HOMEWORK 8: GRADER’S NOTES AND SELECTED SOLUTIONS

Grader’s Notes:

- In general when you are trying to show that two groups $G$ and $H$ are not isomorphic, it’s never enough to pick one particular map and show it’s not an isomorphism. For example, in showing that the rational numbers $\mathbb{Q}$ under addition are not isomorphic to any proper sub-group you can’t just show that the map $\phi(x) = x^2$ is not an isomorphism because there could, conceivably be some other map that is an isomorphism.

- The proper way to show that $G \not\cong H$ in general is usually either:
  1. Find some property that you can prove / have proved is preserved under isomorphism and show one of $G, H$ have this property while the other group does not.
  2. Assume that $\phi : G \rightarrow H$ is an isomorphism (and assume nothing more about $\phi$) and somehow arrive at a contradiction.

- The properties “one-to-one”, “onto”, and “operation preserving” are properties that functions have, it is impossible for groups to have these properties, when groups $G, H$ are isomorphic the property that holds of the groups is that there exists a function $f : G \rightarrow H$ such that $f$, the function is one-to-one, onto, and operation preserving.

Chapter 6, page 135, no 28

Prove the quaternion group is not isomorphic to the dihedral group $D_4$.

Example Solution:

Use Theorem 6.2.7 and note that the Quaternions have 1 element of order 2 while $D_4$ has 5 elements of order 2. There are variants on this method of proof all related to the fact that if $\phi$ is an isomorphism then $|x| = |\phi(x)|$.

Grader’s Notes:

- Note that when two groups $G, H$ are given by Cayley tables, an isomorphism $\phi : G \rightarrow H$ need not take the $n \times m$-th entry of the Cayley table for $G$ to the $n \times m$-th entry of the Cayley table for $H$.

Chapter 6, page 136, no. 38

Let $G = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$, and $H = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$. Show that $G$ and $H$ are isomorphic groups under addition. Does your isomorphism preserve multiplication? Generalize to the case where $G = \langle m \rangle$ and $H = \langle n \rangle$.

Grader’s Notes:

- Just because $G, H$ are not groups under multiplication, this doesn’t necessarily mean that a map $\phi : G \rightarrow H$ cannot satisfy $\phi(ab) = \phi(a)\phi(b)$.
- There was a very common miscalculation of $\phi(ab)$ for the map $\phi(ab)$ used in the example solution.

Example Solution:

Let $\phi : G \rightarrow H$ by $\phi(2n) = 3n$. We claim that $\phi$ is an isomorphism, it should be clear that $\langle \phi \rangle = G$ and $\langle \phi \rangle = H$.

We check that $\phi$ is one-to-one: If $\phi(2n) = 3n = \phi(2m) = 3m$ then it must be the case that $n = m$, so $\phi$ is one-to-one.

We check that $\phi$ is onto: Given $3n \in H$, then $\phi(2n) = 3n$ and $2n \in G$, so $\phi$ is onto.

And lastly, we check that $\phi$ preserves the group operation:

$$\phi(2n + 2m) = \phi(2(n + m)) = 2(n + m) = 2n + 2m = \phi(2n) + \phi(2m)$$
so $\phi$ preserves that group operation.

$\phi$ does not, however, preserve multiplication:

$$\phi(2n2m) = \phi(2(2nm)) = 3(2nm) = 6nm \neq 9nm = 3nm(3nm) = \phi(2nm)\phi(2m)$$

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In general you can show that $\phi(km) = km$ is a group isomorphism from $\langle n \rangle$ to $\langle m \rangle$, that does not preserve multiplication when $n \neq m$, this proof is essentially the same as the proof above and I'm not going to write out the details. If this seems non-obvious you should try and write out a detailed proof. 

An alternate, but similar, and perhaps more elegant proof would be to show that $\mathbb{Z} \cong \langle n \rangle$ for any non-zero $n \in \mathbb{Z}$, by the map $\phi(k) = nk$, then by transitivity of isomorphism (problem (6)) $\langle n \rangle \cong \langle m \rangle$.

**Chapter 6, page 136 No. 43**

*Prove that $\mathbb{Q}^+$, the group of positive rational numbers under multiplication, is isomorphic to a proper subgroup.*

**Grader’s Notes:**

- In general, any function $f : X \rightarrow Y$ is *always* maps onto it’s image.
- If $\phi : G \rightarrow (G)$ is an isomorphism, that is $\phi$ is one-to-one, onto (by the remark above $\phi$ is neccessarily onto), and operation preserving, then $(G)$ must be a group.

**Proof.** Let $H = (G)$, let $e \in G$ be the identity for $G$, note that since $\phi$ is onto, any $h \in H$ has the form $h = \phi(g)$ for some $g \in G$, so we may always write elements of $H$ in the form $\phi(g)$ where $g \in G$.

$\phi(e)$ is an identity for $H$, because $\phi(e)\phi(g) = \phi(eg) = \phi(g) = \phi(e)\phi(g)$. $H$ is closed under the operation since if $\phi(g), \phi(h) \in H$ the $\phi(g)\phi(h) = \phi(gh) \in H$. $H$ has inverses because given $\phi(g) \in H$, $\phi(g)\phi(g^{-1}) = \phi(gg^{-1}) = \phi(e) = \phi(g^{-1}g)\phi(g)\phi(g^{-1})$ so $\phi(g^{-1})$ is an inverse for $\phi(g)$.

- If you think about it, the proof of the above gives you way to give a group structure to to any set $Y$ when you have any bijection $f : G \rightarrow Y$, where $G$ is a group, by defining $x \cdot y = f(f^{-1}(x)f^{-1}(y))$.

**Example Solution:**

Let $\phi(x) = x^2$ (actually we could take $\phi(x) = x^n$ for natural number $n \geq 2$), note that $\phi(x)$ is onto it’s image, furthermore $\phi(x)$ is one to one since $\phi(x) = x^2 = \phi(y) = y^2$ implise that $x = y$ since both $x$ and $y$ are positive rational numbers. Note that $\phi$ is operation preserving, $\phi(xy) = (xy)^2 = x^2 y^2 = \phi(x)\phi(y)$.

Note that $\phi$ is proper because numbers such as 2 with no rational square root are not in $\langle \phi \rangle$. So $\phi$ is an isomorphism from $\mathbb{Q}$ to a proper subgroup of $\mathbb{Q}$.

**Chapter 6, page 136 No 44**

*Show that $\mathbb{Q}$, the group of rational numbers under addition, is not isomorphic to a proper subgroup of itself.*

**Grader’s Notes:**

- I originally was going to grade this problem but I decided not to because there is a subtle logic error in what I believe is the intended solution, and I didn’t want to nitpick about it.
- below I present what I believe is the “intended” solution, assuming (42) (Any automorphism $\phi$ of $\mathbb{Q}$ has the form $\phi(x) = x\phi(1)$).

**Example Solution:**

Assume for contradiction that $\phi : \mathbb{Q} \rightarrow H$ is an isomorphism of $\mathbb{Q}$ to a proper subgroup $H \leq \mathbb{Q}$, then by (42), $\phi(x) = x\phi(1)$ for all $x \in \mathbb{Q}$, but clearly $\phi(1) \neq 0$, since $\phi$ is one-to-one and $\phi(0) = 0$ must hold for $\phi$ to be an isomorphism. So given any $y \in \mathbb{Q} \setminus H$ (note that since $0$ is the identity for $\mathbb{Q}$, $0 \in H$, so $y \neq 0$), note that $y = \frac{\phi(1)}{\phi(1)} y = \frac{\phi(1)}{\phi(1)} \phi\left(\frac{y}{\phi(1)}\right)$, so we’ve shown that $y \in H$, but this is a contradiction, so it must be that case that $\mathbb{Q}$ is not isomorphic to any of it’s proper subgroups. 

\[\text{many students incorrectly calculated } \phi(2n2m) = 3nm.\]
Did you spot the error? The problem is that \( \phi \) is not an automorphism of \( \mathbb{Q} \); it’s an isomorphism of \( \mathbb{Q} \) onto a proper subgroup, so it doesn’t follow from the statement of (42) that \( \phi \) must have the form \( \phi(x) = \phi(1)x \). In fact this problem can be solved but you have to use this slightly strong proposition:

**Proposition 0.1.** If \( \phi : \mathbb{Q} \to \mathbb{Q} \) is any operation preserving map (for all \( x, y \in \mathbb{Q} \phi(x+y) = \phi(x) + \phi(y) \) then \( \phi \) has the form \( \phi(x) = \phi(1)x \).

Note that the proposition is stronger than (42) because it applies to maps that need not be automorphisms of \( \mathbb{Q} \), like for instance the constantly 0 map: \( \phi(x) = 0 \).

**Proof.** Let \( \phi : \mathbb{Q} \to \mathbb{Q} \) be operation preserving. We need to show that \( \phi(x) = \phi(1)x \) for any \( x \in \mathbb{Q} \), we may assume \( x = \frac{n}{m} \). Note that \( 1 = n \left( \frac{1}{m} \right) = \sum_{i=1}^{n} \frac{1}{m} \) so:

\[
\phi\left(\frac{n}{m}\right) = \phi\left( \sum_{i=1}^{n} \frac{1}{m} \right) = \sum_{i=1}^{n} \phi\left( \frac{1}{m} \right) = n \cdot \phi\left( \frac{1}{m} \right)
\]

Now note that \( 1 = m \left( \frac{1}{m} \right) = \sum_{i=1}^{m} \frac{1}{m} \) so:

\[
\phi(1) = \phi\left( \sum_{i=1}^{m} \frac{1}{m} \right) = \sum_{i=1}^{m} \phi\left( \frac{1}{m} \right) = m \left( \phi\left( \frac{1}{m} \right) \right)
\]

From which it follows that \( \phi\left( \frac{1}{m} \right) = \frac{\phi(1)}{m} \), putting these two derived identities together we get:

\[
\phi\left( \frac{n}{m} \right) = \frac{n}{m} \phi(1)
\]

\( \square \)