MCS 261, March 29, 2002 Name: ______________________

Hour Exam 2

- ○ Each problem is worth 20 points.
- ○ Write your solutions in your exam book.
- ○ Turn in this sheet along with your solutions.
- ○ Show and explain all of your work.
- ○ **An unjustified answer is not correct!**

---

1. On a recent Saturday night 69 people attended the late show at Hollywood Cinemas. The manager of the concession stand recorded the following.
   - 30 people bought popcorn.
   - 29 people bought soda.
   - 25 people bought candy.
   - 14 people bought popcorn and soda.
   - 9 people bought popcorn and candy.
   - 12 people bought soda and candy.
   - 4 people bought all three.

   Determine how many people did not buy anything at the concession stand that night.

   Solution. Let P, S, and C be the sets of people who bought popcorn, soda, and candy, respectively. We are looking for the number of people who did not buy any of these snacks, which is $|(P \cup S \cup C)^c| = 69 - |P \cup S \cup C|$. By the Principle of Inclusion-Exclusion,

$$\begin{aligned} |P \cup S \cup C| &= |P| + |S| + |C| - |P \cap S| - |P \cap C| - |S \cap C| + |P \cap S \cap C| \\ &= 30 + 29 + 25 - 14 - 9 - 12 + 4 \\ &= 53 \end{aligned}$$

   Thus the number we want is $69 - 53 = 16$.

2. Find the number of license plates that can be manufactured if a license plate consists of three letters followed by three digits where the letters must be distinct but the digits can be arbitrary. Be sure to explain how you get your answer.

   Solution. We have 26 choices for the first letter. We can use any letter but the first for the second, so we have 25 choices for the second. The third can be any letter but the first two so we have 24 choices for the third. We have 10 choices for the first digit. Since we are allowed to repeat digits, we have 10 choices for the second and third digits as well. Thus by the multiplication principle there are $26 \times 25 \times 24 \times 10 \times 10 \times 10 = 15,600,000$ possible license plates.

3. Solve the recursion relation
$$a_n = 6a_{n-1} - 9a_{n-2} + 6^n$$
with initial conditions $a_0 = 1$ and $a_1 = 3$.

Solution. For the particular solution, try $p_n = a6^n$. We must have $p_n = 6p_{n-1} - 9p_{n-2} + 6^n$ so $a6^n = 6a6^{n-1} - 9a6^{n-2} + 6^n$. Divide this by 6^{n-2} to get $36a = 36a - 9a + 36$. This is easily solved to get $a = 4$. So $p_n = 4 \cdot 6^n$.

For the homogeneous solution, the characteristic polynomial is $x^2 - 6x + 9 = (x-3)^2$ which has a double root of 3. So $q_n = c_1 3^n + c_2 n 3^n$.

Thus $a_n = p_n + q_n = 4 \cdot 6^n + c_1 3^n + c_2 n 3^n$. We use the initial conditions to solve for c_1 and c_2. We have that $a_0 = 1 = 4 \cdot 6^0 + c_1 3^0 + c_2 \cdot 0 \cdot 3^0$ so $1 = 4 + c_1$ and $c_1 = -3$. We have that $a_1 = 3 = 4 \cdot 6^1 + c_1 3^1 + c_2 \cdot 1 \cdot 3^1$ so $3 = 24 + 3c_1 + 3c_2$ so $3c_2 = -12$ and $c_2 = -4$. So the solution is
$$a_n = 4 \cdot 6^n - 3 \cdot 3^n - 4n3^n.$$

4. Use mathematical induction to prove the following identity for all natural numbers n.
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution. Base Case. $n = 1$. The left hand side is $\sum_{i=1}^{1} i^2 = 1^2 = 1$. The right hand side is $\frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1$. These are equal so the statement is true for $n = 1$.

Inductive Step. Assume that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and prove that $\sum_{i=1}^{n+1} = \frac{(n+1)(n+2)(2n+3)}{6}$. Using the inductive hypothesis we have
$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n+1}{6}\left(n(2n+1) + 6(n+1)\right) \\ &= \frac{n+1}{6}\left(2n^2 + 7n + 6\right) \\ &= \frac{n+1}{6}(n+2)(2n+3)\end{aligned}$$
This is what we were trying to prove. So we have show that if it is true for n, then it is true for $n+1$, so the statement is true by the Principle of Mathematical Induction.

5. (a) Let a, b, x, y, and n be integers with $n > 1$. Given that $a \equiv x \pmod{n}$ and $b \equiv y \pmod{n}$, prove that
$$a + b \equiv x + y \pmod{n}.$$

Solution. $a \equiv x \pmod{n}$ means $n|(a-x)$ so $a - x = nk$ for some integer k. $b \equiv y \pmod{n}$ means $n|(b-y)$ so $b - y = nl$ for some integer l. Consider

$(a+b)-(x+y)$. This is equal to $(a-x)+(b-y)=nk+nl=n(k+l)$. So $n|(a+b)-(x+y)$ which means that $a+b \equiv x+y \pmod{n}$.

(b) Find $987654321+123456789 \pmod{7}$.

Solution. $987654321 = 141093474 \cdot 7 + 3$ so $987654321 \equiv 3 \pmod{7}$. $123456789 = 17636684 \cdot 7 + 1$ so $123456789 \equiv 1 \pmod{7}$. So using the result of part (a),

$$987654321 + 123456789 \equiv 3 + 1 \equiv 4 \pmod{7}.$$