1. Find the radius of convergence for each power series below.

(a) \[ \sum_{n=2}^{\infty} n^2(z - 3)^n \]

(b) \[ \sum_{n=4}^{\infty} e^n(z + i)^n \]

Solution:

(a) Using the Ratio Test we have

\[
L = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n \to \infty} \frac{|(n + 1)^2(z - 3)^{n+1}|}{|n^2(z - 3)^n|} = \lim_{n \to \infty} \frac{(n + 1)^2}{n^2} |z - 3| = |z - 3| \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^2 = |z - 3|
\]

The series converges when \( L = |z - 3| < 1 \). Therefore, the radius of convergence is 1.

(b) Using the Ratio Test we have

\[
L = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n \to \infty} \frac{|e^{n+1}(z + i)^{n+1}|}{|e^n(z + i)^n|} = \lim_{n \to \infty} e|z + i| = e|z + i|
\]

The series converges when \( L = e|z + i| < 1 \implies |z + i| < \frac{1}{e} \). Therefore, the radius of convergence is \( \frac{1}{e} \).
2. What is the radius of convergence of the Taylor Series of \( f(z) = \frac{1}{z^2 - 3z + 2} \) about \( z = 0 \)\? about \( z = 3i \)\? 

**Solution:** The singular points of \( f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} \) are \( z = 1 \) and \( z = 2 \). Therefore, since \( f(z) \) is analytic at \( z = 0 \), it has a Taylor Series representation for all \( z \) satisfying \( |z| < R \) where \( R \) is the distance between \( z = 0 \) and the nearest singular point which is \( z = 1 \). Therefore, \( R = |1 - 0| = 1 \).

Since \( f(z) \) is analytic at \( z = 3i \), it has a Taylor Series representation for all \( z \) satisfying \( |z - 3i| < R \) where \( R \) is the distance between \( z = 3i \) and the nearest singular point which is \( z = 1 \). Therefore, \( R = |1 - 3i| = \sqrt{10} \).

Region of convergence about \( z = 0 \). Region of convergence about \( z = 3i \).

3. Find the Taylor Series of \( f(z) = \frac{z}{1 + z^2} \) about \( z = 0 \) and state the region of validity. Write your answer in summation form.

**Solution:** The singular points of \( f(z) \) are \( z = i \) and \( z = -i \). Since \( f(z) \) is analytic at \( z = 0 \), it has a Taylor Series representation for all \( z \) satisfying \( |z| < R \) where \( R \) is the distance between \( z = 0 \) and the closest singular point. Both singular points are at a distance of 1 from the origin. Therefore, the region of validity is \( |z| < 1 \).

We are looking for a series representation in the form

\[
f(z) = \sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots
\]

To get the Taylor Series we will write \( f(z) \) as

\[
f(z) = \frac{z}{1 + z^2} = z \cdot \frac{1}{1 + z^2}
\]
and then use the Maclaurin Series for $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$ and replace z with $z^2$ to get

$$f(z) = z \cdot \frac{1}{1 + z^2}$$
$$f(z) = z \cdot (1 - z^2 + (z^2)^2 - (z^2)^3 + \cdots)$$
$$f(z) = z - z^3 + z^5 - z^7 + \cdots$$
$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}$$

4. Find the Laurent Series of $f(z) = \frac{z}{1+z}$ about $z = 0$ in the region $1 < |z| < \infty$. Write your answer in summation form.

**Solution:** We are looking for a series representation in the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n = \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

To obtain this series we will rewrite $f(z)$ as

$$f(z) = z \cdot \frac{1}{1 + z}$$
$$f(z) = z \cdot \frac{1}{z \left( \frac{1}{z} + 1 \right)}$$
$$f(z) = \frac{1}{1 + \frac{1}{z}}$$

and then use the Maclaurin Series for $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$ and replace z with $\frac{1}{z}$ to get

$$f(z) = \frac{1}{1 + \frac{1}{z}}$$
$$f(z) = 1 - \frac{1}{z} + \left( \frac{1}{z} \right)^2 - \left( \frac{1}{z} \right)^3 + \cdots$$
$$f(z) = 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots$$
$$f(z) = \sum_{n=0}^{0} (-1)^n z^n \text{ or } \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^n}$$

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5. Determine all regions for which $f(z)$ has a Taylor Series expansion about $z = 2$. Then determine all regions for which $f(z)$ has a Laurent Series expansion about $z = 2$.

DO NOT FIND THE SERIES EXPANSIONS!

(a) $f(z) = e^z$
(b) $f(z) = \frac{1}{z^2 + 1}$
(c) $f(z) = \frac{1}{z(z + 1)(z + 2i)}$

Solution:

(a) The function is entire so it has a Taylor Series expansion that is valid for $|z - 2| < \infty$.

(b) The function has singular points at $z = i$ and $z = -i$. Since $f(z)$ is analytic at $z = 2$ it has a Taylor Series expansion for all $z$ satisfying $|z - 2| < R$ where $R$ is the distance between $z = 2$ and the nearest singular point. Both singular points are at a distance of $R = \sqrt{5}$ from $z = 2$. Therefore, $f(z)$ has a Taylor Series expansion in the region $|z - 2| < \sqrt{5}$ and a Laurent Series expansion in the region $\sqrt{5} < |z - 2| < \infty$. 
(c) The function has singular points at \( z = 0 \), \( z = -1 \), and \( z = -2i \). Since \( f(z) \) is analytic at \( z = 2 \) it has a Taylor Series expansion for all \( z \) satisfying \( |z - 2| < R \) where \( R \) is the distance between \( z = 2 \) and the nearest singular point which is \( z = 0 \). The distance between these points is \( R = 2 \) so \( f(z) \) has a Taylor Series expansion in the region \( |z - 2| < 2 \).

The next closest singular point is \( z = -2i \). The distance between \( z = 2 \) and \( z = -2i \) is \( R = |-2i - 2| = 2\sqrt{2} \). Therefore, \( f(z) \) has a Laurent Series expansion in the region \( 2 < |z - 2| < 2\sqrt{2} \).

The distance between \( z = 2 \) and the last singular point \( z = -1 \) is \( R = |1 - 2| = 3 \). Therefore, \( f(z) \) has another Laurent Series expansion in the region \( 2\sqrt{2} < |z - 2| < 3 \).

Finally, \( f(z) \) has a third Laurent Series expansion in the region \( 3 < |z - 2| < \infty \).

If we were interested in finding the series expansions for \( f(z) = \frac{1}{z(z + 1)(z + 2i)} \) about \( z = 2 \), we would perform a Partial Fraction Decomposition of \( f(z) \) to get

\[
f(z) = \frac{1}{z(z + 1)(z + 2i)} = \frac{c_1}{z} + \frac{c_2}{z + 1} + \frac{c_3}{z + 2i}
\]

where \( c_1 \), \( c_2 \), and \( c_3 \) are complex numbers. Then, on each interval we would write either a Taylor or Laurent Series for each function and it would go as follows:
6. Find the Laurent Series of \( f(z) = \frac{1}{z^2 - 4} \) about \( z = -1 \) in the region \( 1 < |z + 1| < 3 \).

It is not necessary to write your answer in summation form. However, you should write out sufficiently many terms so that the pattern is clear.

**Solution:** First, we use the Method of Partial Fractions to rewrite the function as

\[
f(z) = \frac{1}{z^2 - 4} = \frac{1}{4} \cdot \frac{1}{z - 2} - \frac{1}{4} \cdot \frac{1}{z + 2}
\]

The function \( f_1(z) = \frac{1}{z - 2} \) has a singular point at \( z = 2 \). Since \( f_1(z) \) is analytic at \( z = -1 \) and the distance between \( z = -1 \) and \( z = 2 \) is 3, \( f_1(z) \) has a Taylor Series expansion in the region \( |z + 1| < 3 \). Since we are looking for a series expansion for \( f(z) \) in the annulus \( 1 < |z + 1| < 3 \), we will write the Taylor Series for \( f_1(z) \) around \( z = -1 \).
\[
\begin{align*}
  f_1(z) &= \frac{1}{z - 2} \\
  f_1(z) &= \frac{1}{(z + 1) - 3} \\
  f_1(z) &= \frac{1}{3 \left( \frac{z + 1}{3} - 1 \right)} \\
  f_1(z) &= -\frac{1}{3} \cdot \frac{1}{1 - \frac{z + 1}{3}} \\
  f_1(z) &= -\frac{1}{3} \left( 1 + \frac{z + 1}{3} + \left( \frac{z + 1}{3} \right)^2 + \left( \frac{z + 1}{3} \right)^3 + \cdots \right) \\
  f_1(z) &= -\frac{1}{3} - \frac{z + 1}{3^2} - \frac{(z + 1)^2}{3^3} - \frac{(z + 1)^3}{3^4} - \cdots \\
\end{align*}
\]

The function \( f_2(z) = \frac{1}{z + 2} \) has a singular point at \( z = -2 \). Since \( f_2(z) \) is analytic at \( z = -1 \) and the distance between \( z = -1 \) and \( z = -1 \) is 1, \( f_2(z) \) has a Taylor Series expansion in the region \( |z + 1| < 1 \). However, we are interested in the series expansion of \( f(z) \) in the annulus \( 1 < |z + 1| < 3 \). Therefore, we want to write the Laurent Series of \( f_2(z) \) around \( z = -1 \).

\[
\begin{align*}
  f_2(z) &= \frac{1}{z - 2} \\
  f_2(z) &= \frac{1}{(z + 1) - 3} \\
  f_2(z) &= \frac{1}{(z + 1) \left( 1 - \frac{3}{z + 1} \right)} \\
  f_2(z) &= \frac{1}{z + 1} \cdot \frac{1}{1 - \frac{z + 1}{z + 1}} \\
  f_2(z) &= \frac{1}{z + 1} \left( 1 + \frac{3}{z + 1} + \left( \frac{3}{z + 1} \right)^2 + \left( \frac{3}{z + 1} \right)^3 + \cdots \right) \\
  f_2(z) &= \frac{1}{z + 1} + \frac{3}{(z + 1)^2} + \frac{3^2}{(z + 1)^3} + \frac{3^3}{(z + 1)^4} + \cdots \\
\end{align*}
\]
Putting the series expansions for \( f_1(z) \) and \( f_2(z) \) back into the formula for \( f(z) \) we get

\[
f(z) = \frac{1}{4} f_1(z) - \frac{1}{4} f_2(z)
\]

\[
f(z) = \frac{1}{4} \left[ -\frac{1}{3} + \frac{z + 1}{3^2} - \frac{(z + 1)^2}{3^3} - \cdots \right] - \frac{1}{4} \left[ \frac{1}{z + 1} + \frac{3}{(z + 1)^2} + \frac{3^2}{(z + 1)^3} + \cdots \right]
\]

\[
f(z) = \cdots - \frac{3^2}{4(z + 1)^3} - \frac{3^1}{4(z + 1)^2} - \frac{3^0}{4(z + 1)} - \frac{3^{-1}}{4(z + 1)} - \frac{3^{-2}}{4(z + 1)^2} - \cdots
\]