Math 181, Exam 1, Fall 2012
Problem 1 Solution

1. Let \( R \) be the region enclosed by the curves \( y = x^4 \) and \( y = x \).

(a) Sketch the region \( R \).

(b) Write down the integral representing the volume of the solid obtained by revolving \( R \) about the \( x \)-axis.

(c) Compute the volume of the solid.

Solution:

(b) The solid of revolution has cross-sections that are washers. We then use the formula:

\[
V = \int_a^b \pi \left[ f(x)^2 - g(x)^2 \right] \, dx
\]

where \( f(x) = x \) and \( g(x) = x^4 \) and the interval is \([a, b] = [0, 1]\). Thus, the volume integral is:

\[
V = \int_0^1 \pi \left( x^2 - x^8 \right) \, dx
\]

(c) The volume calculation is as follows:

\[
V = \pi \left[ \frac{x^3}{3} - \frac{x^9}{9} \right]_0^1
\]

\[
V = \pi \left[ \frac{1}{3} - \frac{1}{9} \right]
\]

\[
V = \frac{2\pi}{9}
\]
2. Compute the arc length of the curve given by \( y(x) = 5 - 2x^{3/2} \) between \( x = 0 \) and \( x = 9 \).

**Solution:** The arc length is computed via the formula

\[
L = \int_a^b \sqrt{1 + [y'(x)]^2} \, dx
\]

Since \( y(x) = 5 - 2x^{3/2} \) we know that \( y'(x) = -3x^{1/2} \) and \( y'(x)^2 = 9x \). Therefore, the arc length is

\[
L = \int_0^9 \sqrt{1 + 9x} \, dx
\]

\[
L = \left[ \frac{2}{27} (1 + 9x^{3/2}) \right]_0^9
\]

\[
L = \left[ \frac{2}{27} (1 + 9 \cdot 9)^{3/2} \right] - \left[ \frac{2}{27} (1 + 9 \cdot 0)^{3/2} \right]
\]

\[
L = \frac{2}{27} (82^{3/2} - 1)
\]
3. Compute the following integrals:

(a) \( \int x^3 \sin(x^2) \, dx \)

(b) \( \int x^2 4^x \, dx \)

Solution:

(a) We begin by letting \( u = x^2, \, \frac{1}{2} \, du = x \, dx \). Using these substitutions, the integral is transformed as follows:

\[
\int x^3 \sin(x^2) \, dx = \int x^2 \sin(x^2) \cdot x \, dx \\
= \int u \sin(u) \cdot \frac{1}{2} \, du \\
= \frac{1}{2} \int u \sin(u) \, du
\]

The resulting integral may be evaluated using integration by parts. Letting \( w = u \) and \( dv = \sin(u) \, du \) we have \( dw = du \) and \( v = -\cos(u) \). Thus, using the integration by parts formula

\[
\int w \, dv = wv - \int v \, dw
\]

we obtain

\[
\int u \sin(u) \, du = u(-\cos(u)) - \int (-\cos(u)) \, du \\
= -u \cos(u) + \int \cos(u) \, du \\
= -u \cos(u) + \sin(u) + C
\]

Finally, we use the fact that \( u = x^2 \) to write our answer as

\[
\int x^3 \sin(x^2) \, dx = -x^2 \cos(x^2) + \sin(x^2) + C
\]
(b) The integral may be evaluated using tabular integration.

\[
\int x^2 4^x \, dx = + \frac{1}{\ln(4)} \cdot 4^x \cdot x^2 - \frac{1}{(\ln(4))^2} \cdot 4^x \cdot 2x + \frac{1}{(\ln(4))^3} \cdot 4^x \cdot 2 + C
\]
4. Compute the integral \( \int \frac{dx}{(4 - x^2)^{3/2}} \).

**Solution:** The computation requires the trigonometric substitution \( x = 2 \sin \theta \), \( dx = 2 \cos \theta \, d\theta \). Plugging these into the integral gives us

\[
\int \frac{dx}{(4 - x^2)^{3/2}} = \int \frac{2 \cos \theta \, d\theta}{(4 - (2 \sin \theta)^2)^{3/2}}
\]

\[
= \int \frac{2 \cos \theta}{(4 - 4 \sin^2 \theta)^{3/2}} \, d\theta
\]

\[
= \int \frac{2 \cos \theta}{(4 \cos^2 \theta)^{3/2}} \, d\theta
\]

\[
= \int \frac{2 \cos \theta}{4^{3/2} \cos^3 \theta} \, d\theta
\]

\[
= \int \frac{2 \cos \theta}{8 \cos^3 \theta} \, d\theta
\]

\[
= \frac{1}{4} \int \sec^2 \theta \, d\theta
\]

\[
= \frac{1}{4} \tan \theta + C
\]

Since \( x = 2 \sin \theta \) we know that \( \sin \theta = \frac{x}{2} \). We can then construct a right triangle where the side opposite the angle \( \theta \) is \( x \) and the hypotenuse is 2. Using the Pythagorean Theorem, the side adjacent to \( \theta \) is \( \sqrt{4 - x^2} \).

![Right Triangle Diagram]

Thus, the tangent of \( \theta \) is the ratio of the side opposite \( \theta \) to the adjacent side.

\[
\tan \theta = \frac{x}{\sqrt{4 - x^2}}
\]
The integral is then

\[ \int \frac{dx}{(4 - x^2)^{3/2}} = \frac{1}{4} \cdot \frac{x}{\sqrt{4 - x^2}} + C \]
5. Compute the following integrals:

(a) \( \int \frac{5}{x^2 + 3x - 4} \, dx \),

(b) \( \int \frac{dx}{x^2 + 4x + 5} \).

Solution:

(a) The integrand is a rational function and the denominator factors into \((x + 4)(x - 1)\). Thus, we may use the method of partial fraction decomposition. Since the denominator has two distinct roots we decompose the integrand as follows:

\[
\frac{5}{(x + 4)(x - 1)} = \frac{A}{x + 4} + \frac{B}{x - 1}.
\]

Clearing denominators gives us

\[
5 = A(x - 1) + B(x + 4).
\]

Letting \( x = 1 \) leads to \( B = 1 \) and letting \( x = -4 \) leads to \( A = -1 \). Replacing \( A \) and \( B \) in the decomposition and evaluating the integral gives us:

\[
\int \frac{5}{x^2 + 3x - 4} \, dx = \int \left( \frac{-1}{x + 4} + \frac{1}{x - 1} \right) \, dx,
\]

\[
\int \frac{5}{x^2 + 3x - 4} \, dx = -\ln |x + 4| + \ln |x - 1| + C.
\]

(b) Once again, the integrand is a rational function but the denominator does not factor nicely. Therefore, we resort to completing the square:

\[
x^2 + 4x + 5 = (x + 2)^2 + 1
\]

We then introduce the substitution \( u = x + 2, \, du = dx \) to convert the integral into:

\[
\int \frac{dx}{x^2 + 4x + 5} = \int \frac{du}{u^2 + 1} = \arctan(u) + C.
\]
Replacing $u$ with $x + 2$ gives us our result:

\[
\int \frac{dx}{x^2+4x+5} = \arctan(x + 2) + C
\]