Math 181, Exam 1, Spring 2006
Problem 1 Solution

1. The graph of a function \( g(x) \) is given below. Let \( G \) be an antiderivative for \( g \) on the interval \([0, 5]\) with \( G(1) = 2 \). Compute \( G(0) \), \( G(3) \), and \( G(5) \).

![](image.png)

**Solution:** Since \( G \) is an antiderivative of \( g \) we know that \( G'(x) = g(x) \). That is,

\[
G(x) = \int g(x) \, dx
\]

where

\[
g(x) = \begin{cases} 
  x - 1 & \text{if } 0 \leq x < 3 \\
  2 & \text{if } 3 \leq x < 4 \\
  -2x + 10 & \text{if } 4 \leq x \leq 5
\end{cases}
\]

- On the interval \( 0 \leq x < 3 \), we have \( g(x) = x - 1 \). Therefore,

\[
G(x) = \int (x - 1) \, dx
\]

\[
G(x) = \frac{1}{2}x^2 - x + C_1
\]

The value of \( C_1 \) is found by using the fact that \( G(1) = 2 \).

\[
G(1) = 2 \\
\frac{1}{2}(1)^2 - 1 + C_1 = 2 \\
C_1 = \frac{5}{2}
\]

Therefore, on the interval \( 0 \leq x < 3 \) we have \( G(x) = \frac{1}{2}x^2 - x + \frac{5}{2} \) and, thus, \( G(0) = \frac{5}{2} \).
• On the interval $3 \leq x < 4$, we have $g(x) = 2$. Therefore,

$$G(x) = \int 2 \, dx$$
$$G(x) = 2x + C_2$$

The value of $C_2$ is found by ensuring continuity of $G(x)$ at $x = 3$. That is, we need:

$$\lim_{x \to 3^-} G(x) = \lim_{x \to 3^+} G(x)$$
$$\lim_{x \to 3^-} \left( \frac{1}{2}x^2 - x + \frac{5}{2} \right) = \lim_{x \to 3^+} (2x + C_2)$$
$$\frac{1}{2}(3)^2 - 3 + \frac{5}{2} = 2(3) + C_2$$
$$C_2 = -2$$

Therefore, on the interval $3 \leq x < 4$ we have $G(x) = 2x - 2$ and, thus, $G(3) = 4$.

• On the interval $4 \leq x \leq 5$, we have $g(x) = -2x + 10$. Therefore,

$$G(x) = \int (-2x + 10) \, dx$$
$$G(x) = -x^2 + 10x + C_3$$

The value of $C_3$ is found by ensuring continuity of $G(x)$ at $x = 4$. That is, we need:

$$\lim_{x \to 4^-} G(x) = \lim_{x \to 4^+} G(x)$$
$$\lim_{x \to 4^-} (2x - 2) = \lim_{x \to 4^+} (-x^2 + 10x + C_3)$$
$$2(4) - 2 = -(4)^2 + 10(4) + C_3$$
$$C_3 = -18$$

Therefore, on the interval $4 \leq x \leq 5$ we have $G(x) = -x^2 + 10x - 18$ and, thus, $G(5) = 7$.

The graphs of both $G(x)$ (left) and $g(x)$ (right) are shown below.
2. On the planet Penthesilea IV, the gravitational acceleration is $-20 \text{ ft/sec}^2$. A stone is thrown upwards from a height of 60 ft with initial velocity 100 ft/sec.

   i) When will the stone reach its maximum height?

   ii) What is the maximum height reached by the stone?

Solution:

   i) The velocity of the stone is given by:

   $$v(t) = -gt + v_0,$$

   where $g = -20 \text{ ft/sec}^2$ is the gravitational acceleration and $v_0 = 100 \text{ ft/sec}$ is the initial velocity. When the stone reaches its maximum height, its velocity is 0. The time when this happens is then:

   $$v(t) = 0$$
   $$-20t + 100 = 0$$
   $$t = 5 \text{ sec}$$

   ii) The height of the stone is given by:

   $$x(t) = -\frac{1}{2}gt^2 + v_0t + x_0$$

   where $x_0 = 60 \text{ ft}$ is the initial height. Evaluating $x(t)$ at $t = 5 \text{ sec}$ we get:

   $$x(5) = -\frac{1}{2}(20)(5)^2 + 100(5) + 60 = 310 \text{ ft}$$
3. Differentiate the function:

\[ T(x) = \int_{1}^{\cos x} e^{t^2} \, dt \]

**Solution:** Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative \( T'(x) \) is:

\[
T'(x) = \frac{d}{dx} \int_{1}^{\cos x} e^{t^2} \, dt \\
= e^{(\cos x)^2} \cdot \frac{d}{dx} \cos x \\
= e^{(\cos x)^2} \cdot (-\sin x)
\]
4. Compute the definite integral:

\[ \int_0^1 xe^{2x} \, dx \]

**Solution:** We will evaluate the integral using Integration by Parts. Let \( u = x \) and \( v' = e^{2x} \). Then \( u' = 1 \) and \( v = \frac{1}{2} e^{2x} \). Using the Integration by Parts formula:

\[
\int_a^b uv' \, dx = [uv]_a^b - \int_a^b u'v \, dx
\]

we get:

\[
\int_0^1 xe^{2x} \, dx = \left[ \frac{1}{2} xe^{2x} \right]_0^1 - \int_0^1 \frac{1}{2} e^{2x} \, dx
\]

\[
= \left[ \frac{1}{2} xe^{2x} \right]_0^1 - \left[ \frac{1}{4} e^{2x} \right]_0^1
\]

\[
= \left( \frac{1}{2} e^2 - 0 \right) - \left( \frac{1}{4} e^2 - \frac{1}{4} \right)
\]

\[
= \frac{1}{4} e^2 + \frac{1}{4}
\]
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Problem 5 Solution

5. Find:
\[ \int \frac{x}{\sqrt{x+4}} \, dx \]

Solution: We will evaluate the integral using the \( u \)-substitution method. Let \( u = x + 4 \) so that \( du = dx \) and \( x = u - 4 \). Substituting these expressions into the given integral and evaluating we get:

\[
\int \frac{x}{\sqrt{x+4}} \, dx = \int \frac{u - 4}{\sqrt{u}} \, du \\
= \int \left( \frac{u}{\sqrt{u}} - \frac{4}{\sqrt{u}} \right) \, du \\
= \int \left( u^{1/2} - 4u^{-1/2} \right) \, du \\
= \frac{2}{3} u^{3/2} - 8u^{1/2} + C \\
= \frac{2}{3} (x + 4)^{3/2} - 8(x + 4)^{1/2} + C
\]
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Problem 6 Solution

6. Find:
\[ \int \frac{dx}{x^2 - 3x + 2} \]

Solution: We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

\[ \frac{1}{x^2 - 3x + 2} = \frac{1}{(x - 1)(x - 2)} = \frac{A}{x - 1} + \frac{B}{x - 2} \]

Next, we multiply the above equation by \((x - 1)(x - 2)\) to get:

\[ 1 = A(x - 2) + B(x - 1) \]

Then we plug in two different values for \(x\) to create a system of two equations in two unknowns \((A,B)\). We select \(x = 1\) and \(x = 2\) for simplicity.

\[ x = 1: \quad A(1 - 2) + B(1 - 1) = 1 \quad \Rightarrow \quad A = -1 \]
\[ x = 2: \quad A(2 - 2) + B(2 - 1) = 1 \quad \Rightarrow \quad B = 1 \]

Finally, we plug these values for \(A\) and \(B\) back into the decomposition and integrate.

\[
\begin{align*}
\int \frac{dx}{x^2 - 3x + 2} &= \int \left( \frac{A}{x - 1} + \frac{B}{x - 2} \right) dx \\
&= \int \left( \frac{-1}{x - 1} + \frac{1}{x - 2} \right) dx \\
&= -\ln |x - 1| + \ln |x - 2| + C
\end{align*}
\]
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Problem 7 Solution

7. Compute the definite integral:
\[ \int x^6 \ln x \, dx \]

Solution: We will evaluate the integral using Integration by Parts. Let \( u = \ln x \) and \( v' = x^6 \). Then \( u' = \frac{1}{x} \) and \( v = \frac{1}{7}x^7 \). Using the Integration by Parts formula:

\[
\int uv' \, dx = uv - \int u'v \, dx
\]

we get:

\[
\int x^6 \ln x \, dx = \frac{1}{7}x^7 \ln x - \int \left( \frac{1}{x} \right) \left( \frac{1}{7}x^7 \right) \, dx
\]

\[
= \frac{1}{7}x^7 \ln x - \frac{1}{7} \int x^6 \, dx
\]

\[
= \frac{1}{7}x^7 \ln x - \frac{1}{49}x^7 + C
\]
8. Find:
\[ \int \arctan x \, dx \]

**Solution:** We will evaluate the integral using Integration by Parts. Let \( u = \arctan x \) and \( v' = 1 \). Then \( u' = \frac{1}{x^2 + 1} \) and \( v = x \). Using the Integration by Parts formula:

\[ \int uv' \, dx = uv - \int u'v \, dx \]

we get:

\[ \int \arctan x \, dx = x \arctan x - \int \frac{1}{x^2 + 1} \, dx. \]

Use the substitution \( w = x^2 + 1 \) to evaluate the integral on the right hand side. Then \( dw = 2x \, dx \implies \frac{1}{2} \, dw = x \, dx \) and we get:

\[ \int \arctan x \, dx = x \arctan x - \int \frac{x}{x^2 + 1} \, dx \]
\[ = x \arctan x - \frac{1}{2} \int \frac{1}{w} \, dw \]
\[ = x \arctan x - \frac{1}{2} \ln |w| + C \]
\[ = x \arctan x - \frac{1}{2} \ln(x^2 + 1) + C \]

Note that the absolute value signs aren’t needed because \( x^2 + 1 > 0 \) for all \( x \).