Spring 2003 Name _______________

MCS 261:01 Final Exam (175 + 5 points)

1. A table of logic rules is provided.
2. Write you solutions in your exam book.
3. Turn in this sheet along with your exam book.
4. Show and explain all your work. An unjustified answer may not receive credits.

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1. Simplify $(p \wedge (\neg q)) \rightarrow q$. (15 points)

2. Is $\mathcal{R} = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$ an equivalence relation? Explain your answer. (20 points)

3. Let $\mathbf{N} = \{1,2,3,4,\ldots\}$, and $S = \{1,4,9,25,\ldots\} \subset \mathbf{N}$ be the set of perfect squares. Show that $\mathbf{N}$ and S has the same cardinality. (20 points)

4. Suppose $a_0, a_1, a_2, \ldots$ is a sequence such that $a_0 = a_1 = 1$ and for $n > 1$, $a_n = n(a_{n-1} + a_n)$. Proved that $a_n = n!$ for all $n \geq 1$. (20 points)

5. (a) In how many ways can letters a, b, c, d, e, f form a line such that a is next to b and c is NOT next to d? (10 points)
   (b) In how many ways 4 students are chosen out of a class of 20 students such that Micelle must be selected and Paul must not be selected? (10 points)

6. (a) In how many ways can 10 red balls and 10 blue balls be placed in 30 different boxes, at most one ball in a box? (10 points)
   (b) Answer (a) when we allow any number of balls in a box. (10 points)

7. Show that the complexity function of multiplying an n-digit number by a single-digit number is $O(n)$. Assumed that the addition of two single-digit numbers is the unit operation and the multiplication of two single-digit numbers is considered as 3 unit operations. (20 points)

8. Draw all non-isomorphic trees on 5 vertices with degree sequences. (20 points)

9. Given a graph $G = (V, E)$ with $|V| = n$ and $|E| = n - 1$.
   (a) What is the sum of the degrees of the vertices? (6 points)
   (b) One version of Pigeon-Hole principle says, if n objects are put into m boxes, then some box must contain AT MOST _______ objects. What is this blank space? (6 points)
   (c) Use (a) and (b) to show that G has a vertex of degree at most 1. (8 points)

10. (**BONUS**, 5 extra points) Show that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$ for all $n \in \mathbf{N}$.

SOME BASIC LOGICAL EQUIVALENCES

1. Idempotence
   (a) $(p \vee p) \Longleftrightarrow p$
   (b) $(p \wedge p) \Longleftrightarrow p$
2. Commutativity
   (a) $(p \vee q) \Longleftrightarrow (q \vee p)$
   (b) $(p \wedge q) \Longleftrightarrow (q \wedge p)$
3. Associativity
   (a) $((p \vee q) \vee r) \Longleftrightarrow (p \vee (q \vee r))$
   (b) $((p \wedge q) \wedge r) \Longleftrightarrow (p \wedge (q \wedge r))$
4. Distributivity
   (a) $(p \vee (q \wedge r)) \Longleftrightarrow ((p \vee q) \wedge (p \vee r))$
   (b) $(p \wedge (q \vee r)) \Longleftrightarrow ((p \wedge q) \vee (p \wedge r))$
5. Double Negation $\neg(\neg p) \Longleftrightarrow p$
6. De Morgan's Laws
   (a) $\neg(p \vee q) \Longleftrightarrow ((\neg p) \wedge (\neg q))$
   (b) $\neg(p \wedge q) \Longleftrightarrow ((\neg p) \vee (\neg q))$
7. (a) $(p \vee \mathbf{1}) \Longleftrightarrow \mathbf{1}$
   (b) $(p \wedge \mathbf{1}) \Longleftrightarrow p$
8. (a) $(p \vee \mathbf{0}) \Longleftrightarrow p$
   (b) $(p \wedge \mathbf{0}) \Longleftrightarrow \mathbf{0}$
9. (a) $(p \vee (\neg p)) \Longleftrightarrow \mathbf{1}$
   (b) $(p \wedge (\neg p)) \Longleftrightarrow \mathbf{0}$
10. (a) $\neg\mathbf{1} \Longleftrightarrow \mathbf{0}$
    (b) $\neg\mathbf{0} \Longleftrightarrow \mathbf{1}$
11. $(p \rightarrow q) \Longleftrightarrow [(\neg q) \rightarrow (\neg p)]$
12. $(p \leftrightarrow q) \Longleftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$
13. $(p \rightarrow q) \Longleftrightarrow [(\neg p) \vee q]$