 - 133 - 95 - 57 + 66 + 47 + 28 + 19 - 9 = 1542$.

- (3) (a) A palindrome is a word that reads the same backwards and forwards. *Sununus* is a palindrome.

- (b) All of these words will be of the form $S\alpha\beta\gamma\beta\alpha S$, where there is some repetition of letters. This is a case where it is easier to count what we *don't* want. We will count the number of ways to choose α , β , and γ so there is no repetition and S is not used. We have 25 choices for α (can't be S), 24 choices for β (can't be S or α) and 23 for γ . So there are $25 \times 24 \times 23$ ways to choose α , β , and γ that we don't want. We need to subtract this from the total number of ways to choose them, which is 26^3 , since for each letter we have free choice for the whole alphabet. The answer is $26^3 - 25 \times 24 \times 23 = 3776$.
- (c) Here our words will be of the form $\alpha\beta\gamma\delta\gamma\beta\alpha$ and there are no restrictions on any of our choices for α , β , γ , and δ so there are $26^4 = 456976$ palindromes.

- (4) There are the same number of palindromes with ten letters as there are with nine. A palindrome with ten letters has the form $\alpha\beta\gamma\delta\epsilon\delta\gamma\beta\alpha$ while a palindrome with nine letters is of the form $\alpha\beta\gamma\delta\epsilon\delta\gamma\beta\alpha$. In both cases we need to choose α , β , γ , δ , and ϵ , which can be done in 26^5 ways.

- (5) We want $|A_1 \cup A_2 \cup A_3 \cup A_4|$. By the Principle of Inclusion-Exclusion,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - |A_1 + A_2 + A_3 + A_4|.$$

We know that $|A_i| = 5$, $|A_i \cap A_j| = 1$, and $|A_i \cap A_j \cap A_k| = 0$ for all i, j, k . So we have

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 4 \cdot 5 - 6 \cdot 1 + 4 \cdot 0 - 0 = 14.$$

- (6) Let S , A , and R be the sets of cars with stereo systems, air conditioners, and sun roofs, respectively. We are given that $|S| = 30$, $|A| = 30$, $|R| = 40$, $|S \cap A \cap R| = 10$, and $|(S \cap A) \cup (S \cap R) \cup (A \cap R)| = 30$. We want to find $|S \cup A \cup R| = |S| + |A| + |R| - |S \cap A| - |S \cap R| - |A \cap R| + |S \cap A \cap R|$.

To do this we need to find $|S \cap A| + |S \cap R| + |A \cap R|$. The information about how many cars have at least two of the options.

$$\begin{aligned} 30 &= |(S \cap A) \cup (S \cap R) \cup (A \cap R)| \\ &= |S \cap A| + |S \cap R| + |A \cap R| - |(S \cap A) \cap (S \cap R)| - |(S \cap A) \cap (A \cap R)| \\ &\quad - |(S \cap R) \cap (A \cap R)| + |(S \cap A) \cap (S \cap R) \cap (A \cap R)| \\ &= |S \cap A| + |S \cap R| + |A \cap R| - 2|S \cap A \cap R| \\ &= |S \cap A| + |S \cap R| + |A \cap R| - 2 \cdot 10 \end{aligned}$$

So $|S \cap A| + |S \cap R| + |A \cap R| = 30 + 20 = 50$. We then have

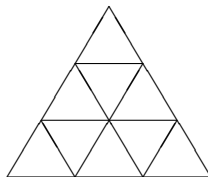
$$|S \cup A \cup R| = 30 + 30 + 40 - 50 + 10 = 60.$$

So 60 of the cars have at least one option.

The number of cars that have exactly one option is the number that have at least one option minus the number that have at least two options, that is, $60 - 30 = 30$.

- (7) If n objects are put into m boxes and $n > m$, then some box must contain at least $\lceil \frac{n}{m} \rceil$ objects.

- (8) Let the 18 integers be the objects and let the boxes be the congruence classes (remainders) $(\text{mod } 17)$. There are 17 boxes $(0, 1, \dots, 16)$ and 18 objects so by the Pigeon-Hole Principle, there must be a box with at least two objects. This means that two of the integers have the same remainder when divided by 17. Let them be $x = 17q_1 + r$ and $y = 17q_2 + r$. Their difference is $x - y = (17q_1 + r) - (17q_2 + r) = 17(q_1 - q_2)$ which is divisible by 17.
- (9) The proof is by contradiction. Assume that \mathbf{Z} is finite. Then there is a one-to-one correspondence $f : \mathbf{Z} \rightarrow \{1, 2, \dots, n\}$ for some n . Let the boxes be the values $1, 2, \dots, n$, and the objects be the integers. We assign objects to boxes by applying f , so k goes to box i if $f(k) = i$. We then end up assigning the $n + 1$ integers $1, 2, \dots, n, n + 1$ to n boxes so by the Pigeon-Hole Principle, there must be a box with two objects, that is, there must be $k_1 \neq k_2$ such that $f(k_1) = f(k_2)$. But this contradicts our assumption that f is a one-to-one correspondence. Thus such a correspondence can not exist and \mathbf{Z} is infinite.
- (10) Divide the triangle into 9 smaller triangles with side lengths $\frac{1}{3}$ as shown below.



Think of the smaller triangles as the boxes and the ten points chosen in the larger triangle as the objects. Then by the Pigeon-Hole Principle, there must be some smaller triangle with at least two of the points. The longest distance inside one of the smaller triangles is the side length, $\frac{1}{3}$. Thus the two points that lie within the same smaller triangle must be within $\frac{1}{3}$ of each other.