We may use the following derivative rules now:

\[ \frac{d}{dx} b^x = b^x \ln(b) \quad \frac{d}{dx} \ln x = \frac{1}{x} \quad \frac{d}{dx} \ln |u(x)| = \frac{u'(x)}{u(x)} \quad \frac{d}{dx} \log_b x = \frac{1}{x \ln b} \]

And the technique of logarithmic differentiation: take a log of both sides of the equation, then take the derivative using implicit differentaition to solve for \( f'(x) \).

1. Find the following derivatives

(a) \( \frac{d}{dx} (x^2 \ln x) \)

**SOLUTION:**

\[ = 2x \ln x + x^2 \frac{1}{x} \]

(b) \( \frac{d}{dx} x^3 3^x \)

**SOLUTION:**

\[ = 3x^2 3^x + x^3 (3^x \ln 3) \]

(c) \( \frac{d}{dx} (\ln |\sin x|) \)

**SOLUTION:**

\[ = \frac{\cos x}{\sin x} = \cot x \]

(d) \( \frac{d}{dx} \ln(10^x) \)

**SOLUTION:**

\[ = \frac{10^x \ln 10}{10^x} = \ln 10 \]

Or you may use hte fact that \( \ln(10^x) = x \ln 10 \)

and simply take that derivative (\( \ln 10 \) is just a constant).

(e) \( \frac{d}{dx} (\ln(\ln x)) \)

**SOLUTION:**

\[ = \frac{1}{x \ln x} = \frac{1}{x \ln x} \]

2. Find the derivatives

(a) \( s(t) = \cos(2^t) \)

**SOLUTION:**

\[ s'(t) = -\sin(2^t)(2^t \ln 2) \]

(b) \( f(x) = \ln [(x^3 + 1)^\pi] \)

**SOLUTION:**

\[ f'(x) = \frac{\pi(x^3 + 1)^{\pi-1}(3x^2)}{(x^3 + 1)^\pi} \]
3. Evaluate the derivative of \( h(x) = x^{\sqrt{x}} \) at \( x = 4 \).

**SOLUTION:** Using logarithmic differentiation, we take the natural log of both sides first

\[
\ln h(x) = \ln \left( x^{\sqrt{x}} \right)
\]

\[
\ln h(x) = \sqrt{x} \ln x
\]

by log property

\[
\frac{h'(x)}{h(x)} = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \frac{1}{x}
\]

take derivative of both sides

\[
h'(x) = h(x) \left( \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)
\]

solve for \( h'(x) \)

\[
h'(x) = x^{\sqrt{x}} \left( \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)
\]

substitute in \( h(x) \)

\[
h'(4) = 4^{\sqrt{4}} \left( \frac{\ln 4}{2\sqrt{4}} + \frac{1}{\sqrt{4}} \right)
\]

plug in 4

\[
= 16 \left( \frac{\ln 4}{4} + \frac{1}{2} \right)
\]

\[
= 4 \ln 4 + 8
\]

4. Find the horizontal tangent line equation for \( y = x^{\ln x} \)

**SOLUTION:** Using logarithmic differentiation,

\[
\ln y = \ln \left( x^{\ln x} \right)
\]

\[
\ln y = (\ln x)(\ln x) = (\ln x)^2
\]

by log property

\[
\frac{y'}{y} = 2(\ln x) \frac{1}{x}
\]

take derivative of both sides

\[
y' = y 2\frac{\ln x}{x}
\]

solve for \( y' \)

\[
y' = x^{\ln x} 2\frac{\ln x}{x}
\]

substitute in \( y \)

Because \( x = 0 \) is not in the domain, the only way this derivative can be zero is for \( \ln x = 0 \). This happens when \( x = 1 \) (this is true for all logs, no matter what base). Plugging in 1 for \( x \) we get

\[
y = (1)^{\ln 1} = 1^0 = 1
\]

So the equation of the horizontal tangent line is

\[
y = 1
\]

5. Use logarithmic differentiation to find the derivative of

\[
f(x) = \frac{x^8 \cos^3 x}{\sqrt{x - 1}}
\]
SOLUTION:

\[
\ln f(x) = \ln \left( \frac{x^8 \cos^3 x}{\sqrt{x-1}} \right)
\]
take log of both sides

\[
\ln f(x) = \ln(x^8) + \ln(\cos^3 x) - \ln(\sqrt{x-1})
\]
by log properties

\[
\ln f(x) = 8 \ln x + 3 \ln(\cos x) - \frac{1}{2} \ln(x-1)
\]
by log properties

\[
\frac{f'(x)}{f(x)} = \frac{8 - 3 \sin x}{\cos x} - \frac{1}{2} \frac{1}{x-1}
\]
take derivative of both sides

\[
= \frac{8}{x} - 3 \tan x + \frac{1}{2} \frac{1}{-2x}
\]

\[
f'(x) = f(x) \left( \frac{8}{x} - 3 \tan x + \frac{1}{2 - 2x} \right)
\]
solve for \(f'(x)\)

\[
= \left( \frac{x^8 \cos^3 x}{\sqrt{x-1}} \right) \left( \frac{8}{x} - 3 \tan x + \frac{1}{2 - 2x} \right)
\]
substitute back \(f(x)\)

6. Find the derivative \(y'\) of

\[y = (x^2 + 1)^x\]

using two methods:

(1) Use the fact that

\[b^x = e^{x \ln b}\]

(2) Use logarithmic differentiation.

SOLUTION:
Method (1):

\[y = e^{x \ln(x^2 + 1)}\]

So

\[y' = e^{x \ln(x^2 + 1)} \left( \ln(x^2 + 1) + x \frac{2x}{x^2 + 1} \right) = (x^2 + 1)^x \left( \ln(x^2 + 1) + \frac{2x}{x^2 + 1} \right)\]

Method (2):

\[
\ln y = \ln \left( (x^2 + 1)^x \right)
= x \ln(x^2 + 1)
\]

\[
\frac{y'}{y} = \ln(x^2 + 1) + \frac{x}{x^2 + 1} (2x)
\]

\[
y' = y \left( \ln(x^2 + 1) + \frac{2x}{x^2 + 1} \right)
= (x^2 + 1)^x \left( \ln(x^2 + 1) + \frac{2x}{x^2 + 1} \right)\]