Abstract Algebra, Homework 9 Solutions

Chapter 10

12.) We need to refer First Isomorphism Theorem to solve this problem: if $\phi$ is a homomorphism between $G$ and $\overline{G}$, then we have

$$G/\text{Ker}\phi \cong \phi(G).$$

So we require to introduce an onto homomorphism $\phi: Z_n \to Z_k$ such that $\text{Ker}(\phi) = \langle k \rangle$. We need an onto homomorphism since we want to obtain $\phi(Z_n) = Z_k$. Then we let $G = Z_n$, $\text{Ker}\phi = \langle Z_k \rangle$ and $\phi(G) = Z_k$ and apply First Isomorphism theorem. Namely, we get

$$Z_n/\langle k \rangle \cong Z_k.$$

It is given that $k$ is a divisor of $n$. If we denote the identity in $Z_n$ with $1_n$ and the identity in $Z_k$ with $1_k$, there exists a homomorphism $\phi: Z_n \to Z_k$ such that $\phi(1_n) = 1_k$. There exists such a homomorphism since $|\phi(1_n)| = k$ divides $|1_n| = n$. In fact, in general, we define $\phi(x) = x \mod k$ for every $x \in Z_n$. This homomorphism is onto since we have $k \leq n$, which implies that $\phi(0) = 0$, $\phi(1_n) = 1_k$, $\ldots$, $\phi(k - 1) = k - 1$ in $Z_k$. The kernel of $\phi$ are those elements in $Z_n$ which are multiples of $k$ (and hence are equivalent to $0 \mod k$). By the First Isomorphism theorem, we conclude that $Z_n/\langle k \rangle \cong Z_k$.

16.) Let us assume otherwise that there exists an onto homomorphism $\phi$ from $Z_8 \oplus Z_2$ onto $Z_4 \oplus Z_4$. Note that we have $|Z_8 \oplus Z_2| = 16$ and $|Z_4 \oplus Z_4| = 16$. Since these group has the same number of elements and $\phi$ is onto, $\phi$ has to be one-one. We assumed that $\phi$ is a homomorphism. So it is operation-preserving. This means that $\phi$ is an isomorphism. By Theorem 6.2(5), we know that $|\phi(g)| = |g|$ for every $g \in Z_8 \oplus Z_2$. Let us consider the order of the element $(1, 0) \in Z_8 \oplus Z_2$. Since we know that $Z_8 = \langle 1 \rangle$, we get $|(1, 0)| = 8$. But note that if $(g_1, g_2)$ is an element of $Z_4 \oplus Z_4$, we have $|(g_1, g_2)| = \text{lcm}(|g_1|, |g_2|) \leq 4$ because $|g_1|$ and $|g_2|$ can only be 1 or 2. In other words, $Z_4 \oplus Z_4$ doesn’t has an element of order 8. On the contrary, $Z_8 \oplus Z_2$ has an element of order 8. This contradicts with Theorem 6.2(5). Hence there is no isomorphism between $Z_8 \oplus Z_2$ and $Z_4 \oplus Z_4$.

20.) We claim that there is no homomorphism from $Z_{20}$ onto $Z_8$. Suppose for contradiction that there exists an onto homomorphism $\phi$ between $Z_{20}$ and $Z_8$. Since $\phi$ is onto, there is an element $g \in Z_{20}$ such that $\phi(g) = 1 \in Z_8$. By Theorem 10.1(3), $|\phi(g)| = 1$ divides $|g|$. Since we have that $Z_8 = \langle 1 \rangle$, we get $|1| = 8$. Then we see that 8 divides $|g|$. But $|g|$ divides $|Z_{20}| = 20$ by Corollary 2 of Lagrange’s Theorem (Theorem 7.1). Hence, we get 8 divides 20. This is a contradiction. Therefore there is no onto homomorphism between $Z_{20}$ and $Z_8$.

If we drop the requirement onto homomorphism, there exists homomorphisms from $Z_{20}$ (in) to $Z_8$. Remember that the image of 1 determines the homomorphism $\phi$ between $Z_{20}$ and $Z_8$. So, we need to follow the steps:
STEP 1: Find the possible orders for $\phi(1)$ in $Z_8$: We know that $|1| = 20$ in $Z_{20}$. By Theorem 10.1(3), $|\phi(1)|$ must be a divisor of 20. Since elements of $Z_8$ have order 1, 2, 4, or 8, $|\phi(1)|$ must be 1, 2, or 4.

STEP 2: Find all the elements in $Z_8$ with order 1, 2, or 4: Note that we have $|0| = 1$, $|4| = 2$, $|2| = 4$, and $|6| = 4$ in $Z_8$. So, 0, 4, 2, 6 in $Z_8$ are possible images for 1 in $Z_{20}$.

STEP 3: Determine all homomorphisms: As we mentioned, the image of 1 determines the homomorphism between $Z_{20}$ and $Z_8$. If we map $1 \mapsto 0$, we get a homomorphism. Let us call $\phi_1$. If we map $1 \mapsto 4$, we get another homomorphism. Let us call $\phi_2$. If we map $1 \mapsto 2$, we get another homomorphism. Let us call $\phi_3$. If we map $1 \mapsto 6$, we get another homomorphism. Let us call $\phi_4$. Therefore, there are 4 homomorphisms $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ from $Z_{20}$ (in)to $Z_8$.

22.) It is given that $\overline{G}$ has an element of order 8. Let us call this element $\overline{g}$. So we get $|\overline{g}| = 8$. Since $\phi$ is an onto homomorphism, there is an element $g \in G$ so that $\phi(g) = \overline{g}$. By Theorem 10.2(3) we know that $|\phi(g)| = |\overline{g}|$ divides $|g|$. This means 8 divides $|g|$ because $|\overline{g}| = 8$. So we obtain that $|g| = 8t$ for some positive integer $t$. Then if we consider the element $g^t$ in $G$, we get

$$(g^t)^8 = g^t \cdot g^t \cdots g^t \quad \text{8 copies of } g^t = g^{8t} = e.$$ 

Thus $G$ has an element $g^t$ of order 8.

As a generalization, If $G$ is finite and $\overline{G}$ has an element of order $n$, then so does $G$.

26.) We follow the steps in the question 20. We know that $Z_4 = \{1\}$. Let $\phi$ be a homomorphism between $Z_4$ and $Z_2 \oplus Z_2$.

STEP 1: Find the possible orders for $\phi(1)$: We know that $|1| = 4$ in $Z_4$. By Theorem 10.2(3), $|\phi(1)|$ must be a divisor of 4. We know that $|(g_1, g_2)| = \text{lcm}(|g_1|, |g_2|)$ for every $(g_1, g_2) \in Z_2 \oplus Z_2$. Since we have $|g_1| = 1$ or 2 and $|g_2| = 1$ or 2, we get that $|(g_1, g_2)| = 1$ or 2. So $|\phi(1)|$ must have order 1 or 2.
STEP 2: Find all the elements in $Z_2 \oplus Z_2$ of order 1 or 2: We have $|(0,0)| = 1$, $|(1,0)| = 2$, $|(0,1)| = 2$ and $|(1,1)| = 2$. So, $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ are possible images for 1 in $Z_4$.

STEP 3: Determine all homomorphisms: the image of 1 determines the homomorphism between $Z_4$ and $Z_2 \oplus Z_2$. If we map

$$1 \mapsto (0,0),$$

we get a homomorphism. Let us call $\phi_1$. If we map

$$1 \mapsto (1,0),$$

we get another homomorphism. Let us call $\phi_2$. If we map

$$1 \mapsto (0,1),$$

we get another homomorphism. Let us call $\phi_3$. If we map

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we get another homomorphism. Let us call $\phi_4$. Therefore, there are 4 homomorphisms $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ from $Z_4$ to $Z_2 \oplus Z_2$. 

