Math 181, Exam 1, Fall 2008
Problem 1 Solution

1. Compute the following integrals.

(a) \( \int \frac{\sin x}{1 - 2\cos x} \, dx \)

(b) \( \int \frac{dx}{\sqrt{1 - 4x^2}} \)

Solution:

(a) The integral is computed using the \( u \)-substitution method. Let \( u = 1 - 2\cos x \). Then \( du = 2\sin x \, dx \Rightarrow \frac{1}{2} du = \sin x \, dx \). Substituting these into the integral and evaluating we get:

\[
\int \frac{\sin x}{1 - 2\cos x} \, dx = \int \frac{1}{1 - 2 \cos x} \cdot \sin x \, dx \\
= \int \frac{1}{u} \cdot \frac{1}{2} du \\
= \frac{1}{2} \int \frac{1}{u} du \\
= \frac{1}{2} \ln |u| + C \\
= \frac{1}{2} \ln |1 - 2 \cos x| + C
\]

(b) The integral is computed using the \( u \)-substitution method. Let \( u = 2x \). Then \( du = 2 \, dx \Rightarrow \frac{1}{2} \, du = dx \) and we get:

\[
\int \frac{dx}{\sqrt{1 - 4x^2}} = \int \frac{dx}{\sqrt{1 - (2x)^2}} \\
= \int \frac{\frac{1}{2} du}{\sqrt{1 - u^2}} \\
= \frac{1}{2} \int \frac{1}{\sqrt{1 - u^2}} du \\
= \frac{1}{2} \arcsin u + C \\
= \frac{1}{2} \arcsin(2x) + C
\]
2. Find the volume of the solid of revolution obtained by rotating the region in the first quadrant bounded by \( y = x^2, \ x + y = 6, \) and \( x = 0 \) about the \( y \)-axis.

Solution:

To find the volume we will use the **Shell Method**. The variable of integration is \( x \) and the formula is:

\[
V = 2\pi \int_a^b x (\text{top} - \text{bottom}) \, dx
\]

where the top curve is \( y = 6 - x \) and the bottom curve is \( y = x^2 \). The lower limit of integration is \( a = 0 \). To determine the upper limit we must find the points of intersection of the top and bottom curves. To do this we set the \( y \)'s equal to each other and solve for \( x \).

\[
y = y \\
x^2 = 6 - x
\]

\[
x^2 + x - 6 = 0 \\
(x + 3)(x - 2) = 0 \\
x = -3, \ x = 2
\]

In the problem statement we are told to take the region in the first quadrant. Therefore, we
take $b = 2$. The volume is then:

$$V = 2\pi \int_0^2 x \left[(6 - x) - x^3\right] \, dx$$

$$= 2\pi \int_0^2 (6x - x^2 - x^3) \, dx$$

$$= 2\pi \left[ 3x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^2$$

$$= 2\pi \left[ 3(2)^2 - \frac{1}{3}(2)^3 - \frac{1}{4}(2)^4 \right]$$

$$= \frac{32\pi}{3}$$
Math 181, Exam 1, Fall 2008
Problem 3 Solution

3. Compute each part below.

(a) Compute the area of the region bounded by $y = x^2 - 1$ and $y = 4x - 4$.

(b) Compute $f'(x)$ where

$$f(x) = \int_1^{x^2} \ln(t) \, dt, \quad x > 0.$$ 

Solution:

The formula we will use to compute the area of the region is:

$$\text{Area} = \int_a^b (\text{top} - \text{bottom}) \, dx$$

where the limits of integration are the $x$-coordinates of the points of intersection of the two curves. These are found by setting the $y$'s equal to each other and solving for $x$.

$$y = y \quad x^2 - 1 = 4x - 4 \quad x^2 - 4x + 3 = 0 \quad (x - 1)(x - 3) = 0 \quad x = 1, \ x = 3$$
From the graph we see that the top curve is $y = 4x - 4$ and the bottom curve is $y = x^2 - 1$. Therefore, the area between the curves is:

\[
\text{Area} = \int_a^b (\text{top} - \text{bottom}) \, dx \\
= \int_1^3 [(4x - 4) - (x^2 - 1)] \, dx \\
= \int_1^3 (-x^2 + 4x - 3) \, dx \\
= \left[-\frac{1}{3}x^3 + 2x^2 - 3x\right]_1^3 \\
= \left[-\frac{1}{3}(3)^3 + 2(3)^2 - 3(3)\right] - \left[-\frac{1}{3}(1)^3 + 2(1)^2 - 3(1)\right] \\
= [-9 + 18 - 9] - \left[-\frac{1}{3} + 2 - 3\right] \\
= \frac{4}{3}
\]

(b) Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative is:

\[
F'(x) = \frac{d}{dx} \int_1^{x^2} \ln(t) \, dt \\
= \ln(x^2) \cdot \frac{d}{dx} (x^2) \\
= \ln(x^2) \cdot (2x)
\]
4. Use an integral to compute the volume of a right circular cone whose base has radius $R$ and whose height is $h$.

**Solution:** To find the volume we will use the formula:

$$V = \int_{c}^{d} A(y) \, dy$$

where $A(y)$ is the cross-sectional area of the cone as a function of height $y$ and $0 \leq y \leq h$. The horizontal cross sections are circles, so the cross-sectional area is:

$$A(y) = \pi r^2$$

where $r$ is the radius of the cross section at height $y$ from the base. If we look at the cone from the side, we see a triangle. The cross-section as viewed from the side is a horizontal line segment at height $y$. The radius of the cross section is half of the length of this line segment. Using similar triangles, we have:

$$\frac{\text{base}}{\text{height}} = \frac{R}{h} = \frac{r}{h - y}$$

$$r = \frac{R}{h} (h - y)$$
The volume is then:

\[ V = \int_0^h \pi r^2 \, dy \]

\[ = \int_0^h \pi \left[ \frac{R}{h} (h - y) \right]^2 \, dy \]

\[ = \pi \frac{R^2}{h^2} \int_0^h (y - h)^2 \, dy \]

\[ = \pi \frac{R^2}{h^2} \left[ \frac{1}{3} (y - h)^3 \right]_0^h \]

\[ = \pi \frac{R^2}{h^2} \left[ \frac{1}{3} (h - h)^3 - \frac{1}{3} (0 - h)^3 \right] \]

\[ = \frac{1}{3} \pi R^2 h \]
5. Approximate the value of the definite integral:

\[ \int_{1}^{3} \frac{dx}{x} \]

using

(a) the Midpoint Rule with \( N = 2 \),
(b) the Trapezoidal Rule with \( N = 2 \), and
(c) Simpson’s Rule with \( N = 4 \).

Your answers should be written as a single, reduced fraction.

Solution:

(a) The length of each subinterval of \([1, 3]\) is

\[ \Delta x = \frac{b - a}{N} = \frac{3 - 1}{2} = 1 \]

The estimate \( M_2 \) is:

\[ M_2 = \Delta x \left[ f \left( \frac{3}{2} \right) + f \left( \frac{5}{2} \right) \right] = 1 \cdot \left[ \frac{1}{2} + \frac{1}{5} \right] = \frac{2}{3} + \frac{2}{5} = \frac{16}{15} \]

(b) The length of each subinterval of \([1, 3]\) is \( \Delta x = 1 \) just as in part (a). The estimate \( T_2 \) is:

\[ T_2 = \frac{\Delta x}{2} \left[ f(1) + 2f(2) + f(3) \right] = \frac{1}{2} \left[ \frac{1}{1} + 2 \cdot \frac{1}{2} + \frac{1}{3} \right] = \frac{7}{6} \]
(c) We can use the following formula to find $S_4$:

$$S_4 = \frac{2}{3}M_2 + \frac{1}{3}T_2$$

where $M_2$ and $T_2$ were found in parts (a) and (b). We get:

$$S_4 = \frac{2}{3} \left( \frac{16}{15} \right) + \frac{1}{3} \left( \frac{7}{6} \right)$$

$$= \frac{32}{45} + \frac{7}{18}$$

$$= \frac{11}{10}$$