1. Find all values of each expression below.

(a) \((1 - i)^i\)

(b) \(\cos(1 - i)\)

(c) \(\sin^{-1}(2)\)

Solution:

(a) Here we use the formula

\[ z^c = e^{c \log z} \]

\[ (1 - i)^i = e^{i \log(1 - i)} \]

The modulus of \(1 - i\) is \(r = \sqrt{2}\) and the principal argument is \(\Theta = -\frac{\pi}{4}\). Therefore,

\[ \log(1 - i) = \ln \sqrt{2} + i \left( -\frac{\pi}{4} + 2k\pi \right), \quad k = 0, \pm 1, \pm 2, \ldots \]

Multiplying \(\log(1 - i)\) by \(i\) we get

\[ i \log(1 - i) = i \left[ \ln \sqrt{2} + i \left( -\frac{\pi}{4} + 2k\pi \right) \right] \]

\[ i \log(1 - i) = \left( \frac{\pi}{4} + 2k\pi \right) + i \left( \ln \sqrt{2} \right) \]

Finally, we exponentiate \(i \log(1 - i)\) to get

\[ (1 - i)^i = e^{\pi/4 + 2k\pi + i \ln \sqrt{2}} \]

\[ (1 - i)^i = e^{\pi/2 + 2k\pi} e^{i \ln \sqrt{2}} \]

\[ (1 - i)^i = e^{\pi/4 + 2k\pi} \left[ \cos \left( \ln \sqrt{2} \right) + i \sin \left( \ln \sqrt{2} \right) \right] \]

where \(k = 0, \pm 1, \pm 2, \ldots\)

(b) Here we can use either

\[ \cos z = \frac{e^{iz} + e^{-iz}}{2} \]

\[ \cos(1 - i) = \frac{e^{i(1-i)} + e^{-i(1-i)}}{2} \]

\[ \cos(1 - i) = \frac{e^{1+i} + e^{-1-i}}{2} \]
\[
\cos z = \cosh x \cos y - i \sinh x \sin y \\
\cos(1 - i) = \cosh 1 \cos(-1) - i \sinh 1 \sin(-1) \\
\cos(1 - i) = \cosh 1 \cos 1 + i \sinh 1 \sin 1
\]

(c) Here we use the formula
\[
\sin^{-1} z = -i \log \left[ iz + (1 - z^2)^{1/2} \right] \\
\sin^{-1} 2 = -i \log \left[ 2i \pm \sqrt{3}i \right] \\
\sin^{-1} 2 = -i \log \left[ (2 \pm \sqrt{3})i \right]
\]

If we take the positive root, then we have
\[
\sin^{-1} 2 = -i \log \left[ (2 + \sqrt{3})i \right] \\
\sin^{-1} 2 = -i \left[ \ln(2 + \sqrt{3}) + i \left( \frac{\pi}{2} + 2k\pi \right) \right] \\
\sin^{-1} 2 = \frac{\pi}{2} + 2k\pi - i \ln(2 + \sqrt{3})
\]

If we take the negative root, then we have
\[
\sin^{-1} 2 = -i \log \left[ (2 - \sqrt{3})i \right] \\
\sin^{-1} 2 = -i \left[ \ln(2 - \sqrt{3}) + i \left( \frac{\pi}{2} + 2k\pi \right) \right] \\
\sin^{-1} 2 = \frac{\pi}{2} + 2k\pi - i \ln(2 - \sqrt{3})
\]

where \( k = 0, \pm 1, \pm 2, \ldots \).

2. Prove that \( \sin(2z) = 2 \sin z \cos z \) by using the definitions of \( \sin z \) and \( \cos z \).

**Solution:** Using the definition of \( \sin z \) we have
\[
\sin(2z) = \frac{e^{i(2z)} - e^{-i(2z)}}{2i} \\
\sin(2z) = \frac{(e^{iz} - e^{-iz})(e^{iz} + e^{-iz})}{2i} \\
\sin(2z) = 2 \left( \frac{e^{iz} - e^{-iz}}{2i} \right) \left( \frac{e^{iz} + e^{-iz}}{2} \right) \\
\sin(2z) = 2 \sin z \cos z
\]

where in the last step, we used the definitions of \( \sin z \) and \( \cos z \).
3. Find the values of $z$ for which $\cos z = 0$ by using the fact that

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

where $\sinh y = \frac{e^y - e^{-y}}{2}$

**Solution:** If $\cos z = 0$ then $|\cos z| = 0$. So it must be the case that both $\cos x = 0$ and $\sinh y = 0$ happen simultaneously. From the first equation we have

$$x = \frac{(2k + 1)\pi}{2}, \quad k = 0, \pm 1, \pm 2, \ldots$$

From the second equation we have $y = 0$. Therefore, $\cos z = 0$ when

$$z = \frac{(2k + 1)\pi}{2}, \quad k = 0, \pm 1, \pm 2, \ldots$$

4. Show that $f(z) = \sin(\bar{z})$ is analytic nowhere.

**Solution:** The function can be written as

$$\sin(\bar{z}) = \sin(x - iy)$$

$$\sin(\bar{z}) = \sin(x \cosh(-y) + i \cos x \sin(-y))$$

$$\sin(\bar{z}) = \sin x \cosh y - i \cos x \sinh y$$

Letting $u = \sin x \cosh y$ and $v = -\cos x \sinh y$ and computing their first partial derivatives we get

$$u_x = \cos x \cosh y, \quad v_y = -\cos x \cosh y$$

$$u_y = \sin x \sinh y, \quad v_x = \sin x \sinh y$$

In order for the Cauchy-Riemann equations ($u_x = v_y, \ u_y = -v_x$) to be satisfied, we need

$$\cos x \cosh y = 0 \quad \text{and} \quad \sin x \sinh y = 0$$

to occur simultaneously. From the first equation we can only have $\cos x = 0$ since $\cosh y > 0$ for all $y$. Therefore,

$$x = \frac{(2k + 1)\pi}{2}, \quad k = 0, \pm 1, \pm 2, \ldots$$

From the second equation we must have $\sinh y = 0$ because $\sin x$ and $\cos x$ cannot be 0 simultaneously. Therefore, $y = 0$.

Thus, since the first partial derivatives of $u$ and $v$ are continuous everywhere in the complex plane and the Cauchy-Riemann equations are satisfied for $z = \frac{(2k + 1)\pi}{2}$, $f'(z)$ exists for these values of $z$. However, at each point there is no neighborhood throughout which $f(z)$ is analytic. Therefore, $f(z) = \sin(\bar{z})$ is analytic nowhere.
5. Evaluate the integral
\[ \int_C e^z \, dz \]
where \( C \) is the contour consisting of the two straight-line segments: (1) from \( z = i \) to \( z = 1 + i \) and (2) from \( z = 1 + i \) to \( z = 1 - 2i \).

**Solution:** To evaluate the integral we integrate over each line segment and then add the results. On the first segment we have the parametrization
\[ z(t) = t + i, \quad 0 \leq t \leq 1 \]
Therefore, the integral of \( f(z) \) over this segment is
\[
\int_{C_1} f(z) \, dz = \int_0^1 f(z(t)) z'(t) \, dt = \int_0^1 e^{t+i} (1) \, dt = e^{t+i} \bigg|_0^1 = e^{1+i} - e^i
\]
On the second segment we have the parametrization
\[ z(t) = 1 + it, \quad -2 \leq t \leq 1 \]
Therefore, the integral of \( f(z) \) over this segment is
\[
\int_{C_2} f(z) \, dz = \int_1^{-2} f(z(t)) z'(t) \, dt = \int_1^{-2} e^{1+it} (i) \, dt = e^{1+it} \bigg|_1^{-2} = e^{1-2i} - e^{1+i}
\]
The value of the integral is then
\[
\int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz = e^{1+i} - e^i + e^{1-2i} - e^{1+i} = e^{1-2i} - e^i = e((\cos 2 - i \sin 2) + e(\cos 1 + i \sin 1)) = e[(\cos 2 + \cos 1) + i(\sin 1 - \sin 2)]
\]
Note: Instead of using parametrizations, we could have said that \( f(z) \) is entire so it has an antiderivative \( F(z) = e^z \) and the value of the integral is

\[
\int_C e^z \, dz = F(1-2i) - F(i) = e^{1-2i} - e^i
\]

which is exactly what we obtained above.

6. Evaluate the integral

\[
\int_C (z^2 - 1) \, dz
\]

where \( C \) is the semicircle \( z = e^{it}, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \) oriented counterclockwise.

Solution: The value of the integral is

\[
\int_C f(z) \, dz = \int_a^b f(z(t))z'(t) \, dt = \int_{\pi/2}^{\pi/2} (e^{2it} - 1) ie^{it} \, dt
\]

\[
= i \int_{\pi/2}^{\pi/2} (e^{3it} - e^{it}) \, dt
\]

\[
= i \left[ \frac{1}{3i}e^{3it} - \frac{1}{i}e^{it} \right]_{\pi/2}^{\pi/2}
\]

\[
= \left( \frac{1}{3}e^{i(3\pi/2)} - e^{i(\pi/2)} \right) - \left( \frac{1}{3}e^{i(-3\pi/2)} - e^{i(-\pi/2)} \right)
\]

\[
= -\frac{1}{3}i - i - \frac{1}{3}i - i
\]

\[
= -\frac{8}{3}i
\]

Note: Instead of using the parametrization, we could have said that \( f(z) \) is entire so it has an antiderivative \( F(z) = \frac{1}{3}z^3 - z \) and the value of the integral is

\[
\int_C (z^2 - 1) \, dz = F(i) - F(-i) = \left( \frac{1}{3}i^3 - i \right) - \left( \frac{1}{3}(-i)^3 - (-i) \right) = -\frac{1}{3}i - \frac{1}{3}i - i = -\frac{8}{3}i
\]

which is exactly what we obtained above.

7. Show that

\[
\left| \int_C \frac{2z + 1}{z^2 - 4} \, dz \right| \leq \pi
\]
where \( C \) is the upper half of the circle \(|z| = 1\) oriented counterclockwise. Justify your answer.

**Solution:** The length of the contour is \( L = \pi \). Now we must find an upper bound on \(|f(z)|\). Using the triangle inequality \(|z_1 + z_2| \leq |z_1| + |z_2|\) on the numerator we have

\[
|2z + 1| \leq 2|z| + 1 = 2 + 1 = 3
\]

Using the triangle inequality \(|z_1 - z_2| \geq ||z_1| - |z_2||\) on the denominator we have

\[
|z^2 - 4| \geq ||z|^2 - 4| = |1 - 4| = 3
\]

Thus, the modulus of \( f(z) \) satisfies the inequality

\[
|f(z)| = \left| \frac{2z + 1}{z^2 - 4} \right| \leq \frac{3}{3} = 1
\]

Choosing \( M = 1 \) and using the formula for the \( ML \)-Bound we have

\[
\left| \int_C \frac{2z + 1}{z^2 - 4} \, dz \right| \leq ML = \pi
\]

8. Find an upper bound on

\[
\left| \int_C \frac{dz}{z^2 + 1} \right|
\]

where \( C \) is the circle \(|z - i| = 1\) oriented counterclockwise. Justify your answer.

**Solution:** The length of the contour is \( L = 2\pi \). To find an upper bound on \(|f(z)|\) we’ll factor \( z^2 + 1 \) and take the modulus to get

\[
\left| \frac{1}{z^2 + 1} \right| = \frac{1}{|z - i||z + i|} = \frac{1}{1 \cdot |(z - i) + 2i|} = \frac{1}{|(z - i) + 2i|}
\]

Now we use the triangle inequality \(|z_1 + z_2| \geq ||z_1| - |z_2||\) on the denominator to get

\[
|(z - i) + 2i| \geq ||z - i| - |2i|| = |1 - 2| = 1
\]

Thus, we have

\[
\left| \frac{1}{z^2 + 1} \right| \leq \frac{1}{1} = 1
\]

Choosing \( M = 1 \) and using the \( ML \)-Bound formula we have

\[
\left| \int_C \frac{dz}{z^2 + 1} \right| \leq ML = 2\pi
\]