1. Find the value of constant $c$ for which the function given by

$$f(x) = \begin{cases} 
    cx + 5, & x \geq 1 \\
    x^2 + x - 3c, & x < 1
\end{cases}$$

is continuous at all points on the real line.

**Solution:** First we note that $cx + 5$ and $x^2 + x - 3c$ are polynomials and are continuous on the intervals $x > 1$ and $x < 1$, respectively. We must determine the constant $c$ so that $f(x)$ is continuous at $x = 1$. Recall that for continuity at $x = 1$ we need $\lim_{x \to 1} f(x)$ to exist.

The one-sided limits of $f(x)$ at $x = 1$ are:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (cx + 5) = c + 5$$
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + x - 3c) = 2 - 3c$$

In order for $\lim_{x \to 1} f(x)$ to exist we need the one-sided limits to be the same. That is, we need:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$
$$c + 5 = 2 - 3c$$
$$4c = -3$$

**Answer** $c = \frac{-3}{4}$
2. Find an equation for the tangent line to the graph of the function \( f(x) = \sin(x) \) at the point \( x = \pi/4 \).

Solution: The derivative of \( f(x) \) at \( x = \pi/4 \) is the slope of the tangent line. The derivative of \( f \) is \( f'(x) = \cos(x) \). At \( t = \pi/4 \) we have

\[ f'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \]

Thus, the slope of the tangent line is \( m_{\text{tan}} = \frac{\sqrt{2}}{2} \). The \( y \)-coordinate of the point on the tangent line is obtained by evaluating \( f(x) \) at \( x = \pi/4 \).

\[ f'(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \]

Therefore, the point on the tangent line is \( \left( \frac{\pi}{4}, \frac{\sqrt{2}}{2} \right) \) and the equation for the tangent line in point-slope form is:

**Answer** \[ y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \left( x - \frac{\pi}{4} \right) \]
3. Find the derivative of $f$ if

(a) $f(x) = \sqrt{\cot(e^x)}$

(b) $f(t) = \frac{t + \tan(t)}{\sqrt{t} + 1}$

Solution:

(a) The derivative is obtained using the Chain Rule and the fact that $\frac{d}{dx} \cot(x) = -\csc^2(x)$ and $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$

We obtain

$$f'(x) = \frac{1}{2\sqrt{\cot(e^x)}} \cdot \frac{d}{dx} \cot(e^x)$$

$$f'(x) = \frac{1}{2\sqrt{\cot(e^x)}} \cdot (-\csc^2(e^x)) \cdot \frac{d}{dx} e^x$$

**Answer**

$$f'(x) = \frac{1}{2\sqrt{\cot(e^x)}} \cdot (-\csc^2(e^x)) \cdot e^x$$

(b) The derivative is obtained using the Quotient Rule and the fact that $\frac{d}{dx} \tan(x) = \sec^2(x)$ and $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$

We obtain

$$f'(t) = \frac{(\sqrt{t} + 1) \cdot \frac{d}{dt} (t + \tan(t)) - (t + \tan(t)) \cdot \frac{d}{dt} (\sqrt{t} + 1)}{(\sqrt{t} + 1)^2}$$

**Answer**

$$f'(t) = \frac{(\sqrt{t} + 1) \cdot (1 + \sec^2(t)) - (t + \tan(t)) \cdot \left( \frac{1}{2\sqrt{t}} + 0 \right)}{(\sqrt{t} + 1)^2}$$
4. Evaluate the limits

(a) \( \lim_{x \to \infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} \)

(b) \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) \)

Solution:

(a) We compute this limit by multiplying and dividing by \( \frac{1}{x^2} \).

\[
\lim_{x \to \infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \lim_{x \to \infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} \cdot \frac{1}{1/x^2}
\]

\[
\lim_{x \to \infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \lim_{x \to \infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{\frac{1}{x^2}\sqrt{x^4 + 1}}
\]

\[
\lim_{x \to \infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \lim_{x \to \infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{\sqrt{\frac{1}{x^4} \cdot (x^4 + 1)}}
\]

\[
\lim_{x \to \infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \lim_{x \to \infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{\sqrt{1 + \frac{1}{x^4}}}
\]

The value of the limit is obtained using the fact that

\[
\lim_{x \to \infty} \frac{1}{x^n} = 0, \quad n > 0
\]

We then obtain

**Answer**

\[
\lim_{x \to \infty} \frac{x^2 - x + 1}{\sqrt{x^4 + x}} = \frac{1 - 0 + 0}{\sqrt{1 + 0}} = 1
\]

(b) First we identify the fact that the function \( g(x) = \sin \left( \frac{1}{x} \right) \) fluctuates between \(-1\) and \(1\) as \( x \to 0 \). Thus, the limit of this function does not exist as \( x \to 0 \). However, the function is bounded for all \( x \) while the function \( f(x) = x \) tends to 0 as \( x \to 0 \). Therefore, the limit of the product \( f(x)g(x) = x \sin \left( \frac{1}{x} \right) \) is 0.
To be more precise about the value of this limit, we use the Squeeze Theorem. To do this we begin by noting that

\[-|u| \leq u \leq |u|\]

for all \( u \). By replacing \( u \) with the function \( x \sin \left( \frac{1}{x} \right) \) we obtain

\[-|x| \leq x \sin \left( \frac{1}{x} \right) \leq |x|\]

where we used the fact that \(|ab| = |a||b|\). We now use the fact that

\[\left| \sin \left( \frac{1}{x} \right) \right| \leq 1\]

to obtain the inequality

\[-|x| \leq x \sin \left( \frac{1}{x} \right) \leq |x|\]

which is valid for all \( x \). Furthermore, we know that

\[\lim_{x \to 0} (-|x|) = \lim_{x \to 0} |x| = 0\]

Thus, by the Squeeze Theorem we obtain

\[\text{Answer} \quad \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0\]
Math 180, Exam 1, Spring 2013
Problem 5 Solution

5. An object is thrown vertically upward. The position of the object after \( t \) seconds is given by the function \( s(t) = -3t^2 + 2t + 1 \) in the units of feet.

(a) Find the velocity and acceleration of the object at time \( t \).

(b) What is the highest point the object will reach, and at what time?

(c) Calculate the point of time when the object hits the ground.

Solution:

(a) By definition, the velocity is \( s'(t) \) and the acceleration is \( s''(t) \). These derivatives are

\[
\text{Answer} \quad s'(t) = -6t + 2
\]

\[
\text{Answer} \quad s'(t) = -6
\]

(b) The object will reach its highest point when the velocity is zero. That is,

\[
s'(t) = 0
\]
\[-6t + 2 = 0\]

\[
\text{Answer} \quad t = \frac{1}{3}
\]

(c) The object will hit the ground when the position is zero. That is,

\[
-3t^2 + 2t + 1 = 0
\]
\[3t^2 - 2t - 1 = 0\]
\[(3t + 1)(t - 1) = 0\]

\[t = -\frac{1}{3}, \ t = 1\]

Since \( t \geq 0 \) we take the positive root. Therefore, the time when the object hits the ground is

\[
\text{Answer} \quad t = 1
\]