1. Evaluate the limits, if they exist:

(a) \( \lim_{x \to 1} \frac{x + 1}{x^2 + 1} \)

(b) \( \lim_{x \to 3} \frac{2x^2 - 7x + 3}{3x - x^2} \)

Solution:

(a) The function \( f(x) = \frac{x + 1}{x^2 + 1} \) is continuous at \( x = 1 \). In fact, \( f(x) \) is continuous at all \( x \) in the interval \((-\infty, \infty)\). Therefore, we can evaluate the limit using substitution.

\[
\lim_{x \to 1} \frac{x + 1}{x^2 + 1} = \frac{1 + 1}{1^2 + 1} = 1
\]

(b) When substituting \( x = 3 \) into the function \( f(x) = \frac{2x^2 - 7x + 3}{3x - x^2} \) we find that

\[
\frac{2x^2 - 7x + 3}{3x - x^2} = \frac{2(3)^2 - 7(3) + 3}{3(3) - 3^2} = \frac{0}{0}
\]

which is indeterminate. We can resolve this indeterminacy by factoring.

\[
\lim_{x \to 3} \frac{2x^2 - 7x + 3}{3x - x^2} = \lim_{x \to 3} \frac{(x - 3)(2x - 1)}{-x(x - 3)}
\]

\[
= \lim_{x \to 3} \frac{2x - 1}{-x}
\]

\[
= \frac{2(3) - 1}{-3}
\]

\[
= -\frac{5}{3}
\]
2. Use the Intermediate Value Theorem to show that the function

\[ f(x) = xe^{x-1} - \frac{1}{2} \]

has a zero in the interval \([0, 1]\).

**Solution:** First we recognize that \(f(x) = xe^{x-1} - \frac{1}{2}\) is continuous on the interval \([0, 1]\). In fact, \(f(x)\) is continuous everywhere. Next, we evaluate \(f(x)\) at the endpoints of the interval.

\[
\begin{align*}
  f(0) &= 0 \cdot e^{0-1} - \frac{1}{2} = -\frac{1}{2} \\
  f(1) &= 1 \cdot e^{1-1} - \frac{1}{2} = \frac{1}{2}
\end{align*}
\]

Since \(f(0) < 0\) and \(f(1) > 0\), the Intermediate Value Theorem tells us that \(f(c) = 0\) for some \(c\) in the interval \([0, 1]\).

![Graph of \(f(x) = xe^{x-1} - \frac{1}{2}\) on the interval \([0, 1]\).](image)
3. Compute the derivatives of the following functions.

(a) \( f(x) = \frac{x^2 - 1}{x^2 + 1} \)

(b) \( f(x) = 3x^5 - 6x^{-4/3} \)

(c) \( f(x) = (x - 1)e^x \)

Solution:

(a) Use the Quotient Rule.

\[
\begin{align*}
  f'(x) &= \frac{(x^2 + 1)(x^2 - 1)' - (x^2 - 1)(x^2 + 1)'}{(x^2 + 1)^2} \\
  &= \frac{2x(x^2 + 1) - 2x(x^2 - 1)}{(x^2 + 1)^2} \\
  &= \frac{2x^3 + 2x^2 - 2x^3 + 2x}{x^2 + 1} \\
  &= \frac{2x}{x^2 + 1}
\end{align*}
\]

(b) Use the Power Rule.

\[
  f'(x) = 15x^4 + 8x^{-7/3}
\]

(c) Use the Product Rule.

\[
\begin{align*}
  f'(x) &= (x - 1)(e^x)' + (x - 1)'e^x \\
  &= xe^x - e^x + e^x \\
  &= xe^x
\end{align*}
\]
4. Consider the function \( f(x) = x^2 - 2x \).

(a) Use the definition of the derivative as a limit of a difference quotient to compute \( f'(3) \).

(b) Write an equation for the line tangent to the graph of \( f \) at \( x = 3 \).

Solution:

(a) There are two possible difference quotients we can use to evaluate \( f'(3) \). One is:

\[
\begin{align*}
f'(3) &= \lim_{h \to 0} \frac{f(h + 3) - f(3)}{h} \\
&= \lim_{h \to 0} \frac{[(h + 3)^2 - 2(h + 3)] - [3^2 - 2(3)]}{h}.
\end{align*}
\]

The other is:

\[
\begin{align*}
f'(3) &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \\
&= \lim_{x \to 3} \frac{(x^2 - 2x) - [3^2 - 2(3)]}{x - 3}.
\end{align*}
\]

Evaluating the first limit above we have:

\[
\begin{align*}
f'(3) &= \lim_{h \to 0} \frac{[(h + 3)^2 - 2(h + 3)] - [3^2 - 2(3)]}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 6h + 9 - 2h - 6}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 4h}{h} \\
&= \lim_{h \to 0} (h + 4) \\
&= 0 + 4 \\
&= 4
\end{align*}
\]

(b) The slope of the tangent line is \( f'(3) = 4 \). At \( x = 3 \) we have \( f(3) = 3^2 - 2(3) = 3 \) so \((3, 3)\) is a point on the line. Therefore, an equation for the tangent line is:

\[
y - 3 = 4(x - 3)
\]
5. Consider the piecewise-defined function below:

\[ f(x) = \begin{cases} 
  x^2 & \text{if } x < 1 \\
  4 - kx & \text{if } x \geq 1 
\end{cases} \]

(a) Find the value of \( k \) for which \( f(x) \) is continuous for all values of \( x \). Justify your answer.

(b) For the value of \( k \) you found in part (a), is \( f(x) \) differentiable at \( x = 1 \)? Explain your answer.

Solution:

(a) The functions \( x^2 \) and \( 4 - kx \) are continuous for all \( x \). In order for \( f(x) \) to be continuous for all \( x \), we must select \( k \) so that \( f(x) \) is continuous at \( x = 1 \). To do this, we must compute the one-sided limits at \( x = 1 \).

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1
\]

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4 - kx) = 4 - k(1) = 4 - k
\]

In order to have continuity at \( x = 1 \), the one-sided limits must be equal there. Thus, we need:

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \\
1 = 4 - k
\]

\[ k = 3 \]

Therefore, \( \lim_{x \to 1} f(x) = 1 \) for this value of \( k \). Furthermore, we have \( f(1) = 4 - 3(1) = 1 \). Thus, since \( \lim_{x \to 1} f(x) = f(1) \) we know that \( f(x) \) is continuous at \( x = 1 \).

(b) \( f(x) \) is differentiable at \( x = 1 \) if \( f'(x) \) is continuous there. The derivative \( f'(x) \) when \( k = 3 \) is:

\[
f'(x) = \begin{cases} 
  2x & \text{if } x < 1 \\
  -3 & \text{if } x > 1 
\end{cases}
\]

The one-sided limits of \( f'(x) \) at \( x = 1 \) are:

\[
\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} 2x = 2(1) = 2
\]

\[
\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} -3 = -3
\]

Therefore, since the one-sided limits are not equal at \( x = 1 \), \( f(x) \) is not differentiable there.