1. Find the modulus and conjugate of each complex number below.

(a) \(-2 + i\)

(b) \((2 + i)(4 + 3i)\)

(c) \(\frac{3 - i}{\sqrt{2} + 3i}\)

Solution:

(a) For the complex number \(z = -2 + i\), the modulus is

\[|z| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}\]

and the conjugate is

\(\bar{z} = -2 - i\)

(b) For the complex number \(z = (2 + i)(4 + 3i) = 5 + 10i\), the modulus is

\[|z| = \sqrt{5^2 + 10^2} = 5\sqrt{5}\]

and the conjugate is

\(\bar{z} = 5 - 10i\)

(c) For the complex number \(z = \frac{3 - i}{\sqrt{2} + 3i}\), the modulus is

\[|z| = \frac{|3 - i|}{|\sqrt{2} + 3i|} = \frac{\sqrt{3^2 + (-1)^2}}{\sqrt{(\sqrt{2})^2 + 3^2}} = \frac{\sqrt{10}}{\sqrt{11}}\]

and the conjugate is

\[\bar{z} = \frac{3 - i}{\sqrt{2} + 3i} = \frac{3 + i}{\sqrt{2} - 3i} = \frac{(3 + i)(\sqrt{2} + 3i)}{(\sqrt{2} - 3i)(\sqrt{2} + 3i)} = \frac{3\sqrt{2} - 3}{11} + i\frac{\sqrt{2} + 9}{11}\]

2. Express each complex number below in exponential form. In each case, use the principal argument of the number.

(a) \(2i\)

(b) \(1 + i\)

(c) \(-2 + i\sqrt{12}\)
Solution:

(a) The modulus of $z = 2i$ is

$$|z| = \sqrt{0^2 + 2^2} = 2$$

The principal argument $\Theta$ is found from the equations

$$\cos \Theta = \frac{x}{|z|} = \frac{0}{2} = 0$$
$$\sin \Theta = \frac{y}{|z|} = \frac{2}{2} = 1$$

and is $\Theta = \frac{\pi}{2}$. Therefore, the exponential form of $z$ is

$$z = 2i = 2e^{i(\pi/2)}$$

(b) The modulus of $z = 1 + i$ is

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

The principal argument $\Theta$ is found from the equations

$$\cos \Theta = \frac{x}{|z|} = \frac{1}{\sqrt{2}}$$
$$\sin \Theta = \frac{y}{|z|} = \frac{1}{\sqrt{2}}$$

and is $\Theta = \frac{\pi}{4}$. Therefore, the exponential form of $z$ is

$$z = 1 + i = \sqrt{2}e^{i(\pi/4)}$$

(c) The modulus of $z = -2 + i\sqrt{12}$ is

$$|z| = \sqrt{(-2)^2 + (\sqrt{12})^2} = \sqrt{4 + 12} = 4$$

The principal argument $\Theta$ is found from the equations

$$\cos \Theta = \frac{x}{|z|} = \frac{-2}{4} = -\frac{1}{2}$$
$$\sin \Theta = \frac{y}{|z|} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$$

and is $\Theta = \frac{2\pi}{3}$. Therefore, the exponential form of $z$ is

$$z = -2 + i\sqrt{12} = 4e^{i(2\pi/3)}$$
3. Use DeMoivre’s Theorem to expand \((1 + i)^6\). Write your answer in the form \(a + bi\).

Solution: The modulus and principal argument of \(1 + i\) are \(|z| = r = \sqrt{2}\) and \(\Theta = \frac{\pi}{4}\). Then, using DeMoivre’s Theorem we have

\[
z^6 = r^6 [\cos (6\Theta) + i \sin (6\Theta)]
\]

\[
(1 + i)^6 = (\sqrt{2})^6 \left[ \cos \left( 6 \cdot \frac{\pi}{4} \right) + i \sin \left( 6 \cdot \frac{\pi}{4} \right) \right]
\]

\[
= 8(0 - i)
\]

= \(-8i\)

4. Show that \(e^{i\theta} = e^{-i\theta}\).

Solution: Starting with the left hand side we have

\[
\overline{e^{i\theta}} = \cos \theta + i \sin \theta = \cos \theta - i \sin \theta
\]

Now use the negative angle identities

\[\cos(-\theta) = \cos \theta, \quad \sin(-\theta) = -\sin \theta\]

to write \(\cos \theta - i \sin \theta\) as

\[
\overline{e^{i\theta}} = \cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta) = e^{(-i\theta)} = e^{-i\theta}
\]

5. Find all solutions to \(z^4 = -16\).

Solution: The modulus and principal argument of \(-16\) are \(|z| = r = 16\) and \(\Theta = \pi\), respectively. The fourth roots of \(-16\) are given by the formula

\[
z = r^{1/4} \left[ \cos \left( \frac{\Theta}{4} + \frac{2k\pi}{4} \right) + i \sin \left( \frac{\Theta}{4} + \frac{2k\pi}{4} \right) \right], \quad \text{for } k = 0, 1, 2, 3
\]

The roots are then

\[
z_1 = 16^{1/4} \left[ \cos \left( \frac{\pi}{4} + \frac{2(0)\pi}{4} \right) + i \sin \left( \frac{\pi}{4} + \frac{2(0)\pi}{4} \right) \right] = 2 \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = \sqrt{2} + i \sqrt{2}
\]

\[
z_2 = 16^{1/4} \left[ \cos \left( \frac{\pi}{4} + \frac{2(1)\pi}{4} \right) + i \sin \left( \frac{\pi}{4} + \frac{2(1)\pi}{4} \right) \right] = 2 \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = -\sqrt{2} + i \sqrt{2}
\]

\[
z_3 = 16^{1/4} \left[ \cos \left( \frac{\pi}{4} + \frac{2(2)\pi}{4} \right) + i \sin \left( \frac{\pi}{4} + \frac{2(2)\pi}{4} \right) \right] = 2 \left( -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) = -\sqrt{2} - i \sqrt{2}
\]

\[
z_4 = 16^{1/4} \left[ \cos \left( \frac{\pi}{4} + \frac{2(3)\pi}{4} \right) + i \sin \left( \frac{\pi}{4} + \frac{2(3)\pi}{4} \right) \right] = 2 \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) = \sqrt{2} - i \sqrt{2}
\]
6. Solve the equation 

\[ z^{4/3} + 2i = 0 \]

for \( z \) and plot the roots in the complex plane.

**Solution:** The solutions to the equation are given by

\[ z = (-2i)^{3/4} = \left[ (-2i)^3 \right]^{1/4} = (8i)^{1/4} \]

The modulus and principal argument of \( 8i \) are \(|z| = r = 8\) and \( \Theta = \frac{\pi}{2} \), respectively. The fourth roots of \( 8i \) are given by the formula

\[ z = r^{1/4} \left[ \cos \left( \frac{\Theta + 2k\pi}{4} \right) + i \sin \left( \frac{\Theta + 2k\pi}{4} \right) \right], \quad \text{for} \quad k = 0, 1, 2, 3 \]

The roots are then

\[
\begin{align*}
z_1 &= 8^{1/4} \left[ \cos \left( \frac{\pi/2 + 2(0)\pi}{4} \right) + i \sin \left( \frac{\pi/2 + 2(0)\pi}{4} \right) \right] = 4 \left( \cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \right) \\
z_2 &= 8^{1/4} \left[ \cos \left( \frac{\pi/2 + 2(1)\pi}{4} \right) + i \sin \left( \frac{\pi/2 + 2(1)\pi}{4} \right) \right] = 4 \left( \cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8} \right) \\
z_3 &= 8^{1/4} \left[ \cos \left( \frac{\pi/2 + 2(2)\pi}{4} \right) + i \sin \left( \frac{\pi/2 + 2(2)\pi}{4} \right) \right] = 4 \left( \cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8} \right) \\
z_4 &= 8^{1/4} \left[ \cos \left( \frac{\pi/2 + 2(3)\pi}{4} \right) + i \sin \left( \frac{\pi/2 + 2(3)\pi}{4} \right) \right] = 4 \left( \cos \frac{13\pi}{8} + i \sin \frac{13\pi}{8} \right)
\end{align*}
\]

and lie on a circle of radius \( 4\sqrt{8} \) centered at the origin, \( \frac{\pi}{2} \) radians apart.
7. Write the function \( f(z) = z^3 + z + 1 \) in the form \( f(x, y) = u(x, y) + i v(x, y) \).

**Solution:** To write \( f(z) \) in terms of \( x \) and \( y \) we substitute \( z = x + iy \) and simplify.

\[
\begin{align*}
    f(z) &= (x + iy)^3 + (x + iy) + 1 \\
    &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\
    &= x^3 + i(3x^2y) - 3xy^2 - i(y^3) + x + i(y) + 1 \\
    &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) \\
\end{align*}
\]

8. Suppose that \( f(z) = x^2 - y^2 - 2y + i(2x - 2xy) \), where \( z = x + iy \). Use the expressions

\[
    x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}
\]

to write \( f(z) \) in terms of \( z \) and simplify the result.

**Solution:** Substituting the above expressions into \( f(z) \) and simplifying we get

\[
\begin{align*}
    f(z) &= \left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 - 2 \left(\frac{z - \bar{z}}{2i}\right) + i \left[ 2 \left(\frac{z + \bar{z}}{2}\right) - 2 \left(\frac{z + \bar{z}}{2}\right) \left(\frac{z - \bar{z}}{2i}\right) \right] \\
    &= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) + \frac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) + i(z - \bar{z}) + i \left[ z + \bar{z} + i \left( \frac{1}{2}z^2 - \frac{1}{2}\bar{z}^2 \right) \right] \\
    &= \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 - \frac{1}{2}z^2 + \frac{1}{2}\bar{z}^2 + i(z - \bar{z} + z + \bar{z}) \\
    &= \bar{z}^2 + i(2z) \\
\end{align*}
\]

9. Find the image of the semi-infinite strip \( x \geq 0, 0 \leq y \leq \pi \) under the transformation \( w = e^z \) and label corresponding portions of the boundaries.

**Solution:** We did this in class.