1. Find the derivatives of the following functions.

(a) \( f(x) = 6x - 2xe^{x} \)

**SOLUTION:** Use the sum/difference rule, followed by a product rule

\[
f'(x) = \left( \frac{d}{dx} 6x \right) - \left( \frac{d}{dx} \frac{2x}{u} e^{\frac{x}{v}} \right) = 6 - (u'v + v'u) = 6 - \left( (2e^{x} + (2x)e^{x}) \right)
\]

(b) \( f(x) = (1 + \frac{1}{x}) (x^{2} + 1) \)

**SOLUTION:** It’s easier to write \( \frac{1}{x} = x^{-2} \). Use product rule again

\[
\frac{d}{dx} \left( 1 + x^{-2} \right) \left( x^{2} + 1 \right) = u'v + v'u \quad \text{where} \quad u' = -2x^{-3}, v' = 2x.
\]

\[
= (-2x^{-3})(x^{2} + 1) + (2x)(1 + x^{-2})
\]

\[
= -2x^{-1} - 2x^{-3} + 2x + 2x^{-1}
\]

\[
= 2x - 2x^{-3}
\]

(c) \( f(x) = \frac{2e^{x} - 1}{2e^{x} + 1} \)

**SOLUTION:** Use the quotient rule

\[
\frac{d}{dx} \frac{2e^{x} - 1}{2e^{x} + 1} \left\{ u'v - v'u \right\} = u'v' = 2e^{x}
\]

\[
= \frac{(2e^{x})(2e^{x} - 1) - (2e^{x})(2e^{x} - 1)}{(2e^{x} + 1)^{2}}
\]

Stop here; don’t simplify.

(d) \( f(t) = 2500e^{0.075t} \)

**SOLUTION:** This just follows from the derivative of \( e^{kx} \) for some constant \( k \).

\[
f'(t) = (.075)2500e^{0.075t} = 1875e^{0.075t}
\]

(e) \( f(x) = \frac{1}{x} \)

**SOLUTION:** Write it as \( f(x) = x^{-5} \) and it becomes easy.

\[
f'(x) = -5x^{-5} - 1 = -5x^{-6}
\]

2. Find the \( x \) values such that the slope of \( f(x) = xe^{2x} \) is zero.

**SOLUTION:** We first find the derivative, \( f'(x) \) using the product rule:

\[
\frac{d}{dx} \left( \frac{xe^{2x}}{u} \right) = u'v + v'u = (1)(e^{2x}) + (e^{2x})(x) = e^{2x} + xe^{2x}
\]
Now we set this equal to zero and solve for \( x \).
\[
0 = e^{2x} + xe^{2x}
\]
\[
0 = e^{2x}(1 + 2x) \quad \text{and we can divide by } e^{2x} \text{ because it is nonzero for all } x.
\]
\[
0 = 1 + 2x
\]
The only solution is \( x = -\frac{1}{2} \).

3. True or false:
(a) \( \frac{d}{dx}(e^5) = 5e \)

**SOLUTION:** False! \( e^5 \) is not an exponential function - it is a constant function (about 148.413), its derivative is zero. Same goes for part b.

(b) \( \frac{d}{dx}(e^5) = e^5 \)

4. Based on the following table of \( x \) values and function/derivative values, evaluate the following:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( g(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
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<td>5</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

a) \( \frac{d}{dx}(f(x)g(x)) \bigg|_{x=1} \)

b) \( \frac{d}{dx}\left(\frac{xf(x)}{g(x)}\right) \bigg|_{x=4} \)

**a) SOLUTION:**
\[
\frac{d}{dx}(f(x)g(x)) \bigg|_{x=1} = (f'(x)g(x) + g'(x)f(x)) \bigg|_{x=1} = (3)(4) + (2)(5) = 10
\]

**b) SOLUTION:** We use the quotient rule, with \( u = xf(x) \) and \( v = g(x) \).
\[
\frac{d}{dx}\left(\frac{xf(x)}{g(x)}\right) \bigg|_{x=4} = \left(\frac{u'v - v'u}{v^2}\right) \bigg|_{x=4}
\]
\[
= \left(\frac{(f(x) + xf'(x))g(x) - g'(x)(xf(x))}{g(x)^2}\right) \bigg|_{x=4}
\]
\[
= \left(\frac{(f(4) + 4f'(4))g(4) - g'(4)(4f(4))}{g(4)^2}\right)
\]
\[
= \frac{(2 + 4 \cdot 1)(3) - (1)(4 \cdot 2)}{3^2}
\]
\[
= \frac{10}{9}
\]

5. Find \( f'(x) \), \( f''(x) \), and \( f'''(x) \).

(a) \( f(x) = \frac{1}{x} \)

**SOLUTION:** It is much easier to first write \( f(x) = x^{-1} \) and use the power rule.
\[
f'(x) = -1x^{-1-1} = -x^{-2}
\]
\[
f''(x) = -(-2)x^{-2-1} = 2x^{-3}
\]
\[
f'''(x) = (-3)2x^{-3-1} = -6x^{-4}
\]
(b) \( f(x) = x^2 e^{3x} \)

**SOLUTION:** We just have to use product rule.

\[
\frac{d}{dx} f(x) = \frac{d}{dx} \left( x^2 e^{3x} \right) = u'v + v'u = (2x)(e^{3x}) + (3e^{3x})(x^2) = 2xe^{3x} + 3x^2e^{3x}
\]

It is helpful when evaluating the second derivative to write \( f''(x) = 2xe^{3x} + 3f(x) \).

\[
\frac{d}{dx} f'(x) = \frac{d}{dx} \left( \left( \frac{2x}{u} \right) e^{3x} + 3f(x) \right) = u'v + v'u + 3f'(x) = (2)(e^{3x}) + (3e^{3x})(2x) + 3(2xe^{3x} + 3x^2e^{3x}) = 2e^{3x} + 6xe^{3x} + 6xe^{3x} + 9x^2e^{3x} = 2e^{3x} + 12xe^{3x} + 9x^2e^{3x}
\]

Again, it may be helpful to write this as \( f''(x) = 2e^{3x} + 6f'(x) - 9f(x) \) for the third derivative.

\[
\frac{d}{dx} f''(x) = \frac{d}{dx} \left( 2e^{3x} + 6f'(x) - 9f(x) \right) = (3)2e^{3x} + 6f''(x) - 9f'(x) = 6e^{3x} + 6(2e^{3x} + 12xe^{3x} + 9x^2e^{3x}) - 9(2xe^{3x} + 3x^2e^{3x}) = 6e^{3x} + 12e^{3x} + 72xe^{3x} + 54x^2e^{3x} - 18xe^{3x} - 27x^2e^{3x} = 18e^{3x} + 54xe^{3x} + 27x^2e^{3x}
\]

6. Find some \( f \) and \( g \) non-constant functions such that \( \frac{d}{dx} f(x)g(x) = f'(x)g'(x) \)

**SOLUTION:** This is very tricky. We wouldn’t expect you to get this one, but here’s a solution, see if you understand it.

In general, \( \frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x) \) (this is the Product Rule for derivatives). If we find \( f \) and \( g \) as desired, then they satisfy the equation

\[
f'(x)g(x) + f(x)g'(x) = f'(x)g'(x)
\]

Let’s put all terms with \( f'(x) \) as a factor on the right. We get

\[
f(x)g(x) = f'(x)(g'(x) - g(x))
\]

For any \( x \) such that \( g'(x) \neq 0 \) we could divide by \( g(x) \) and get

\[
f(x) = f'(x) \frac{g'(x) - g(x)}{g'(x)} \tag{6.1}
\]

What does this tell us about the \( f \) we are looking for? Well, for one thing if it is possible that its derivative is a scalar multiple of itself, then it might do the trick. The same goes for \( g \) (since we could have done the
same thing swapping the places of $f$ and $g$. So we want functions whose derivatives are scalar multiples of themselves. One such candidate is $e^{kx}$, whose slope is never equal to zero. Let us try such functions. Assume

$$f(x) = e^{ax}, g(x) = e^{bx} \text{ for some } a, b \in \mathbb{R}.$$ 

Plugging these into equation 6.1, we get

$$e^{ax} = \frac{(ae^{ax})b e^{bx} - e^{bx}}{e^{bx}} = a e^{ax} \frac{b - 1}{b}$$

Which is true only when

$$a = \frac{b}{b - 1}$$

So pick some $b \neq 1$ or 0 and you will get your $a$. It turns out that $a = b = 2$ works, so one solutions is $f(x) = g(x) = e^{2x}$. 