1. Compute the indefinite integrals:

(a) \[ \int (x^3 - 4x^2 + 3x + 5) \, dx \]

(b) \[ \int \sqrt{x} (x^2 - 1) \, dx \]

Solution:

(a) Using the linearity and power rules we have:

\[
\int (x^3 - 4x^2 + 3x + 5) \, dx = \int x^3 \, dx - 4 \int x^2 \, dx + 3 \int x \, dx + 5 \int 1 \, dx
\]

\[= \frac{1}{4} x^4 - 4 \left( \frac{1}{3} x^3 \right) + 3 \left( \frac{1}{2} x^2 \right) + 5x + C\]

\[= \frac{1}{4} x^4 - \frac{4}{3} x^3 + \frac{3}{2} x^2 + 5x + C\]

(b) Using some algebra and the linearity and power rules we have:

\[
\int \sqrt{x} (x^2 - 1) \, dx = \int (x^{5/2} - x^{1/2}) \, dx
\]

\[= \int x^{5/2} \, dx - \int x^{1/2} \, dx\]

\[= \frac{2}{7} x^{7/2} - \frac{2}{3} x^{3/2} + C\]
2. Use L'Hôpital’s Rule to compute \( \lim_{x \to 0} \frac{e^{7x} - 1}{e^{3x} - 1} \).

**Solution:** Upon substituting \( x = 0 \) into the function \( \frac{e^{7x} - 1}{e^{3x} - 1} \) we get

\[
\frac{e^{7(0)} - 1}{e^{3(0)} - 1} = \frac{0}{0}
\]

which is indeterminate. We resolve the indeterminacy using L'Hôpital’s Rule.

\[
\lim_{x \to 0} \frac{e^{7x} - 1}{e^{3x} - 1} \overset{\text{L'Hôpital's Rule}}{=} \lim_{x \to 0} \frac{(e^{7x} - 1)'}{(e^{3x} - 1)'}
\]

\[
= \lim_{x \to 0} \frac{7e^{7x}}{3e^{3x}}
\]

\[
= \frac{7e^{7(0)}}{3e^{3(0)}}
\]

\[
= \frac{7}{3}
\]
3. Let \( f(x) = x^3 - 2x^2 + x \).

(a) Find the critical point(s) of \( f \) and classify each as a local maximum, local minimum, or neither. Determine the intervals of monotonicity of \( f \).

(b) Find the inflection point(s) of \( f \). Determine the intervals where \( f \) is concave up and concave down.

(c) Sketch the graph \( y = f(x) \), labeling the critical points and inflection points.

Solution:

(a) The critical points of \( f(x) \) are the values of \( x \) for which either \( f'(x) \) does not exist or \( f'(x) = 0 \). Since \( f(x) \) is a polynomial, \( f'(x) \) exists for all \( x \in \mathbb{R} \) so the only critical points are solutions to \( f'(x) = 0 \).

\[
\begin{align*}
f'(x) &= 0 \\
(x^3 - 2x^2 + x)' &= 0 \\
3x^2 - 4x + 1 &= 0 \\
(3x - 1)(x - 1) &= 0 \\
x &= \frac{1}{3}, \ x = 1
\end{align*}
\]

Thus, \( x = \frac{1}{3} \) and \( x = 1 \) are the critical points of \( f \).

We will use the First Derivative Test to classify the critical points. The domain of \( f \) is \(( -\infty, \infty )\). We now split the domain into the three intervals \(( -\infty, \frac{1}{3} )\), \(( \frac{1}{3}, 1 )\), and \(( 1, \infty )\). We then evaluate \( f'(x) \) at a test point in each interval.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Point, ( c )</th>
<th>( f'(c) )</th>
<th>Sign of ( f'(c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(( -\infty, \frac{1}{3} ))</td>
<td>0</td>
<td>( f'(0) = 1 )</td>
<td>+</td>
</tr>
<tr>
<td>(( \frac{1}{3}, 1 ))</td>
<td>( \frac{2}{3} )</td>
<td>( f'(\frac{2}{3}) = -\frac{1}{3} )</td>
<td>-</td>
</tr>
<tr>
<td>(( 1, \infty ))</td>
<td>2</td>
<td>( f'(2) = 5 )</td>
<td>+</td>
</tr>
</tbody>
</table>

Since the sign of \( f'(x) \) changes from + to − at \( x = \frac{1}{3} \), the First Derivative Test implies that \( f(\frac{1}{3}) = \frac{1}{27} \) is a local maximum. Since the sign of \( f'(x) \) changes from − to + at \( x = 1 \), the First Derivative Test implies that \( f(1) = 0 \) is a local minimum. Furthermore, from the table we conclude that \( f \) is increasing on \(( -\infty, \frac{1}{3} )\) \( \cup \ (1, \infty) \) because \( f'(x) > 0 \) for all \( x \in \ ( -\infty, \frac{1}{3} )\) \( \cup \ (1, \infty) \) and \( f \) is decreasing on \(( \frac{1}{3}, 1 )\) because \( f'(x) < 0 \) for all \( x \in \ (\frac{1}{3}, 1) \).
(b) The inflection points of \( f(x) \) are the points where \( f''(x) \) changes sign. To determine these points we start by finding solutions to the equation \( f''(x) = 0 \).

\[
\begin{align*}
f''(x) &= 0 \\
(3x^2 + 4x + 1)' &= 0 \\
6x - 4 &= 0 \\
x &= \frac{2}{3}
\end{align*}
\]

We now split the domain of \( f \) into the two intervals \((-\infty, \frac{2}{3})\) and \( (\frac{2}{3}, \infty) \). We then evaluate \( f''(x) \) at a test point in each interval to determine the intervals of concavity.

<table>
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<th>( f''(c) )</th>
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</tr>
</thead>
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<tr>
<td>((-\infty, \frac{2}{3}))</td>
<td>0</td>
<td>(-4)</td>
<td>-</td>
</tr>
<tr>
<td>( (\frac{2}{3}, \infty) )</td>
<td>1</td>
<td>(2)</td>
<td>+</td>
</tr>
</tbody>
</table>

Since there is a sign change in \( f''(x) \) at \( x = \frac{2}{3} \), the point \( x = \frac{2}{3} \) is an inflection point. Furthermore, from the table we conclude that \( f \) is concave up on \( (\frac{2}{3}, \infty) \) because \( f''(x) > 0 \) for all \( x \in (\frac{2}{3}, \infty) \) and \( f \) is concave down on \((-\infty, \frac{2}{3})\) because \( f''(x) < 0 \) for all \( x \in (-\infty, \frac{2}{3}) \).

(c)
4. Find the absolute minimum and the absolute maximum of \( f(x) = \frac{x^3}{3} - \frac{x^2}{2} + 2 \) on the interval \([-1, 2]\).

**Solution:** The minimum and maximum values of \( f(x) \) will occur at a critical point in the interval \([-1, 2]\) or at one of the endpoints. The critical points are the values of \( x \) for which either \( f'(x) = 0 \) or \( f'(x) \) does not exist. Since \( f(x) \) is a polynomial, \( f'(x) \) exists for all \( x \in \mathbb{R} \). Therefore, the only critical points are solutions to \( f'(x) = 0 \).

\[
f'(x) = 0 \\
\left(\frac{x^3}{3} - \frac{x^2}{2} + 2\right)' = 0 \\
x^2 - x = 0 \\
x(x - 1) = 0 \\
x = 0, \ x = 1
\]

Both critical points \( x = 0 \) and \( x = 1 \) lie in \([-1, 2]\). Therefore, we check the value of \( f(x) \) at \( x = -1, \ 0, \ 1, \) and \( 2 \).

\[
f(-1) = (-1)^3/3 - (-1)^2/2 + 2 = \frac{7}{6} \\
f(0) = 0^3/3 - 0^2/2 + 2 = 2 \\
f(1) = 1^3/3 - 1^2/2 + 2 = \frac{11}{6} \\
f(2) = 2^3/3 - 2^2/2 + 2 = \frac{8}{3}
\]

The minimum value of \( f(x) \) on \([-1, 2]\) is \( \boxed{\frac{7}{6}} \) because it is the smallest of the above values of \( f \). The maximum is \( \boxed{\frac{8}{3}} \) because it is the largest.
5. Design a rectangular box with square base (as in the diagram below) and a total surface area of 6 square feet that encloses the maximum possible volume. Determine both the dimensions of the box and the volume enclosed.

Solution: We begin by letting \( w \) be the length of one side of the base and \( h \) be the height of the box. The function we seek to minimize is the volume of the box.

Function: \[ \text{Volume} = w^2h \] (1)

The constraint in the problem is that the total surface area is 6. This gives us the equation

Constraint: \[ 2w^2 + 4wh = 6 \] (2)

Solving this equation for \( h \) we get

\[
2w^2 + 4wh = 6 \\
w^2 + 2wh = 3 \\
h = \frac{3 - w^2}{2w} \] (3)

We then plug this into the volume equation (1) to write the volume in terms of \( w \) only.

\[
\text{Volume} = w^2h \\
\text{Volume} = w^2 \left( \frac{3 - w^2}{2w} \right) \\
f(w) = \frac{3}{2}w - \frac{1}{2}w^3 \] (4)

We want to find the absolute maximum of \( f(w) \) on the interval \( (0, \sqrt{3}] \). We know that \( w > 0 \) because \( w \) must be positive and nonzero (otherwise, the surface area would be 0 and it must be 6). It is possible that \( h = 0 \) in which case the surface area constraint would give us \( 2w^2 + 4w(0) = 6 \Rightarrow w^2 = 3 \Rightarrow w = \sqrt{3} \).
The absolute maximum of \( f(w) \) will occur either at a critical point of \( f(w) \) in \((0, \sqrt{3}]\), at \( x = \sqrt{w} \), or it will not exist. The critical points of \( f(w) \) are solutions to \( f'(x) = 0 \).

\[
f'(w) = 0
\]
\[
\frac{3}{2} - \frac{3}{2}w^2 = 0
\]
\[
w^2 = 1
\]
\[
w = \pm 1
\]

However, since \( w = -1 \) is outside \((0, \sqrt{3}]\), the only critical point is \( w = 1 \). Plugging this into \( f(w) \) we get:

\[
f(1) = \frac{3}{2}(1) - \frac{1}{2}(1)^2 = 1
\]

Evaluating \( f(w) \) at \( w = \sqrt{3} \) and taking the limit of \( f(w) \) as \( w \) approaches \( w = 0 \) we get:

\[
\lim_{w \to 0^+} f(w) = \lim_{w \to 0^+} \left( \frac{3}{2}w - \frac{1}{2}w^3 \right) = 0
\]
\[
f(\sqrt{3}) = \frac{3}{2}(\sqrt{3}) - \frac{1}{2}(\sqrt{3})^3 = 0
\]

both of which are smaller than 1. We conclude that the volume is an absolute maximum at \( w = 1 \) and that the resulting volume is 1 ft\(^3\). The height of the box when \( w = 1 \) is found using equation (3).

\[
h = \frac{3 - 1^2}{2(1)} = 1
\]
6. Compute the area of the region defined by $2 \leq x \leq 5, \ 0 \leq y \leq x^2$.

Solution: The area of the region is given by the formula:

$$\text{Area} = \int_2^5 x^2 \, dx$$

Using the Fundamental Theorem of Calculus, Part I to evaluate the integral we get:

$$\text{Area} = \left[ \frac{1}{3} x^3 \right]_2^5$$

$$= \frac{1}{3} 5^3 - \frac{1}{3} 2^3$$

$$= \frac{125}{3} - \frac{8}{3}$$

$$= 39$$