1. Find the limit, \( \lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - x} \).

**Solution**: Upon substituting \( x = 1 \) into the function \( f(x) = \frac{x^2 + 2x - 3}{x^2 - x} \) we find that

\[
\frac{x^2 + 2x - 3}{x^2 - x} = \frac{1^2 + 2(1) - 3}{1^2 - 1} = \frac{0}{0}
\]

which is indeterminate. We can resolve the indeterminacy by factoring the numerator and denominator of \( f(x) \).

\[
\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - x} = \lim_{x \to 1} \frac{(x + 3)(x - 1)}{(x - 1)x} = \lim_{x \to 1} \frac{x + 3}{x} = \frac{1 + 3}{1} = 4
\]

In the final step above we were able to plug in \( x = 1 \) by using the fact that the function \( \frac{x + 3}{x} \) is continuous at \( x = 1 \). In fact, \( \frac{x + 3}{x} \) is continuous at all values of \( x \) in its domain \( (x \neq 0) \).
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Problem 2 Solution

2. Find the derivatives of the following functions using the basic rules. Do not simplify your answer.

(a) $4x^3 - 5x^{1/3} + 3x^{-2}$  
(b) $(x^2 - 3x)e^x$  
(c) $\frac{x - 3}{x^2 + x + 1}$

Solution:

(a) Use the Power Rule.

$$(4x^3 - 5x^{1/3} + 3x^{-2})' = 12x^2 - \frac{5}{3}x^{-2/3} - 6x^{-3}$$

(b) Use the Product Rule.

$$[(x^2 - 3x)e^x]' = (x^2 - 3x)(e^x)' + (x^2 - 3x)'e^x$$
$$= (x^2 - 3x)e^x + (2x - 3)e^x$$

(c) Use the Quotient Rule.

$$\left(\frac{x - 3}{x^2 + x + 1}\right)' = \frac{(x^2 + x + 1)(x - 3)' - (x - 3)(x^2 + x + 1)'}{(x^2 + x + 1)^2}$$
$$= \frac{(x^2 + x + 1) - (x - 3)(2x + 1)}{(x^2 + x + 1)^2}$$
3. Find the equation of the tangent line to $y = x^3 - 3x$ at $x = 2$.

Solution: The derivative $y'$ is found using the Power Rule.

$$y' = (x^3 - 3x)' = 3x^2 - 3$$

At $x = 2$ the values of $y$ and $y'$ are:

$$y(2) = 2^3 - 3(2) = 2$$
$$y'(2) = 3(2)^2 - 3 = 9$$

We now know that the point $(2, 2)$ is on the tangent line and that the slope of the tangent line is 9. Therefore, an equation for the tangent line in point-slope form is:

$$y - 2 = 9(x - 2)$$
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Problem 4 Solution

4. Let \( f(x) = \sqrt{x} \).

(a) Find the average rate of change of \( f(x) \) over the interval \( 4 \leq x \leq 9 \).

(b) Find the instantaneous rate of change of \( f(x) \) at \( x = 4 \).

Solution:

(a) The average rate of change formula is:

\[
\text{average ROC} = \frac{f(b) - f(a)}{b - a}
\]

Using \( f(x) = \sqrt{x} \), \( b = 9 \), and \( a = 4 \) we have:

\[
\text{average ROC} = \frac{\sqrt{9} - \sqrt{4}}{9 - 4} = \frac{3 - 2}{5} = \frac{1}{5}
\]

(b) The instantaneous rate of change at \( x = 4 \) is \( f'(4) \). The derivative \( f'(x) \) is:

\[
f'(x) = \frac{1}{2\sqrt{x}}
\]

At \( x = 4 \) we have:

\[
\text{instantaneous ROC} = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}
\]
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Problem 5 Solution

5. Let \( f(x) = \frac{1}{x} \).

(a) Write the derivative, \( f'(5) \), as the limit of the difference quotient.

(b) Evaluate this limit to find \( f'(5) \).

Solution:

(a) There are two possible difference quotients we can use to evaluate \( f'(5) \). One is:

\[
f'(5) = \lim_{h \to 0} \frac{f(h + 5) - f(5)}{h} = \lim_{h \to 0} \frac{1}{h + 5} - \frac{1}{5}.
\]

The other is:

\[
f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \to 5} \frac{1}{x} - \frac{1}{5}.
\]

(b) Evaluating the first limit above we have:

\[
f'(5) = \lim_{h \to 0} \frac{1}{h} \cdot \frac{5(h + 5) - 5}{5(h + 5)}
\]

\[
= \lim_{h \to 0} \frac{5 - (h + 5)}{5h(h + 5)}
\]

\[
= \lim_{h \to 0} \frac{-h}{5h(h + 5)}
\]

\[
= \lim_{h \to 0} \frac{-1}{5(h + 5)}
\]

\[
= \frac{-1}{5(0 + 5)}
\]

\[
= \frac{-1}{25}
\]
Evaluating the second limit we have:

\[
f'(5) = \lim_{x \to 5} \frac{1 - \frac{1}{5}}{\frac{5x}{x - 5}} \cdot \frac{5x}{5x}
\]

\[
= \lim_{x \to 5} \frac{5 - x}{5x(x - 5)}
\]

\[
= \lim_{x \to 5} \frac{-(x - 5)}{5x(x - 5)}
\]

\[
= \lim_{x \to 5} \frac{-1}{5x}
\]

\[
= \frac{-1}{5(5)}
\]

\[
= -\frac{1}{25}
\]
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Problem 6 Solution

6. Use the table below, which shows values of \( f(x) \) for \( x \) near 2.5,

\[
\begin{array}{cccccc}
 x & 2.3 & 2.4 & 2.5 & 2.6 & 2.7 \\
 f(x) & 1.41 & 1.40 & 1.38 & 1.35 & 1.31 \\
\end{array}
\]

to find the slope of a secant line that is an estimate for \( f'(2.5) \). Why did you choose the line you did?

**Solution:** An approximate value for \( f'(2.5) \) is

\[
f'(2.5) \approx \frac{f(2.6) - f(2.5)}{2.6 - 2.5} = \frac{1.35 - 1.38}{0.1} = -0.3
\]

This formula was used because the exact value of \( f'(2.5) \) is:

\[
f'(2.5) = \lim_{x \to 2.5} \frac{f(x) - f(2.5)}{x - 2.5}
\]

As we approach \( x = 2.5 \) from the right, we can plug in either \( x = 2.7 \) or \( x = 2.6 \) to estimate the value of \( f'(2.5) \). We used \( x = 2.6 \) because the estimate is generally more accurate as \( x \) gets closer and closer to 2.5.