Math 180, Exam 1, Fall 2011
Problem 1 Solution

1. Evaluate the following limits. Show your work.

   (a) \( \lim_{x \to 0} \frac{2 \cos x}{\sqrt{x + 1} - 2} \)

   (b) \( \lim_{x \to +\infty} \frac{2x^2 + 4x - 1}{x^3 + 1} \)

Solution:

(a) The function \( f(x) = \frac{2 \cos x}{\sqrt{x + 1} - 2} \) is continuous for all \( x \in (-1, 3) \cup (3, +\infty) \). Therefore, since \( f(x) \) is continuous at \( x = 0 \), we can evaluate the limit using substitution.

\[
\lim_{x \to 0} \frac{2 \cos x}{\sqrt{x + 1} - 2} = \frac{2 \cos 0}{\sqrt{0 + 1} - 2} = \boxed{-2}
\]

(b) This is a limit at infinity of a rational function. Our approach is to multiply the function by \( \frac{1}{x^3} \) divided by itself and simplify:

\[
\lim_{x \to +\infty} \frac{2x^2 + 4x - 1}{x^3 + 1} = \lim_{x \to +\infty} \frac{2x^2 + 4x - 1}{x^3 + 1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}
\]

\[
= \lim_{x \to +\infty} \frac{\frac{2}{x} + \frac{4}{x^2} - \frac{1}{x^3}}{1 + \frac{1}{x^3}}
\]

Using the fact that \( \lim_{x \to +\infty} \frac{1}{x^n} = 0 \) for \( n > 0 \), we find that:

\[
\lim_{x \to +\infty} \frac{2x^2 + 4x - 1}{x^3 + 1} = \lim_{x \to +\infty} \frac{\frac{2}{x} + \frac{4}{x^2} - \frac{1}{x^3}}{1 + \frac{1}{x^3}}
\]

\[
= 0 + 0 - 0 = \frac{0}{1 + 0} = \boxed{0}
\]
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Problem 2 Solution

2. Compute the derivatives of the following functions AND state where the derivative does not exist. Show your work and do not simplify your answers.

(a) \( \frac{x^2 + 1}{x} \)

(b) \( |x| \)

(c) \( e^{\sin(3x)} \)

Solution:

(a) We begin by rewriting the function as follows:

\[
\frac{x^2 + 1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}
\]

We now use the Power Rule to compute the derivative:

\[
\left( \frac{x^2 + 1}{x} \right)' = \left( x + \frac{1}{x} \right)' = 1 - \frac{1}{x^2}
\]

The derivative exists for all \( x \neq 0 \).

(b) By definition, the absolute value function is the piecewise defined function:

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]

The derivative of \( |x| \) is then:

\[
(|x|)' = \begin{cases} 
  1 & \text{if } x > 0 \\
  -1 & \text{if } x < 0 
\end{cases}
\]

The derivative exists for all \( x \neq 0 \). It does not exist at \( x = 0 \) because the limit

\[
\lim_{h \to 0} \frac{|0 + h| - |0|}{h},
\]

which defines the derivative of \( |x| \) at \( x = 0 \), does not exist (the one-sided limits are 1 and \(-1\)).

(c) We use the Chain Rule.

\[
\left( e^{\sin(3x)} \right)' = e^{\sin(3x)} (\sin(3x))' = e^{\sin(3x)} \cdot 3 \cos(3x)
\]

The derivative exists for all \( x \).
3. (a) Find an equation of the tangent line at \( x_0 = 1 \) to the graph of the following function:

\[
f(x) = x^4 - x^2 + 1
\]

(b) Find all those points \( x_0 \) where the tangent line to the graph is horizontal. Show your work.

Solution: (a) The slope of the tangent line is the derivative \( f'(1) \) and a point on the tangent line is \((1, f(1))\). The derivative of \( f(x) \) is \( f'(x) = 4x^3 - 2x \). Therefore, \( f'(1) = 2 \). We also have \( f(1) = 1 \). Thus, the equation of the tangent line in point-slope form is:

\[
y - 1 = 2(x - 1)
\]

(b) A horizontal line has a slope of 0. Therefore, we seek the values of \( x \) satisfying \( f'(x) = 0 \).

\[
f'(x) = 0
\]

\[
4x^3 - 2x = 0
\]

\[
2x(2x^2 - 1) = 0
\]

We either have \( 2x = 0 \) or \( 2x^2 - 1 = 0 \). The first equation gives us \( x = 0 \) while the second equation gives us \( x = \pm \frac{1}{\sqrt{2}} \).
4. Use the Intermediate Value Theorem to show that there exists a solution to the equation $\cos x = x$ on the interval $[0, \frac{\pi}{2}]$. Show your work.

**Solution:** Let $f(x) = \cos(x) - x$. First we recognize that $f(x)$ is continuous everywhere. Next, we must show that $f(0)$ and $f\left(\frac{\pi}{2}\right)$ have opposite signs.

\[
f(0) = \cos(0) - 0 = 1
\]
\[
f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = -\frac{\pi}{2}
\]

Since $f(0) > 0$ and $f\left(\frac{\pi}{2}\right) < 0$, the Intermediate Value Theorem tells us that $f(c) = 0$ for some $c$ in the interval $\left(0, \frac{\pi}{2}\right)$. 

5. Consider the function \( f \) whose graph appears below and answer the following questions. **You must justify all answers.**

(a) (i) Is \( f(1) \) defined? If so, what is it?
(ii) Does \( \lim_{x \to 1} f(x) \) exist? If so, what is it?
(iii) Is \( f \) continuous at 1?

(b) (i) Is \( f(2) \) defined? If so, what is it?
(ii) Does \( \lim_{x \to 2} f(x) \) exist? If so, what is it?
(iii) Is \( f \) continuous at 2?

(c) (i) Is \( f(4) \) defined? If so, what is it?
(ii) Does \( \lim_{x \to 4} f(x) \) exist? If so, what is it?
(iii) Is \( f \) continuous at 4?

(d) (i) Is \( f(6) \) defined? If so, what is it?
(ii) Does \( \lim_{x \to 6} f(x) \) exist? If so, what is it?
(iii) Is \( f \) continuous at 6?

**Solution:**

(a) (i) \( f(1) = 3 \)
(ii) \( \lim_{x \to 1} f(x) \) does not exist because the one-sided limits are not the same (\( \lim_{x \to 1^-} f(x) = 3 \) but \( \lim_{x \to 1^+} f(x) = 2 \)).
(iii) \( f \) is not continuous at 1 because \( \lim_{x \to 1} f(x) \neq f(1) \)
(b) (i) $f(2)$ is not defined
   (ii) $\lim_{x \to 2} f(x) = 3$ because the one-sided limits are the same ($\lim_{x \to 2^+} f(x) = 3$ and $\lim_{x \to 2^-} f(x) = 3$).
   (iii) $f$ is not continuous at 2 because $\lim_{x \to 2} f(x) \neq f(2)$

(c) (i) $f(4) = 1$
   (ii) $\lim_{x \to 4} f(x) = 1$ because the one-sided limits are the same ($\lim_{x \to 4^+} f(x) = 1$ and $\lim_{x \to 4^-} f(x) = 1$).
   (iii) $f$ is continuous at 4 because $\lim_{x \to 4} f(x) = f(4)$

(d) (i) $f(6) = 3$
   (ii) $\lim_{x \to 6} f(x) = 2$ because the one-sided limits are the same ($\lim_{x \to 6^+} f(x) = 2$ and $\lim_{x \to 6^-} f(x) = 2$).
   (iii) $f$ is not continuous at 6 because $\lim_{x \to 6} f(x) \neq f(6)$