1. Evaluate the following limits, or show they do not exist.

(a) \( \lim_{x \to \pi} 2 \cos x \)

(b) \( \lim_{x \to 2} \frac{x^2 - 4}{x + 2} \)

(c) \( \lim_{x \to 9} \frac{2 - \sqrt{x - 5}}{x - 9} \)

Solution:

(a) The function \( f(x) = 2 \cos x \) is continuous at \( x = \pi \). In fact, \( f(x) \) is continuous at all \( x \) in the interval \(( -\infty, \infty \)\). Therefore, we can evaluate the limit using substitution.

\[ \lim_{x \to \pi} 2 \cos x = 2 \cos \pi = -2 \]

(b) The function \( f(x) = \frac{x^2 - 4}{x + 2} \) is continuous at \( x = 2 \). In fact, \( f(x) \) is continuous at all \( x \neq -2 \). Therefore, we can evaluate the limit using substitution.

\[ \lim_{x \to 2} \frac{x^2 - 4}{x + 2} = \frac{2^2 - 4}{2 + 2} = 0 \]

(c) When substituting \( x = 9 \) into the function \( f(x) = \frac{2 - \sqrt{x - 5}}{x - 9} \) we find that

\[ \frac{2 - \sqrt{9 - 5}}{9 - 9} = \frac{0}{0} \]

which is indeterminate. We can resolve the indeterminacy by multiplying \( f(x) \) by the
“conjugate” of the numerator divided by itself.

\[
\lim_{x \to 9} \frac{2 - \sqrt{x - 5}}{x - 9} = \lim_{x \to 9} \frac{2 - \sqrt{x - 5}}{x - 9} \cdot \frac{2 + \sqrt{x - 5}}{2 + \sqrt{x - 5}}
\]

\[
= \lim_{x \to 9} \frac{4 - (x - 5)}{(x - 9)(2 + \sqrt{x - 5})}
\]

\[
= \lim_{x \to 9} \frac{-(x - 9)}{(x - 9)(2 + \sqrt{x - 5})}
\]

\[
= \lim_{x \to 9} \frac{-1}{2 + \sqrt{x - 5}}
\]

\[
= \frac{-1}{4}
\]

We evaluated the limit above by substituting \(x = 9\) into the function \(\frac{-1}{2 + \sqrt{x - 5}}\). This is possible because the function is continuous at \(x = 9\).
2. Determine the location and type (removable, jump, infinite, or other) of all discontinuities of the function \( \frac{x^2 - 3x + 2}{x^2 - 1} \).

**Solution:** We start by factoring the numerator and denominator.

\[
\frac{x^2 - 3x + 2}{x^2 - 1} = \frac{(x - 2)(x - 1)}{(x + 1)(x - 1)}
\]

As \( x \to -1^+ \), we find that:

\[
\lim_{x \to -1^+} \frac{x^2 - 3x + 2}{x^2 - 1} = \lim_{x \to -1^+} \frac{(x - 2)(x - 1)}{(x + 1)(x - 1)}
\]

\[
= \lim_{x \to -1^+} \frac{x - 2}{x + 1}
\]

\[
= -\infty
\]

Therefore, \( x = -1 \) is an infinite discontinuity.

The limit at \( x = 1 \) is:

\[
\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 1} = \lim_{x \to 1} \frac{(x - 2)(x - 1)}{(x + 1)(x - 1)}
\]

\[
= \lim_{x \to 1} \frac{x - 2}{x + 1}
\]

\[
= \frac{1 - 2}{1 + 1}
\]

\[
= -\frac{1}{2}
\]

However, \( f(1) \) does not exist. Using our textbook’s definitions, \( x = 1 \) cannot be categorized as a removable, jump, or infinite discontinuity. Therefore, \( x = 1 \) falls under the “other” category.
3. Find the equation of the tangent line to \( y = x^3 - 2x^2 + 2 \) at \( x = 1 \).

**Solution:** The derivative \( y' \) is found using the Power Rule.

\[
y' = (x^3 - 2x^2 + 2)' = 3x^2 - 4x
\]

At \( x = 1 \) the values of \( y \) and \( y' \) are:

\[
y(1) = 1^3 - 2(1)^2 + 2 = 1
\]
\[
y'(1) = 3(1)^2 - 4(1) = -1
\]

We now know that the point \((1, 1)\) is on the tangent line and that the slope of the tangent line is \(-1\). Therefore, an equation for the tangent line in point-slope form is:

\[
y - 1 = -(x - 1)
\]
4. Determine the value of $c$ so that the function

$$f(x) = \begin{cases} 
3cx + 1 & \text{if } x < 1 \\
5x^2 + c & \text{if } x \geq 1 
\end{cases}$$

is continuous on $\mathbb{R}$.

**Solution:** The functions $3cx + 1$ and $5x^2 + c$ are continuous for all $x$. In order for $f(x)$ to be continuous on $\mathbb{R}$, we must select $c$ so that $f(x)$ is continuous at $x = 1$. To do this, we must compute the one-sided limits at $x = 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3cx + 1) = 3c(1) + 1 = 3c + 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5x^2 + c) = 5(1)^2 + c = 5 + c$$

In order to have continuity at $x = 1$, the one-sided limits must be equal there. Thus, we need:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$3c + 1 = 5 + c$$

$$2c = 4$$

$$c = 2$$

For this value of $c$ we have $\lim_{x \to 1} f(x) = 7$. Furthermore, we have $f(1) = 5(1)^2 + 2 = 7$. Thus, since $\lim_{x \to 1} f(x) = f(1)$ we know that $f(x)$ is continuous at $x = 1$. 

5. Use the Intermediate Value Theorem in order to show that the equation

\[ x^5 - x + 1 = 0 \]

has at least one real solution.

**Solution:** Let \( f(x) = x^5 - x + 1 \). First we recognize that \( f(x) \) is continuous everywhere because it is a polynomial. Next, we must find an interval \([a, b]\) such that \( f(a) \) and \( f(b) \) have opposite signs. Let’s choose \( a = -2 \) and \( b = -1 \).

\[
\begin{align*}
  f(-2) &= (-2)^5 - (-2) + 1 = -29 \\
  f(-1) &= (-1)^5 - (-1) + 1 = 1
\end{align*}
\]

Since \( f(-2) < 0 \) and \( f(-1) > 0 \), the Intermediate Value Theorem tells us that \( f(c) = 0 \) for some \( c \) in the interval \([-2, -1]\).

Figure 1: Graph of \( f(x) = x^5 - x + 1 \) on the interval \([-2, -1]\).
6. Use the \( \delta - \varepsilon \) definition of the limit to prove that \( \lim_{x \to 3} 3x - 1 = 8 \).

Solution: To show that \( \lim_{x \to 3} 3x - 1 = 8 \) we must find a \( \delta > 0 \) such that \(|(3x - 1) - 8| < \varepsilon\) whenever \(|x - 3| < \delta\) for a given \( \varepsilon > 0 \).

Let’s work with the inequality \(|(3x - 1) - 8| < \varepsilon\).

\[
|3x - 1 - 8| < \varepsilon \\
|3x - 9| < \varepsilon \\
3|x - 3| < \varepsilon \\
|x - 3| < \frac{\varepsilon}{3}
\]

Therefore, we choose \( \delta = \frac{\varepsilon}{3} \).
7. Let \( f(x) = \frac{1}{x+1} \).

(a) Write the derivative, \( f'(3) \), as the limit of the difference quotient.

(b) Evaluate this limit to find \( f'(3) \).

Solution:

(a) There are two possible difference quotients we can use to evaluate \( f'(3) \). One is:

\[
 f'(3) = \lim_{h \to 0} \frac{f(h + 3) - f(3)}{h} = \lim_{h \to 0} \frac{1}{h+3} - \frac{1}{3+1}.
\]

The other is:

\[
 f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{1}{x+1} - \frac{1}{3+1}
\]

(b) Evaluating the first limit above we have:

\[
 f'(3) = \lim_{h \to 0} \frac{1}{(h + 3) + 1} - \frac{1}{3+1} \cdot \frac{4(h + 4)}{4(h + 4)}
\]

\[
 = \lim_{h \to 0} \frac{4 - (h + 4)}{4h(h + 4)}
\]

\[
 = \lim_{h \to 0} \frac{-h}{4h(h + 4)}
\]

\[
 = \lim_{h \to 0} \frac{-1}{4(h + 4)}
\]

\[
 = \frac{-1}{4(0 + 4)} = \frac{-1}{16}
\]
Evaluating the second limit we have:

\[
f'(3) = \lim_{x \to 3} \frac{\frac{1}{x+1} - \frac{1}{3+1}}{x - 3} \cdot \frac{4(x + 1)}{4(x + 1)}
\]

\[
= \lim_{x \to 3} \frac{4 - (x + 1)}{4(x + 1)(x - 3)}
\]

\[
= \lim_{x \to 3} \frac{-(x - 3)}{4(x + 1)(x - 3)}
\]

\[
= \lim_{x \to 3} \frac{-1}{4(x + 1)}
\]

\[
= \frac{-1}{4(3 + 1)}
\]

\[
= \frac{-1}{16}
\]
8. Find the derivatives of the following functions using the basic rules. Leave your answers in an unsimplified form so that your method is obvious.

(a) \( f(x) = x^3 + x^{-1} - x^{1/3} \)

(b) \( g(x) = x^3 e^x \)

(c) \( h(x) = \frac{3x}{1 + x^2} \)

Solution:

(a) Use the Power Rule.
\[
f'(x) = 3x^2 - x^{-2} - \frac{1}{3}x^{-2/3}
\]

(b) Use the Product Rule.
\[
g'(x) = x^3(e^x)' + (x^3)'e^x
\]
\[
= x^3e^x + 3x^2e^x
\]

(c) Use the Quotient Rule.
\[
h'(x) = \frac{(1 + x^2)(3x)' - (3x)(1 + x^2)'}{(1 + x^2)^2}
\]
\[
= \frac{3(1 + x^2) - (3x)(2x)}{(1 + x^2)^2}
\]
9. The table below shows values of the functions \( f(x), g(x), \) and \( h(x) \) for \( x \) near 0. Based on the data is \( h = f' \) or is \( h = g' \)? Explain your answer by citing some feature of the data.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-0.2)</th>
<th>(-0.1)</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.494</td>
<td>0.498</td>
<td>0.500</td>
<td>0.498</td>
<td>0.494</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>0.460</td>
<td>0.480</td>
<td>0.500</td>
<td>0.519</td>
<td>0.539</td>
</tr>
<tr>
<td>( h(x) )</td>
<td>0.059</td>
<td>0.029</td>
<td>0</td>
<td>(-0.029)</td>
<td>(-0.059)</td>
</tr>
</tbody>
</table>

**Solution:** To estimate the derivative \( f'(0) \) we use the formula:

\[
f'(x) \approx \frac{f(x) - f(2)}{x - 2}
\]

Choosing \( x = 0.1 \) we get the estimate:

\[
f'(0) \approx \frac{f(0.1) - f(0)}{0.1 - 0} = \frac{0.498 - 0.500}{0.1} = -0.02
\]

Choosing \( x = -0.1 \) we get the estimate:

\[
f'(0) \approx \frac{f(-0.1) - f(0)}{-0.1 - 0} = \frac{0.498 - 0.500}{-0.1} = 0.02
\]

The average of these two estimates is:

\[
\text{average estimate of } f'(0) = \frac{-0.02 + 0.02}{2} = 0
\]

Noting that \( h(0) = 0 \), it appears as though \( h = f' \).

To confirm, we estimate \( g'(0) \) using the same technique. We find that

\[
g'(0) \approx \frac{g(0.1) - g(0)}{0.1 - 0} = \frac{0.519 - 0.500}{0.1} = 0.19
\]

\[
g'(0) \approx \frac{g(-0.1) - g(0)}{-0.1 - 0} = \frac{0.480 - 0.500}{-0.1} = 2
\]

\[
\text{average estimate of } g'(0) = \frac{0.19 + 0.20}{2} = 0.195
\]

which is decidedly different from \( h(0) = 0 \) in comparison.
10. Suppose that \( f(2) = 3 \), \( f'(2) = -1 \), \( g(2) = 5 \), and \( g'(2) = -2 \). Find the derivative of the product \( f(x)g(x) \) at \( x = 2 \).

Solution: Using the Product Rule we have:

\[
[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)
\]

At \( x = 2 \), the value of the derivative \( [f(x)g(x)]' \) is:

\[
[f(x)g(x)]' \bigg|_{x=2} = f(2)g'(2) + f'(2)g(2)
\]

\[
= (3)(-2) + (-1)(5)
\]

\[
= -11
\]