Math 181, Exam 1, Spring 2013
Problem 1 Solution

1. Compute the integrals \( \int x e^{4x} \, dx \) and \( \int \arctan x \, dx \).

Solution: We compute the first integral using Integration by Parts. The following table summarizes the elements that make up the technique.

<table>
<thead>
<tr>
<th>( u = x )</th>
<th>( dv = e^{4x} , dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( du = dx )</td>
<td>( v = \frac{1}{4} e^{4x} )</td>
</tr>
</tbody>
</table>

Using the Integration by Parts formula we have

\[
\int u \, dv = uv - \int v \, du
\]

\[
\int x e^{4x} \, dx = \frac{1}{4} x e^{4x} - \frac{1}{4} \int e^{4x} \, dx
\]

Answer: \[
\int x e^{4x} \, dx = \frac{1}{4} x e^{4x} - \frac{1}{16} e^{4x} + C
\]

The second integral is computed by Integration by Parts as well. The following table summarizes the elements that make up the technique.

<table>
<thead>
<tr>
<th>( u = \arctan(x) )</th>
<th>( dv = dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( du = \frac{1}{x^2 + 1} , dx )</td>
<td>( v = x )</td>
</tr>
</tbody>
</table>

Using the Integration by Parts formula we have

\[
\int u \, dv = uv - \int v \, du
\]

\[
\int \arctan(x) \, dx = x \arctan(x) - \int \frac{x}{x^2 + 1} \, dx
\]

Answer: \[
\int \arctan(x) \, dx = x \arctan(x) - \frac{1}{2} \ln(x^2 + 1) + C
\]
2. Compute the integral \( \int \frac{dx}{x^2\sqrt{x^2 + 4}} \).

**Solution:** We compute the integral using the trigonometric substitution \( x = 2\tan \theta, \ dx = 2\sec^2 \theta \, d\theta \). After substituting these expressions into the integral and simplifying we obtain:

\[
\int \frac{dx}{x^2\sqrt{x^2 + 4}} = \int \frac{2\sec^2 \theta \, d\theta}{(2\tan \theta)^2\sqrt{(2\tan \theta)^2 + 4}} \\
= \int \frac{2\sec^2 \theta \, d\theta}{4\tan^2 \theta \cdot \sqrt{4\tan^2 \theta + 4}} \\
= \int \frac{2\sec^2 \theta \, d\theta}{4\tan^2 \theta \cdot \sqrt{4\tan^2 \theta + 4}} \\
= \int \frac{2\sec^2 \theta \, d\theta}{4\tan^2 \theta \cdot \sqrt{4\tan^2 \theta + 4}} \\
= \frac{1}{4} \int \frac{\sec \theta \cdot 2\sec \theta \, d\theta}{\tan^2 \theta} \\
= \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} \, d\theta.
\]

To evaluate the resulting trigonometric integral we rewrite the integrand in terms of sin \( \theta \) and cos \( \theta \) by using the definitions

\[
\sec \theta = \frac{1}{\cos \theta}, \quad \tan \theta = \frac{\sin \theta}{\cos \theta}.
\]

The integral then transforms into

\[
\int \frac{\sec \theta}{\tan^2 \theta} \, d\theta = \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta.
\]

We can either

1. rewrite the integrand as cot \( \theta \) csc \( \theta \) and use the fact that this is the derivative of \(- \csc \theta\)

or

2. use the substitution \( u = \sin \theta, \ du = \cos \theta \, d\theta \).

In either case we obtain the result:

\[
\int \frac{\cos \theta}{\sin^2 \theta} \, d\theta = -\csc \theta + C
\]

Therefore, our integral takes the form

\[
\int \frac{dx}{x^2\sqrt{x^2 + 4}} = -\frac{1}{4} \csc \theta + C
\]
We must finish the problem by writing $\csc \theta$ in terms of $x$. Since we know that $x = 2 \tan \theta$ we have

$$\tan \theta = \frac{x}{2} = \frac{\text{OPP}}{\text{ADJ}}$$

where OPP and ADJ are the opposite and adjacent sides of a right triangle, respectively, where opposite refers to the length of the side across from $\theta$. Using the Pythagorean Theorem, the hypotenuse of this triangle is $\sqrt{x^2 + 4}$. Therefore,

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{HYP}}{\text{OPP}} = \frac{\sqrt{x^2 + 4}}{x}$$

After substituting this expression into our result we find that

$$\text{Answer} \int \frac{dx}{x^2 \sqrt{x^2 + 4}} = -\frac{\sqrt{x^2 + 4}}{4x} + C$$
3. The region between the $x$-axis and the parabola $y = 1 - x^2$ is rotated about the line $x = 2$. Find the volume of the resulting solid.

**Solution:** The region being rotated is plotted below.

In this case, we use the Shell Method because the integration with respect to $x$ is easier to perform. The volume formula is

\[ V = 2\pi \int_a^b \text{(radius})(height) \, dx \]

The height of each shell is given by $1 - x^2$. Since the region is being rotated about the axis $x = 2$, the radius of each shell is given by $2 - x$. The interval over which the integral will take place is $x = -1$ to $x = 1$ since these are the points where the parabola $y = 1 - x^2$ intersects the $x$-axis. Therefore, the volume of the solid is

\[ V = 2\pi \int_{-1}^1 (2-x)(1-x^2) \, dx \]
\[ V = 2\pi \int_{-1}^1 (2-x-2x^2+x^3) \, dx \]
\[ V = 8\pi \int_0^1 (1-x^2) \, dx \]
\[ V = 8\pi \left[ x - \frac{x^3}{3} \right]_0^1 \]

**Answer** \[ V = \frac{16\pi}{3} \]
4. Compute the integrals \( \int \frac{dx}{x^2 - x} \) and \( \int \frac{x + 3}{x^2 + 2x + 5} \, dx \).

**Solution:** The integrand of \( \int \frac{dx}{x^2 - x} \) is a rational function whose denominator factors into \( x(x - 1) \). Thus, we will use the method of partial fractions. The partial fraction decomposition of the integrand is

\[
\frac{1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}
\]

After clearing denominators we find that

\[
1 = A(x - 1) + Bx
\]

When \( x = 0 \) we have \( A = -1 \) and when \( x = 1 \) we have \( B = 1 \). Therefore, we may evaluate the integral as follows:

\[
\int \frac{dx}{x^2 - x} = - \ln |x| + \ln |x - 1| + C
\]

The integrand of the integral \( \int \frac{x + 3}{x^2 + 2x + 5} \, dx \) is a rational function but the denominator is an irreducible quadratic. Therefore we begin by completing the square:

\[
x^2 + 2x + 5 = (x^2 + 2x + 1) + 5 - 1 = (x + 1)^2 + 4
\]

At the same time we can rewrite the numerator as \( x + 3 = (x + 1) + 2 \). The integral can then be split into the sum of two integrals

\[
\int \frac{x + 3}{x^2 + 2x + 5} \, dx = \int \frac{x + 1}{(x + 1)^2 + 4} \, dx + \int \frac{2}{(x + 1)^2 + 4} \, dx
\]

Letting \( u = x + 1, \, du = dx \) we obtain:

\[
\int \frac{x + 3}{x^2 + 2x + 5} \, dx = \int \frac{u}{u^2 + 4} \, du + \int \frac{2}{u^2 + 4} \, du
\]

The first integral on the right hand side may be evaluated using the substitution \( v = u^2 + 4 \), \( \frac{1}{2} \, dv = u \, du \) and the second integral may be evaluated using the trigonometric substitution...
\( u = 2 \tan \theta, \ du = 2 \sec^2 \theta \, d\theta \). The sum of these integrals transforms and evaluates as follows:

\[
\int \frac{x + 3}{x^2 + 2x + 5} \, dx = \int \frac{u}{u^2 + 4} + \frac{2}{u^2 + 4} \, du
\]

\[
= \frac{1}{2} \int \frac{dv}{v} + \int d\theta
\]

\[
= \frac{1}{2} \ln |v| + \theta + C
\]

\[
= \frac{1}{2} \ln(u^2 + 4) + \arctan \left( \frac{u}{2} \right) + C
\]

where we used the fact that \( v = u^2 + 4 \) and \( \theta = \arctan (\frac{u}{2}) \) to write our answer in terms of \( u \). We must take it a step further and write our answer in terms of \( x \). We use the fact that \( u = x + 1 \) to obtain:

\[
\int \frac{x + 3}{x^2 + 2x + 5} \, dx = \frac{1}{2} \ln((x + 1)^2 + 4) + \arctan \left( \frac{x + 1}{2} \right) + C
\]

\[
\text{Answer} \quad \int \frac{x + 3}{x^2 + 2x + 5} \, dx = \frac{1}{2} \ln(x^2 + 2x + 5) + \arctan \left( \frac{x + 1}{2} \right) + C
\]
5. Find the arc length of the graph of \( f(x) = \ln x - \frac{x^2}{8} \) from 1 to e.

**Solution:** The arc length formula we will use is

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx
\]

The derivative \( f'(x) \) is

\[
f'(x) = \frac{1}{x} - \frac{x}{4}
\]

Upon adding 1 to the square of \( f'(x) \) we find that the result is a perfect square. The details are outlined below:

\[
1 + f'(x)^2 = 1 + \left( \frac{1}{x} - \frac{x}{4} \right)^2
\]

\[
1 + f'(x)^2 = 1 + \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}
\]

\[
1 + f'(x)^2 = \frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16}
\]

\[
1 + f'(x)^2 = \left( \frac{1}{x} + \frac{x}{4} \right)^2
\]

Therefore, the arc length is

\[
L = \int_1^e \sqrt{1 + [f'(x)]^2} \, dx
\]

\[
L = \int_1^e \left( \frac{1}{x} + \frac{x}{4} \right) \, dx
\]

\[
L = \left[ \ln(x) + \frac{x^2}{8} \right]_1^e
\]

\[
L = \left[ \ln(e) + \frac{e^2}{8} \right] - \left[ \ln(1) + \frac{1^2}{8} \right]
\]

**Answer** \( L = 1 + \frac{1}{8}(e^2 - 1) \)