Math 220 – Section 10.5 Solutions

1. For the problem:

\[ u_t = 5u_{xx}, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = u(1, t) = 0, \quad t > 0 \]
\[ u(x, 0) = (1 - x)x^2, \quad 0 < x < 1 \]

we have \( \beta = 5 \) and \( L = 1 \). The general solution is:

\[ u(x, t) = \sum_{n=1}^{\infty} c_n e^{-5n^2\pi^2t} \sin(n\pi x) \]

We find the \( c_n \) by solving the integral:

\[ c_n = 2 \int_0^1 (1 - x)x^2 \sin(n\pi x) \, dx \]

Using Maple, the solution is:

\[ c_n = -4 + 8(-1)^n \frac{n\pi}{n^3\pi^3} \]

The formal solution is:

\[ u(x, t) = \sum_{n=1}^{\infty} \frac{4 + 8(-1)^n}{n^3\pi^3} e^{-5n^2\pi^2t} \sin(n\pi x) \]

3. For the problem:

\[ u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0, \]
\[ u_x(0, t) = u_x(\pi, t) = 0, \quad t > 0 \]
\[ u(x, 0) = x, \quad 0 < x < \pi \]

we have \( \beta = 3 \) and \( L = \pi \). The general solution is:

\[ u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-3n^2\pi^2t} \cos(nx) \]

We find \( c_0 \) by solving the integral:

\[ c_0 = \frac{2}{\pi} \int_0^\pi x \, dx = \pi \]

We find the \( c_n \) by solving the integral:

\[ c_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) \, dx \]

Using Maple, the solution is:

\[ c_n = \frac{2(-1 + \cos(n\pi) + \sin(n\pi)n\pi)}{\pi n^2} \]
\[ c_n = -\frac{2 + 2(-1)^n}{\pi n^2} \]
The formal solution is:

\[ u(x, t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{2 + 2(-1)^n}{\pi n^2} e^{-3n^2t} \cos(nx) \]

5. For the problem:

\[ u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0, \]
\[ u_x(0, t) = u_x(\pi, t) = 0, \quad t > 0 \]
\[ u(x, 0) = e^x, \quad 0 < x < \pi \]

we have \(\beta = 1\) and \(L = \pi\). The general solution is:

\[ u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2t} \cos(nx) \]

We find \(c_0\) by solving the integral:

\[ c_0 = \frac{2}{\pi} \int_0^\pi e^x \, dx = \frac{1}{\pi} (e^\pi - 1) \]

We find the \(c_n\) by solving the integral:

\[ c_n = \frac{2}{\pi} \int_0^\pi e^x \cos(nx) \, dx \]

Using Maple, the solution is:

\[ c_n = \frac{2(-1 + e^\pi \cos(n\pi) + e^\pi n \sin(n\pi))}{\pi(1 + n^2)} \]
\[ c_n = \frac{2(-1 + e^\pi(-1)^n)}{\pi(1 + n^2)} \]

The formal solution is:

\[ u(x, t) = \frac{1}{2\pi} (e^\pi - 1) + \sum_{n=1}^{\infty} \frac{2(-1 + e^\pi(-1)^n)}{\pi(1 + n^2)} e^{-n^2t} \cos(nx) \]

7. For the problem:

\[ u_t = 2u_{xx}, \quad 0 < x < \pi, \quad t > 0, \]
\[ u(0, t) = 5, \quad u(\pi, t) = 10, \quad t > 0 \]
\[ u(x, 0) = \sin 3x - \sin 5x, \quad 0 < x < \pi \]

we have \(\beta = 2\) and \(L = \pi\). The general solution is:

\[ u(x, t) = 5 + \frac{5}{\pi} x + \sum_{n=1}^{\infty} c_n e^{-2n^2t} \sin(nx) \]
We find the $c_n$ by solving the integral:

\[
c_n = \frac{2}{\pi} \int_0^\pi \left( \sin 3x - \sin 5x - \frac{5}{\pi} \right) \sin(nx) \, dx
\]

\[
= \frac{2}{\pi} \int_0^\pi \sin 3x \sin(nx) \, dx - \frac{2}{\pi} \int_0^\pi \sin 5x \sin(nx) \, dx - \frac{10}{\pi^2} \int_0^\pi \sin(nx) \, dx - \frac{10}{\pi^2} \int_0^\pi x \sin(nx) \, dx
\]

Using Maple, the solution for the $n = 3$ and $n = 5$ cases are:

\[
c_3 = 1 - \frac{20}{3\pi} - \frac{10}{3\pi} = 1 - \frac{10}{\pi}
\]

\[
c_5 = -1 - \frac{4}{\pi} - \frac{2}{\pi} = -1 - \frac{6}{\pi}
\]

The solution for all other $n$ is:

\[
c_n = \frac{10(-1 + \cos(n\pi))}{n\pi} + \frac{10(-\sin(n\pi) + \cos(n\pi)n\pi)}{n^2\pi^2} = -10 + 20(-1)^n
\]

The formal solution is:

\[
u(x, t) = 5 + \frac{5}{\pi} x + \sum_{n=1}^{\infty} c_n e^{-2n^2t} \sin(nx)
\]

\[
= 5 + \frac{5}{\pi} x + c_1 e^{-2t} \sin x + c_2 e^{-8t} \sin 2x + c_3 e^{-18t} \sin 3x + c_4 e^{-32t} \sin 4x + c_5 e^{-50t} \sin 5x
\]

\[
+ \sum_{n=6}^{\infty} c_n e^{-2n^2t} \sin(nx)
\]

\[
= 5 + \frac{5}{\pi} x - \frac{30}{\pi} e^{-2t} \sin x + \frac{5}{\pi} e^{-8t} \sin 2x + \left( 1 - \frac{10}{\pi} \right) e^{-18t} \sin 3x + \frac{5}{2\pi} e^{-32t} \sin 4x
\]

\[
- \left( 1 + \frac{6}{\pi} \right) e^{-50t} \sin 5x + \sum_{n=6}^{\infty} \frac{-10 + 20(-1)^n}{n\pi} e^{-2n^2t} \sin(nx)
\]

9.